Materials and Design 194 (2020) 108914

Contents lists available at ScienceDirect

Materials and Design

journal homepage: www.elsevier.com/locate/matdes

EXPRESS ARTICLE

An analytical solution for the correct determination of crack lengths *via* cantilever stiffness

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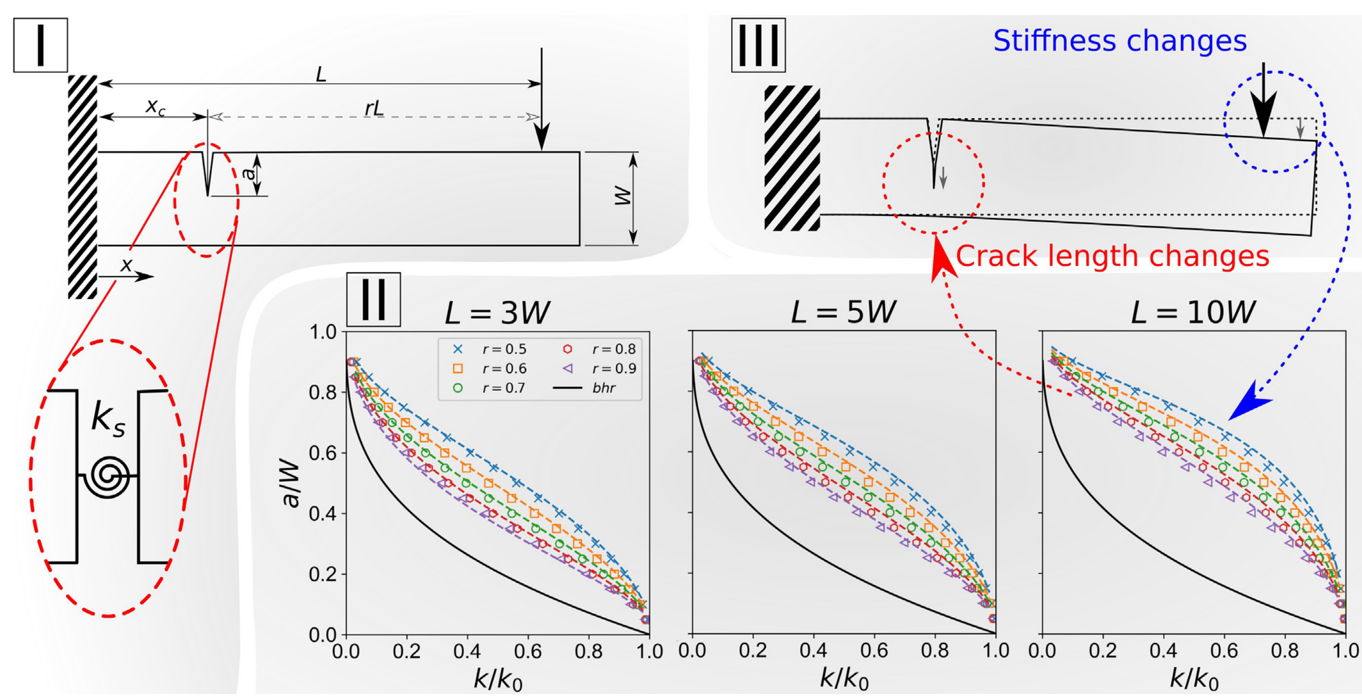
ARTICLE INFO

Article history:
Received 8 May 2020
Received in revised form 10 June 2020
Accepted 20 June 2020
Available online 27 June 2020

Keywords:
Mechanical properties
Fracture
Microcantilever testing
Micromechanical modelling
Analytical methods

ABSTRACT

The present work provides an analytic solution for the stiffness to crack length relation in microscopic cantilever shaped fracture specimens based on classical beam theory and substitution of the crack by a virtual rotational spring element. The resulting compact relationship allows for accounting of the actual beam geometry and agrees very well with accompanying finite element simulations. Compared with the only other model present in literature the proposed relationship reduces the deviation between model and data to a maximum of 1.6% from the previous minimum of 15%. Thus, the novel solution will help to reduce the necessity for individual simulations and aim to increase the comparability of elastic-plastic microcantilever fracture experiments in the future.



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https://doi.org/10.1016/j.matdes.2020.108914
0264-1275/

Most approaches quantify crack extension in an indirect way, through either sequential unloading segments [4] or by an overlaid sinusoidal signal to the applied force [5,6] to measure the change in specimen stiffness. Thereafter, this change in stiffness is used to derive the crack extension. While this seems a trivial elasticity problem and solutions for other geometries are already present in literature [7], for cantilever shaped specimens various different ways were suggested so far. Wurster et al. [4] first assumed a classical Euler-Bernoulli beam with the height being reduced due to the crack extension to describe the stiffness to crack length relation in their experiments. This initial beam height reduction (*bhr*) approach (shown in detail as Supplementary A) gives a straightforward mathematical formulation. However, comparing it to recent results from finite element modelling and *in-situ* experiments it appears to deviate rather distinct from the actual relation between stiffness and crack length, as shown in [6]. The reason for this characteristic is because this analysis results in a globally reduced bending stiffness, whereas the stiffness reduction originating from a sharp crack is of local nature and therefore less pronounced.

Ast et al. [5] later employed finite element modelling for their specific geometry, while Alfreider et al. [6] used a polynomial fit through a wide range of finite element data, validated by experiments on various different materials to ensure a certain degree of geometrical and material independence of their approach. However, the correct physical fundamental translation from experimentally determined stiffness changes to actual crack length is still unknown, therefore giving rise to ambiguity in evaluation of nominally analogous experiments in literature.

To model the realistic situation, a concept in recent works by Biondi and Caddemi [8] as well as Alijani et al. [9] is adopted, where such singularities are addressed analytically through Dirac's delta function $\delta(x)$ as a bending slope discontinuity at the crack position by substitution with a virtual rotational spring k_s, in a two-dimensional Euler-Bernoulli framework, as shown in feature I of the graphical abstract.

The detailed mathematical derivation of the problem can be found as supplementary material (Supplementary B), but the final compact relation states:

$$\int_0^a \frac{a}{W} Y\left(\frac{a}{W}\right)^2 da = \frac{(k_0/k-1)L}{18\pi(1-\nu^2)r^2} \tag{1}$$

where a is the crack length, W and L are geometric parameters shown in the graphical abstract (feature I), k and k_0 are the stiffnesses of the cracked and unnotched beam respectively, ν is Poisson's ratio, $r = (L - x_c) / L$ (with x_c being the offset of the crack from the base) and $Y(a/W)$ is a geometry factor. This factor has previously been calculated for the shown cantilever geometry by various groups, with only slight deviations among each other as shown by Brinckmann et al. [10]. The first term of Eq. (1) cannot be solved analytically in the general case. However, with nowadays computational efficiency it is trivial to compute the integral approximately, *e.g.* trapezoidal rule, for a sensible range of a/W and find the corresponding a by interpolation.

To study the accuracy of the model, it was compared with two-dimensional linear elastic finite element simulations. They were conducted using 4-node plane-stress and plane-strain elements and an isotropic material behaviour with an elastic modulus E_0 = 130 GPa and a Poisson's ratio $\nu = 0.34$. The cantilever base was considered rigid, with a displacement equal to zero, in accordance with the analytical assumptions taken herein. The calculations were conducted for three different cantilever lengths 3W, 5W and 10W with W = 2 μm, while r ranges from 0.5 to 0.9 in increments of 0.1, and a/W spans from 0.05 to 0.9 in 0.05 increments. Thus, a total of 540 different simulations were performed. As expected, no difference in a/W over k/k_0 data was found in comparison between plane stress and plane strain state, respectively. Hence, all the results summarized in feature II of the graphical abstract are shown in plane strain condition. There, the finite element data is depicted by symbols, and the analytical model is shown by the dotted curves in colours corresponding to the given geometries. The continuous black curve depicts the *bhr* model [4]. As shown in Supplementary A the *bhr* model is independent of the cantilever geometry when considered in a normalized manner.

It is evident from the presented data that the proposed analytical model is in very good agreement with the finite element data and the changes in geometry are reflected quite well. To estimate the differences between analytical model and finite element data, the root mean square deviation was calculated for all combinations of r and L, revealing the highest deviation to be 1.6% for $r = 0.5$, $L =$ 10W. In comparison, the *bhr* model would deviate by 15% from the data for $L = 3W$, $r = 0.9$, which represents the minimum discrepancy between finite element data and *bhr* model. Notably, isotropic elasticity was used for convenience. However, due to normalization by the unnotched beam configuration Eq. (1) is independent of elastic properties and therefore, errors originating from elastic anisotropy can be neglected in the given form. In conclusion, the proposed simple and straight forward analytical model describes the observed physical behaviour very well and is recommended to address the stiffness to crack length conversion in the analysis of miniaturized elastic-plastic fracture experiments as schematically depicted in feature III of the graphical abstract.

CRediT authorship contribution statement

Markus Alfreider: Conceptualization, Formal analysis, Writing - original draft. **Stefan Kolitsch:** Software, Writing - review & editing. **Stefan Wurster:** Methodology, Writing - review & editing. **Daniel Kiener:** Supervision, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors gratefully acknowledge the financial support under the scope of the COMET program within the K2 Center “Integrated Computational Material, Process and Product Engineering, IC-MPPE” (Projects A2.12 and A1.24). This program is supported by the Austrian Federal Ministries for Transport, Innovation and Technology (BMVIT) and for Digital and Economic Affairs (BMDW), represented by the Österreichische Forschungsförderungsgesellschaft (Funder ID: 10.13039/501100004955), and the federal states of Styria, Upper Austria, and Tyrol. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 - Research and Innovation Framework Programme (Grant No. 771146 TOUGHIT and No. 757333 SpdTuM).

Data availability statement

The raw/processed data required to reproduce these findings cannot be shared at this time as the data also forms part of an ongoing study.

Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.matdes.2020.108914.

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Supplementary material to:

An analytical solution for the correct determination of crack lengths via cantilever stiffness

DOI: 10.1016/j.matdes.2020.108914

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In this supplementary document, we will provide the derivation of the beam height reduction (*bhr*) model as well as our novel approach. Note that for convenience we will stay in the frame of isotropic elasticity. However, as the final formulation is independent of elastic properties due to the normalization by an unnotched configuration possible errors due to anisotropic crystal structures can be neglected.

Supplementary A: The beam height reduction (*bhr*) approach after Wurster *et al.* [1]

The classical analytical description for the stiffness of an isotropic cantilever, neglecting transverse shear stresses, is the Euler-Bernoulli theory:

$$k = \frac{\Delta P}{\Delta \omega} = \frac{3E_0 I_0}{L^3} = \frac{E_0 B W^3}{4L^3} \quad \text{(A1)}$$

with *B* being the in-plane thickness of the beam and all other variables as defined previously. It is a reasonable first order assumption that a crack of length *a* reduces the initial unnotched beam height *W*, so that virtually a beam of height *W*-*a* remains. Thus, substituting *W* by *W*-*a* and rearranging leads to:

$$a = W - \sqrt[3]{\frac{4kL^3}{BE_0}} \quad \text{(A2)}$$

Furthermore, formulating equation A1 in a normalized manner, *i.e.* k/k_0 results in:

$$\frac{k}{k_0} = \frac{(W-a)^3}{W^3} \tag{A3}$$

which leads to:

$$\frac{a}{W} = 1 - \sqrt[3]{\frac{k}{k_0}} \tag{A4}$$

Thus, when normalized as shown in feature II of the graphical abstract, this approach is independent of the geometric shape of the specimen, which is contradictory to the presented finite element simulations.

Supplementary B:
Detailed derivation of the correlation between crack length and cantilever stiffness

To model the realistic situation, a concept from recent works by Biondi and Caddemi [2] as well as Alijani *et al.* [3] is adopted, where such singularities are addressed analytically through Dirac's delta function $\delta(x)$ as a bending slope discontinuity at the crack position. Applying this to the depicted problem leads to an Euler-Bernoulli beam equation as follows:

$$E_0 I_0 \left(1 - \gamma\delta(x - x_c)\right) \omega''(x) = P(L - x) \tag{B1}$$

where E_0 is the elastic modulus and I_0 the moment of inertia of the unnotched beam, x_c is the position of the crack, ω is the deflection, thus $\omega''(x)$ is the second derivative with respect to x and represents the curvature, and P is the point load acting on the cantilever. Deflection and load are positive in downward direction, as is common for these experiments. The strength of the Dirac delta singularity is described by γ and is a function of the crack length relative to the specimen height a/W for the present problem.

Laplace transformation is a well-known method to address differential mathematical problems that inhibit singularities, e.g. Dirac's delta or step functions, such as the initial mathematical problem of interest herein. Hence, one can start with a transformation of equation B1 as:

$$E_0 I_0\, \boldsymbol{\mathcal{L}}\left(\left(1 - \gamma\delta(x - x_c)\right) \omega''(x)\right) = P\boldsymbol{\mathcal{L}}(L - x) \tag{B2}$$

resulting in:

$$E_0 I_0 [s^2 \boldsymbol{W}(s) - s\omega(0) - \omega'(0) - \gamma\omega''(x_c)e^{-x_c s}] = P\frac{Ls - 1}{s^2} \tag{B3}$$

where s is the complex variable and $\boldsymbol{W}(s) = \boldsymbol{\mathcal{L}}(\omega(x))$ is the Laplace transformation of the deflection. In the classical picture of a beam with an infinitely rigid base it is known that $\omega(0)=0$ and $\omega'(0)=0$. Furthermore, it is assumed that $\omega''(x_c)$ is constant. Thus, one can rearrange equation B3 into:

$$\boldsymbol{W}(s) = \frac{P}{E_0 I_0} \frac{Ls - 1}{s^4} + \gamma \omega''(x_c) \frac{e^{-x_c s}}{s^2} \tag{B4}$$

hence

$$\omega(x) = \boldsymbol{\mathcal{L}}^{-1}\big(\boldsymbol{W}(s)\big) = \frac{P}{E_0 I_0} \frac{3Lx^2 - x^3}{6} + \gamma \omega''(x_c)(L - x_c)\, \sigma(L - x_c) \tag{B5}$$

where $\sigma(x)$ is the Heaviside function, which equals 1 for L-x_c>0 (the only physically reasonable case). As the focus lies solely on the load line displacement, one can evaluate $\omega(L)$ from equation B5, resulting in:

$$\omega(L) = \frac{PL^3}{3EI} + \gamma\, \omega''(x_c)(L - x_c) = \omega_0 + \gamma\, \omega''(x_c)(L - x_c) \tag{B6}$$

It is evident that the deflection is increasing compared to the deflection of the unnotched cantilever ω_0 by a term dependent on the strength of Dirac's delta γ, the curvature at the position of the crack $\omega''(x_c)$, and the lever between load and crack position (L-x_c). Substituting the crack virtually by a discrete rotational spring, as shown schematically in feature I of the graphical abstract, allows a connection between spring stiffness k_s and the strength of the singularity γ [3]:

$$\gamma = \frac{E_0 I_0}{k_s + E_0 I_0 A} \tag{B7}$$

where A=2.013 is a constant value, based on the product of two Dirac delta functions at the same position [2]. The stiffness k_s of this rotational spring is known to be described by [3]:

$$k_s^{-1} = \frac{2B(1 - \nu^2)}{E_0} \int_0^a \left(\frac{K_I}{M}\right)^2 da \tag{B8}$$

where B is the depth of the cantilever in plane, ν is Poisson's ratio, $M=PL$ is the bending moment, and K_I is the stress intensity factor at the notch under mode I loading condition. Given the classical formulation for stress intensity and loading geometry [4,5], one can write:

$$K_I = \sigma\sqrt{\pi a}\, Y\left(\frac{a}{w}\right) = \frac{6PL}{BW^2}\sqrt{\pi a}\, Y\left(\frac{a}{w}\right) \tag{B9}$$

where $Y(a/W)$ is a geometry factor. This has previously been calculated for the shown cantilever geometry by various groups [1,5–7], with only slight deviations from each other as shown by Brinckmann *et. al.* [8]. All calculations shown herein are conducted using the analytical solution for $Y(a/W)$ by Riedl *et al.* [5], which takes into account only bending stresses. Substituting equation B5 into equation B4, and rearranging leads to:

$$k_s^{-1} = \frac{6\pi(1-\nu^2)}{E_0 I_0}\int_0^a \frac{a}{W}\, Y\left(\frac{a}{w}\right)^2 da = \frac{6\pi(1-\nu^2)}{E_0 I_0}\,\frac{1}{G(a,W)} \tag{B10}$$

where $G(a,W)$ is a function of the crack length a and the cantilever height W only, and can be easily evaluated numerically for a given configuration and a sensible range of a/W.

As all of the experimental specimens have slightly different geometries depending on the fabrication route and material features, the stiffness change is commonly not used as an absolute value, but rather as a relative measure, normalized by the stiffness of the unnotched beam k_0:

$$\frac{k}{k_0} = \frac{\omega_0}{\omega(L)} \tag{B11}$$

Thus, substituting equations B6, B7 and B10 into B11 and rearranging leads to:

$$G(a,W) = 6\pi(1-\nu^2)\left(\frac{3r^2}{(k_0/k-1)L} - A\right) \tag{B12}$$

under the assumption of $E_0 I_0 \omega''(x_c)=P(L-x_c)$ and with $r=(L-x_c)/L$.

Notably, the result is a rather compact relation between the stiffness of a cracked cantilever k and the function of its crack length $G(a,W)$ which is only depending on the actual specimen geometry. Furthermore, as A=2.013 is a constant value and L usually in the range of $10^{-4} – 10^{-6}$ m for microcantilever experiments, A does not have a noticeable contribution. For a typical specimen of L=10 µm and r=0.8 the average deviation between equation B8 with and without the term A is 20 ppm. Even up to $L=10^{-3}$ m it is only 0.56% for r=0.9 and increases up to 1.83% for r=0.5, which is already a rather unlikely geometry, thus suggesting that A can be safely neglected for the experiments considered herein. Therefore, equation B12 simplifies to:

$$\int_0^a \frac{a}{W} Y\left(\frac{a}{W}\right)^2 da = \frac{1}{G(a,W)} = \frac{(k_0/k-1)L}{18\pi(1-\nu^2)r^2} \tag{B13}$$

The translation from $G(a,W)$ to the crack length a cannot be solved analytically, as the geometry functions $Y(a/W)$ turn out to be very complex or of higher order polynomial degree, which, to the best of the authors knowledge, makes finding an inverse function impossible. However, with nowadays improved computational efficiency the dependency of a over $G(a,W)$ for a sensible range of a/W can be carried out numerically, for example spanning values from 0.1-0.9, to then conveniently find the corresponding a for a given $G(a,W)$, *i.e.* a given stiffness k, by interpolation.

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