Montanuniversität Leoben

Dissertation

Numerical approximations of nonlinear stochastic partial differential equations appearing in fluid dynamics

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Affidavit

I declare in lieu of oath, that I wrote this thesis and performed the associated research myself, using only literature cited in this volume

Signed:

Date: 05.06.2018

Abstract

In this dissertation, we analyze various discretization of recent mathematical models for turbulent flow modeling. These models share the same complexity. Indeed, they are partial differential, stochastic, and nonlinear equations. By nonlinear, we mean the equations involve terms which are non-globally Lipschitz or/and non-monotone. And stochastic means, we add noise into the model to capture some disturbances which are inherent in nature. These make the model even more realistic.

The results in this work would serve scientist to choose the appropriate numerical methods for their simulations.

In the first part of this dissertation, we consider a stochastic evolution equation in its abstract form. The noise added is a multiplicative noise defined in an infinite Hilbert space. The nonlinear term is non-monotone. Models which fall into this abstract equation are the GOY and Sabra shell models and also nonlinear heat equation, of course in presence of noise. The numerical approximation is based on a semi and fully implicit Euler–Maruyama schemes for the time discretization and a spectral Galerkin method for the space discretization. Our result shows a convergence with rate in probability.

In the second part, we address the very well-known Navier–Stokes equations with an additive noise. A projection method based on the penalized form of the equation is used. We consider only time-discretization since different technicalities appearing after a space-discretization may obscure the main difficulty of the projection method. This method breaks the saddle point character of the Navier–Stokes system which is now a sequence of equations much easier to solve. We show the convergence with rate in probability of the scheme for both variables: velocity and pressure. In addition, we also prove the strong convergence of the scheme.

Zusammenfassung

In dieser Dissertation analysieren wir verschiedene Diskretisierungen neuester mathematischer Modelle für die Modellierung turbulenter Strömungen. Diese Modelle teilen die gleiche Komplexität. Tatsächlich sind sie teilweise differentielle, stochastische und nichtlineare Gleichungen. Mit nichtlinear meinen wir, dass die Gleichungen Terme enthalten, die nicht-global Lipschitz oder / und nicht-monoton sind. Und stochastisch bedeutet, wir fügen Rauschen in das Modell ein, um einige Störungen einzufangen, die der Natur innewohnen. Dies macht das Modell noch realistischer.

Die Ergebnisse dieser Arbeit würden dem Wissenschaftler helfen, die geeigneten numerischen Methoden für ihre Simulationen auszuwählen.

Im ersten Teil dieser Arbeit betrachten wir eine stochastische Evolutionsgleichung in ihrer abstrakten Form. Das hinzugefügte Rauschen ist ein multiplikatives Rauschen, das in einem unendlichen Hilbert-Raum definiert ist. Der nichtlineare Term ist nicht monoton. Modelle, die in diese abstrakte Gleichung fallen, sind die GOY- und Sabra-Schalenmodelle und auch die nichtlineare Wärmeleitungsgleichung, natürlich in Anwesenheit von Rauschen. Die numerische Approximation basiert auf einem halb - und vollständig impliziten Euler - Maruyama - Schema für die Zeitdiskretisierung und einer spektralen Galerkin - Methode für die Raumdiskretisierung. Unser Ergebnis zeigt eine Konvergenz mit der Wahrscheinlichkeitsrate.

Im zweiten Teil werden die sehr bekannten Navier-Stokes-Gleichungen mit additivem Rauschen behandelt. Eine Projektionsmethode basierend auf der bestraften Form der Gleichung wird verwendet. Wir betrachten nur die Zeitdiskretisierung, da verschiedene nach einer Raumdiskretisierung auftretende technische Details die Hauptschwierigkeit der Projektionsmethode verdecken können. Diese Methode bricht den Sattelpunktcharakter des Navier - Stokes - Systems, der jetzt eine viel leichter zu lösende Abfolge von Gleichungen ist. Wir zeigen die Konvergenz mit der Wahrscheinlichkeitsrate des Schemas für beide Variablen: Geschwindigkeit und Druck. Darüber hinaus beweisen wir auch die starke Konvergenz des Systems.

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Abbreviations

DNS	**D**irect **N**umerical **S**imulation
GOY	**G**ledzer **O**khitani **Y**amada
LES	**L**arge **E**ddy **S**imulation
NS	**N**avier **S**tokes **E**quations
PDEs	**P**artial **D**ifferential **E**quations
SNS	**S**tochastic **N**avier **S**tokes **E**quations
SPDEs	**S**tochastic **P**artial **D**ifferential **E**quations
VDTTF	**V**ariable **D**ensity **T**urbulence **T**unnel **F**acility

Chapter 1

Introduction

A fluid flow is usually described by a deterministic PDEs, the *Navier–Stokes equations* (NS). When the fluid is Newtonian, and the flow is incompressible and homogeneous, these equations are defined by the so called incompressible NS

$$\begin{cases} \boldsymbol{v}_t - \nu\Delta\boldsymbol{v} + [\boldsymbol{v}\cdot\nabla]\boldsymbol{v} + \nabla \mathsf{q} = \boldsymbol{f}, \text{ in } \mathbb{R}^d, \\ \operatorname{div}\boldsymbol{v} = 0, \text{ in } \mathbb{R}^d. \end{cases} \tag{0.1}$$

Here $\boldsymbol{v} = \{\boldsymbol{v}(t,\boldsymbol{x}) : t \in [0,T]\}$ and $\mathsf{q} = \{\mathsf{q}(t,\boldsymbol{x}) : t \in [0,T]\}$ are unknown vector fields on $\mathbb{R}^d$, representing, respectively, the velocity and the pressure fields of a fluid with kinematic viscosity ν filling, for instance, the whole space $\mathbb{R}^d$, in each point of $\mathbb{R}^d$ with $d = 2,3$.

We stress that the Navier–Stokes equations are not the only mathematical models used in fluid dynamics. Other models are used depending on the nature of the flow or the property of the fluid. For instance, in rarefied flows when the flow is slower compared with the speed of sound, the Boltzmann equation is often preferred. The Navier–Stokes equation is valid only under the continuum hypothesis. In addition, it suffers from a perpetual competition between the linear diffusion term $\nu\Delta\boldsymbol{v}$ and the nonlinear kinematic term $[\boldsymbol{v}\cdot\nabla]\,\boldsymbol{v}$. This is the reason why problems are still open for the Navier–Stokes equations such as the existence, uniqueness, regularity, and asymptotic behavior of the solution. Nonetheless, it has already proven its worth for the last two centuries. Several applications in physics or engineering have been made thank to the Navier–Stokes equations and many other applications are still based on it. To name but a few, weather forecasting, aeronautic, astrophysics.

However attractive as it is, we cannot cover neither all the mathematical theory of NS nor the beautiful history behind. The reader interested to the mathematical analysis of NS may consult [19, 50, 90, 91] which we will often refer to in this dissertation. These references concern the deterministic NS, that is every noise which may perturb the system is neglected. But at fully developed turbulence, when the inertial effects are dominants compared with the viscous forces, the flow reaches a violent state and considering noise into the system becomes pertinent. Kolmogorov, in a series of papers [64, 65, 66], characterizes this phenomenon by a cascade of energy. This starts from a large scale where the unstable energy is produced and collapses towards smaller scales. This energy is eventually diffused by heat at the smallest scale of the cascade also called *Kolmogorov length scale* and denoted η. This scale depends on the *Reynolds number* denoted $\mathcal{R}e$ which roughly speaking measures the ratio of the inertial effects over the viscous effects in the flow. In fact, $\eta \sim \mathcal{R}e^{-3/4}$.

A direct approach in numerical simulation of turbulence flow requires to solve the NS in a mesh smaller than the scale at which there is no turbulence at all, that is η. This approach is referred to as Direct Numerical Simulation (DNS) and is a well known method in fluid dynamics which consists to solve the NS directly without any turbulence modeling. But, already for moderate Reynolds number $\mathcal{R}e \sim 1000$ we would need to solve the equation in a very thin mesh. Obviously, that would be computationally expensive or time consuming. Due to its cost in terms of computational resources, using DNS is affordable only for research purpose. Engineering applications must rely on turbulence modelings which are computationally cheaper compare to the DNS. These include the Large Eddy Simulation (LES) where only larger scales are considered and the Reynold Averaged Navier–Stokes (RANS) where the flow is decomposed into small and large scales. In the next section, we present two other turbulence modelings, among other: the *Shell models* and the stochastic Navier–Stokes equations (SNS) which is very close the RANS. The first model is derived from the spectral NS and is actually an approximation of this later by retaining with a rigorous way only some wave numbers. Such approximation of the spectral NS has been introduced by Obukhov (1971) and Gledzer (1973). Two examples are briefly described in Section 1.2. In the second model we start with the RANS. Then, the velocity of the flow at small scales is assumed to be a stochastic process. In this last model, Birnir reports in [15, 16] that at fully developed turbulence the random force which governs the flow takes the form of a Lévy noise.

1.1 Stochastic models for fully developed turbulence

There are several turbulence modelings available in the literature at the moment and probably more already used in industry. Two type of models are treated in this dissertation and on which we apply some numerical schemes: the SNS derived from the RANS and the Shell models.

1.1.1 Reynold Averaged Navier–Stokes

Typically, it is common to use the Reynolds decomposition and analyze the flow as two parts: a mean (or average) component $(\boldsymbol{U},\mathsf{P})$ which governs the large scale and a fluctuating component $(\boldsymbol{u},\mathsf{p})$ which governs the small scale. Thus the instantaneous velocity and pressure $(\boldsymbol{v},\mathsf{q})$ can be written as:

$$\boldsymbol{v}=\boldsymbol{U}+\boldsymbol{u}, \qquad \mathsf{q}=\mathsf{P}+\mathsf{p},$$

where the fluctuating velocity and pressure are stochastic processes with vanishing mean.

After substitution of $\boldsymbol{v}$ and q in (0.1) we have

$$\begin{cases} (\boldsymbol{U}+\boldsymbol{u})_t-\nu\Delta(\boldsymbol{U}+\boldsymbol{u})+\operatorname{div}[(\boldsymbol{U}+\boldsymbol{u})\otimes(\boldsymbol{U}+\boldsymbol{u})]+\nabla(\mathsf{P}+\mathsf{p})=\boldsymbol{f}, \text{ in } \mathbb{R}^d, \\ \operatorname{div}(\boldsymbol{U}+\boldsymbol{u})=0, \text{ in } \mathbb{R}^d. \end{cases} \tag{1.2}$$

Since we assume that $\boldsymbol{u}$ is a Gaussian noise with mean denoted by $\overline{\boldsymbol{u}}=0$ and the operation of taking the expectation commutes with differential operators. An equation for the averaged motion, also called the *mean flow*, can be derived by taking the mean of (1.2) which leads to the RANS

$$\begin{cases} \boldsymbol{U}_t - \nu\Delta\boldsymbol{U} + [\boldsymbol{U}\cdot\nabla]\boldsymbol{U} + \nabla\mathsf{P} = \boldsymbol{f} - \operatorname{div}(\overline{\boldsymbol{u}\otimes\boldsymbol{u}}), \text{ in } \mathbb{R}^d, \\ \operatorname{div}\boldsymbol{U} = 0, \text{ in } \mathbb{R}^d. \end{cases} \tag{1.3}$$

Subtracting (1.2) from (1.3) gives an equation for the fluctuating velocity,

$$\begin{cases} \boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + [\boldsymbol{u}\cdot\nabla]\boldsymbol{u} + \nabla\mathsf{p} = -[\boldsymbol{U}\cdot\nabla]\boldsymbol{u} - [\boldsymbol{u}\cdot\nabla]\boldsymbol{U} - \operatorname{div}(\overline{\boldsymbol{u}\otimes\boldsymbol{u}}), \text{ in } \mathbb{R}^d, \\ \operatorname{div}\boldsymbol{u} = 0, \text{ in } \mathbb{R}^d. \end{cases} \tag{1.4}$$

The presence of $\boldsymbol{U}$ in (1.4) characterizes the collapse of energy all the way down to the small scale. While the so called *eddy viscosity* $\operatorname{div}(\overline{\boldsymbol{u}\otimes\boldsymbol{u}})$ in (1.3) represents the force produced from the small scale and acting on the large scale.

Due to the complexity of the flow at fully developed turbulence, simplifications are imposed during the experiment and in theory as well while maintaining as much as possible the main property of a real flow. For instance, in the concept of homogeneous turbulent flow introduced by Taylor [88] (see also [7]), the fluctuating velocity $\boldsymbol{u}$ is statistically homogeneous. This means that the fluctuating velocity is statistically invariant by translation. In practice, this can be easily applied in a Variable Density Turbulence Tunnel Facility or VDTTF [18]. Through a VDTTF, the flow can be adjusted so that no direction are privileged and effects of the boundaries are minimal. An homogeneous turbulent flow is consequently boundless and the mean velocity gradient is spatially uniform [7, 52], i.e. $\nabla\boldsymbol{U} = 0$. This concept implies, among other, that the flow is boundless. However, in experiment, instead of the unbounded space theory we must consider a "turbulent box" which is big enough to capture the integral scales but smaller than the test duct. This leads to the study of a periodic boxed homogeneous fields. Therefore, (1.4) is supplemented with a periodic boundary condition which is even more attractive for mathematical and numerical analysis.

1.1.2 Stochastic Navier–Stokes equations

The SNS has a very long history. The noise added is a term that captures small scales perturbation which is inherent in nature. Different models lead to the stochastic version of the NS including the models developed by Kraichnan [67], Frisch and Lesieur [51], and

Mikulevicius and Rozovskii [59, 75]. In the present subsection, we introduce the model derived recently by Birnir in [14, 16] for fully developed turbulence.

In homogeneous turbulence we can assume that $\nabla \boldsymbol{U} = 0$, and because of the Galilean invariance of the NS, we can restrict the study of (1.4) to the study of the following equation

$$\begin{cases} \boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + [\boldsymbol{u}\cdot\nabla]\boldsymbol{u} + \nabla\mathsf{p} = -\operatorname{div}(\overline{\boldsymbol{u}\otimes\boldsymbol{u}}), \text{ in } \mathbb{R}^d, \\ \operatorname{div}\boldsymbol{u} = 0, \text{ in } \mathbb{R}^d. \end{cases} \tag{1.5}$$

One of the major issues in RANS is to solve the closure problem for the eddy viscosity. Using probability theories, notably the Central Limit Theorem and Large Deviation Principle, Birnir deals with this issue by introducing a stochastic forcing term F defined by

$$d\mathsf{F} = \sum_{k\neq 0}\left[c_k^{1/2} d\beta_t^k + d_k|k|^{1/3}dt + \int_{\mathbb{R}} h_k(t,\boldsymbol{z})\overline{N}^k(dt,d\boldsymbol{z})\right]\mathbf{e}_k(\boldsymbol{x}),$$

where $\mathbf{e}_k(\boldsymbol{x}) = \exp(2\pi i k\boldsymbol{x})$. Here, β_t^k is a standard Brownian motion, $c_k^{1/2}$ and d_k are coefficients that converge sufficiently fast enough to ensure convergence of the entire series. These coefficients are determined to fit the data obtained from experiments [62]. If we let N^k denotes the number of velocity jumps associated to the k-th wave number, $\overline{N}^k$ is the compensated jump, and h_k measures the size of the jump. Hence, instead of (1.5) we consider the stochastic partial differential equations (SPDEs)

$$\begin{cases} d\boldsymbol{u} + [-\nu\Delta\boldsymbol{u} + [\boldsymbol{u}\cdot\nabla]\boldsymbol{u} + \nabla\mathsf{p}]\,dt = d\mathsf{F}, \text{ in } \mathbb{R}^d, \\ \operatorname{div}\boldsymbol{u} = 0, \text{ in } \mathbb{R}^d. \end{cases} \tag{1.6}$$

1.2 Shell models for turbulent flow

In this section we adopt the following formulation of (0.1)

$$\begin{cases} \partial_t v_j - \nu\partial_{kk}v_j + v_k\partial_k v_j + \partial_j\mathsf{q} = f_j, \\ \partial_j v_j = 0, \end{cases} \tag{2.7}$$

where $\partial_j v_k := \partial v_k/\partial x_j, \partial_{jk} v_\ell := \partial^2 v_\ell / \partial x_j \partial x_k$. Moreover, we use the Einstein convention of summing repeated indices; $\partial_{\ell\ell} f := \Delta f$ denotes the Laplacian of f. Assuming the body force f to be rotational, i.e. $\partial_j f_j = 0$, we obtain a Poisson equation for the pressure by applying the divergence operator to the NS

$$\partial_{jj} \mathsf{p} = -\partial_j v_k \partial_k v_j. \tag{2.8}$$

Shell models are based on the Fourier representation of the NS. Thus, it is obvious to define the Fourier transform of the velocity field at $\boldsymbol{\xi}$ by

$$v_j(\boldsymbol{\xi}) = \frac{1}{(2\pi)^3} \int \exp(-i\boldsymbol{\xi} \cdot \boldsymbol{x}) v_j(\boldsymbol{x}) d\boldsymbol{x}, \tag{2.9}$$

where $i = \sqrt{-1}$. Similar to the Fourier representation of NS the time evolution of the shell variables is governed by an infinite system of coupled ordinary differential equations (ODEs).

1.2.1 Spectral Navier–Stokes equation

A Fourier transform of (2.7) and (2.8) gives

$$\begin{cases} \partial_t v_j(\boldsymbol{\xi}) + \nu \xi_k \xi_k v_j(\boldsymbol{\xi}) + i \int v_k(\boldsymbol{\xi} - \boldsymbol{\xi}') v_j(\boldsymbol{\xi}') k_k' d\boldsymbol{\xi}' + i\xi_j \mathsf{q}(\boldsymbol{\xi}) = f_j(\boldsymbol{\xi}), \\ \xi_j' v_j(\boldsymbol{\xi}') = 0, \end{cases} \tag{2.10}$$

and

$$-\xi_k \xi_k \mathsf{q}(\boldsymbol{\xi}) = -\int k_\ell k_m' v_m(\boldsymbol{\xi} - \boldsymbol{\xi}') v_\ell(\boldsymbol{\xi}') d\boldsymbol{\xi}'. \tag{2.11}$$

Inserting (2.11) into (2.10) gives the Fourier representation of NS

$$\begin{cases} \partial_t v_j(\boldsymbol{\xi}) + \nu \xi^2 v_j(\boldsymbol{\xi}) + i\xi_j \int \left(\delta_{jm} - \frac{\xi_i \xi_\ell'}{k^2} \right) v_k(\boldsymbol{\xi}') v_m(\boldsymbol{\xi} - \boldsymbol{\xi}') d\boldsymbol{\xi}' = f_j(\boldsymbol{\xi}), \\ \xi_j' v_j(\boldsymbol{\xi}') = 0. \end{cases} \tag{2.12}$$

where $\xi^2 = \xi_k \xi_k$. In a periodic box $D = [-L, L]^3$, the Fourier transform is substituted by a Fourier series and the integral in (2.12) becomes a sum

$$\partial_t v_j(\boldsymbol{n}) + \nu n^2 v_j(\boldsymbol{n}) + i(2\pi/L) n_k \sum_{\boldsymbol{n}'} \left(\delta_{jm} - \frac{n_j n'_m}{n^2} \right) v_k(\boldsymbol{n}') v_m(\boldsymbol{n} - \boldsymbol{n}') = f_j(\boldsymbol{n}), \tag{2.13}$$

where the wave vectors are $\boldsymbol{\xi}(\boldsymbol{n}) = 2\pi \boldsymbol{n}/L$. As mentioned in the beginning of this Chapter, even for moderate Reynold number the number of waves N necessary to resolve scales larger than η grows with $\mathcal{R}e$ as $N \sim \eta^{-3} \sim \mathcal{R}e^{9/4}$. Some sort of reduction has to be done in practice.

We divide the spectral space into concentric spheres of radii $\xi_n = \lambda^n$, where $\lambda > 1$ is constant.

The set of wave numbers contained in the nth sphere not contained in the $(n-1)$th sphere is called the nth shell, i.e. $\xi_{n-1} < |\boldsymbol{\xi}| < \xi_n$. The equations are

$$\dot{v_n} = a_{n-1} v_{n-1} v_n - a_n v_{n+1}^2 - \nu_n v_n \delta_{n>N} + f \delta_{n,1}, \; n \in \mathbb{N}. \tag{2.14}$$

1.2.2 Gledzer–Okhitani–Yamada model

If only interactions between the first and second neighbor shells are allowed, we obtain the so called GOY model (see [78]),

$$\dot{v}_n = i\xi_n \left(\tilde{a} {v^*}_{n+1} {v^*}_{n+2} + \tilde{b} {v^*}_{n-1} {v^*}_{n+1} + \tilde{c} {v^*}_{n-2} {v^*}_{n-1} \right) - \nu \xi_n^2 v_n + f \delta_{n,1}, \; n \in \mathbb{N}, \tag{2.15}$$

where the body force f acts on the large-scale in order to preserve a statistically stationary dynamical state. The parameters $\tilde{a}, \tilde{b}$, and $\tilde{c}$ are introduced to conserve the energy $E = \sum_n |v_n|^2$, i.e. $\xi_n(\tilde{a} + \tilde{b}\lambda + \tilde{c}\lambda^2) = 0$. The following form of the GOY model can also be found in literature,

$$\dot{v}_n = i\xi_n \left({v^*}_{n+1} {v^*}_{n+2} - \frac{\epsilon}{\lambda} {v^*}_{n-1} {v^*}_{n+1} + \frac{\epsilon - 1}{\lambda^2} {v^*}_{n-2} {v^*}_{n-1} \right) - \nu \xi_n^2 v_n + f \delta_{n,1}, \; n \in \mathbb{N}. \tag{2.16}$$

Therefore, two free parameters define the model, ϵ and λ.

1.2.3 Sabra model

An improvement of the GOY model is proposed in [73] and defined by

$$\dot{v}_n = i\left(\xi_{n+1}v_{n+2}v_{n+1}^* + -\frac{\epsilon}{\lambda}\xi_n v_{n+1}v_{n-1}^* - \frac{\epsilon-1}{\lambda^2}\xi_{n-1}v_{n-1}v_{n-2}\right) - \nu\xi_n^2 v_n + f\delta_{n,4}. \tag{2.17}$$

As in the GOY model, the body force is active only for some small wave numbers. The parameters ϵ and λ are chosen in such a way that the energy is conserved. In addition, (2.17) fulfill the requirement of closing the triads if ξ_n defined a Fibonacci sequence, i.e. $\xi_n = \xi_{n-1} + \xi_{n-2}$.

The aim of this dissertation is to investigate numerical schemes that can be used to solve SPDEs such as (1.6), or (2.16) and (2.17) in presence of noise. Two different approaches are presented for the two group of equations. The first one is based on a fully discretization scheme for the Shell models while the second one is based on a time-discretization scheme which can also be interpreted as a projection method for SNS. In Chapter 2, we treat an abstract and quite general form of (2.16) and (2.17). The convergence with rate of a fully implicit and semi-explicit schemes is proven. In Chapter 3, we present a time-discretization of a stochastic Navier–Stokes equations. An interesting fact of the Navier–Stokes equation is the presence of the pressure which maintains the incompressibility condition. The scheme we use is based on the penalized version of the Navier–Stokes equation and the convergence is obtained for both variables, velocity and pressure.

Chapter 2

Numerical approximation of stochastic evolution equations: Convergence in scale of Hilbert spaces

The present chapter is devoted to the numerical approximation of an abstract stochastic nonlinear evolution equation in a separable Hilbert space H. Examples of equations which fall into our framework include the GOY and Sabra shell models and a class of nonlinear heat equations. The space-time numerical scheme is defined in terms of a Galerkin approximation in space and a semi-implicit Euler–Maruyama scheme in time. We prove the convergence in probability of our scheme by means of an estimate of the error on a localized set of arbitrary large probability. Our error estimate is shown to hold in a more regular space $\mathrm{V}_\beta \subset \mathrm{H}$ with $\beta \in [0, \frac{1}{4})$ and that the explicit rate of convergence of our scheme depends on this parameter β. The results of this chapter will appear in Journal of computational and applied mathematics:

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2.1 Introduction

Throughout this paper we fix a complete filtered probability space $\mathfrak{U}=(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$ with the filtration $\mathbb{F}=\{\mathscr{F}_t;t\geq 0\}$ satisfying the usual conditions. We also fix a separable Hilbert space H equipped with a scalar product $(\cdot,\cdot)$ with the associated norm $|\cdot|$ and another separable Hilbert space $\mathscr{H}$. In this chapter, we analyze numerical approximations for an abstract stochastic evolution equation of the form

$$\begin{cases} d\boldsymbol{u}=-[\mathrm{A}\boldsymbol{u}+\mathrm{B}(\boldsymbol{u},\boldsymbol{u})]dt+G(\boldsymbol{u})dW,\ t\in[0,T], \\ \boldsymbol{u}(0)=\boldsymbol{u}_0, \end{cases} \tag{1.1}$$

where hereafter $T>0$ is a fixed number and A is a self-adjoint positive operators on H. The operators B and G are nonlinear maps satisfying several technical assumptions to be specified later and $W=\{W(t);0\leq t\leq T\}$ is a $\mathscr{H}$-valued Wiener process.

The abstract equation (1.1) can describe several problems from different fields including mathematical finance, electromagnetism, and fluid dynamic. Stochastic models have been widely used to describe small fluctuations or perturbations which arise in nature. For a more exhaustive introduction to the importance of stochastic models and the analysis of stochastic partial differential equations, we refer the reader to [32, 58, 69, 81, 84].

Numerical analysis for stochastic partial differential equations (SPDEs) has known a strong interest in the past decades. Many algorithms which are based on either finite difference or finite element methods or spectral Galerkin methods (for the space discretization) and on either Euler schemes or Crank-Nicholson or Runge-Kutta schemes (for the temporal discretization) have been introduced for both the linear and nonlinear cases and their rate of convergence have been investigated widely. Here we should note that the orders of convergence that are frequently analyzed are the weak and strong orders of convergence. The literature on numerical analysis for SPDEs is now very extensive. Without being exhaustive, we only cite amongst other the recent papers [2, 28, 29, 42, 71], the excellent review paper [61] and references therein. Most of the literature deals with the stochastic heat equations with globally Lipschitz nonlinearities, but there are also several papers that treat abstract stochastic evolution equations. For example, Gyongy and Millet in [57] investigated a general evolution equation with an operator that has the strong monotone and global Lipschitz properties. They were able to implement a space-time discretization and showed a rate of convergence in mean under appropriate assumptions. Similar rate of convergence have been obtained by Bessaih and Schurz in [13] for an equation with globally Lipschitz nonlinearities. When a system of SPDEs with non-globally Lipschitz nonlinearities, such as the stochastic Navier–Stokes equations, is considered the story is completely different. Indeed, in this case the rate of convergence obtained is generally only in probability. This kind of convergence was introduced for the first time by Printems in [82] and is well suited for SPDEs with locally Lipschitz coefficients. When the stochastic perturbation is in an additive form (additive noise), then using a path wise argument one can prove a convergence in mean, we refer to breckner2000galerkin in [20]. Let us mention that in this case, no rate of convergence can be deduced.

Recent literatures involving nonlinear models with nonlinearities which are locally Lipschitz are [10, 22, 30, 44] and references therein. In [22], martingale solutions to the incompressible Navier–Stokes equations with Gaussian multiplicative noise are constructed from a finite element based space-time discretizations. The authors of [30] proved the convergence in probability with rates of an implicit and a semi-implicit numerical schemes by means of a Gronwall argument. The main issue when the term B is not globally Lipschitz lies on its interplay with the stochastic forcing, which prevents a Gronwall argument in the context of expectations. This issue is for example solved in [20, 27] by the introduction of a weight, which when carefully chosen contributes in removing unwanted terms and allows to use Gronwall lemma. In [30], the authors use different approach by computing the error estimates on a sample subset $\Omega_k \subset \Omega$ with large probability. In particular, the set Ω_k is carefully chosen so that the random variables $\|\nabla \boldsymbol{u}^\ell\|_{L^2}$ are bounded as long as the events are taken in Ω_k, and $\lim_{k\searrow 0} \mathbb{P}(\Omega \setminus \Omega_k) = 0$. The result is then obtained using standard arguments based on the Gronwall lemma. Other kinds of numerical algorithms have been used in [10] for a 2D stochastic Navier–Stokes equations. There, a splitting up method has been used and a rate of convergence in probability is obtained. A blending of a splitting scheme and the method of cubature on Wiener space applied to a spectral Galerkin discretisation of degree N is used in [44] to approximate the marginal distribution of the solution of the stochastic Navier–Stokes equations on the two-dimensional torus and rates of convergence are also given. For the numerical analysis of other kind of stochastic nonlinear models that enjoy the local Lipschitz condition, without being exhaustive, we refer to [41, 35, 17, 40] and references therein. They include the stochastic Schrödinder, Burgers, and KdV equations.

In the present chapter, we are interested in the numerical treatment of the abstract stochastic evolution equations (1.1). We first give a simple and short proof of the existence and uniqueness of a mild solution and study the regularity of this solution. The result about the existence of solution is based on a fixed point argument recently developed in [24]. Then, we discretize (1.1) using a coupled Galerkin method and (semi-)implicit Euler scheme and show convergence in probability with rates in $\mathrm{V}_\beta := D(\mathrm{A}^\beta)$. Regarding our approach it is similar to [30] and [82], however, the results are different. Indeed, while [30] and [82] establish their rates of convergence in the space H where the solution lives, we establish our rate of convergence in $\mathrm{V}_\beta \subset \mathrm{H}$ where $\beta \in [0, \frac{1}{4})$ is arbitrary. Hence, our result does not follow from the papers [30] and [82]. In contrast to the nonlinear term of Navier–Stokes equations with periodic boundary condition treated in [30], our nonlinear term does not satisfy the property $\langle \mathrm{B}(\boldsymbol{u},\boldsymbol{u}), \mathrm{A}\boldsymbol{u}\rangle = 0$ which plays a crucial role in the analysis in [30]. We should also point out that our model does not fall into the general framework of the papers [57] and [13], see Remark 2.2.

Examples of semilinear equations which fall into our framework include the GOY and Sabra shell models. These toy models are used to mimic some features of turbulent flows. It seems that our work is the first one rigorously addressing the numerical approximation of such models. Our result also confirm that, in term of numerical analysis, shell models behave far better than the Navier–Stokes equations. On the theoretical point of view, we provide a new and simple proof of the existence of solutions to stochastic shell models driven by Gaussian multiplicative noise. On the physical point of view, it is also worth mentioning that shell models of turbulence are toy models which consist of infinitely many nonlinear differential equations having a structure similar to the Fourier representation of the Navier–Stokes equations, see [43]. Moreover, they capture quite well the statistical properties of three dimensional Navier–Stokes equations, like the Kolmogorov energy spectrum and the intermittency scaling exponents for the high-order structure functions, see [43] and [54]. Due to their success in the study of turbulence, new shell models have been derived by several prominent physicists for the investigation of the turbulence in magnetohydrodynamics, see for instance [79].

Another example of system of equations which falls into our framework is a class of nonlinear heat equations described in Section 2.5. We do not know whether our results can cover the numerical analysis of 1D stochastic nonlinear heat equations driven by additive space-time noise. Despite this fact we believe that our paper is still interesting as we are able to treat a class of 2D stochastic nonlinear heat equations with locally Lipschitz coefficients and we are not aware of results similar to ours. In fact, most of results related to stochastic heat equations are either about 1D model, or d-dimensional, $d\in\{1,2,3\}$, models with globally Lipschitz coefficients and deal with weak convergence or convergence in weaker norm, see for instance [2, 29, 42, 71].

This chapter is organized as follows: in Section 2.2, we introduce the necessary notations and the standing assumptions that will be used in the present work. In Section 2.3, we present our numerical scheme and also discuss the stability and existence of solution at each time step. The convergence of the proposed method is presented in Section 2.4. In Section 2.5 we present the stochastic shell models for turbulence and a class of stochastic nonlinear heat equations as motivating examples.

2.2 Notations, assumptions, preliminary results and the main theorem

In this section we introduce the necessary notations and the standing assumptions that will be used in the present work. We will also introduce our numerical scheme and state our main result.

2.2.1 Assumptions and notations

Throughout this chapter, we fix a separable Hilbert space H with norm $|\cdot|$ and a fixed orthonormal basis $\{\psi_n; n\in\mathbb{N}\}$. We assume that we are given a linear operator $\mathrm{A}: D(\mathrm{A})\subset$

$\mathrm{H}\to\mathrm{H}$ which is a self-adjoint and positive operator such that the fixed orthonormal basis $\{\psi_n; n\in\mathbb{N}\}$ satisfies

$$\{\psi_n; n\in\mathbb{N}\}\subset D(\mathrm{A}),\quad \mathrm{A}\psi_n=\lambda_n\psi_n,$$

for an increasing sequence of positive numbers $\{\lambda_n; n\in\mathbb{N}\}$ with $\lambda_n\to\infty$ as $n\nearrow\infty$. It is clear that $-\mathrm{A}$ is the infinitesimal generator of an analytic semigroup e^{-tA}, $t\geq 0$, on H. For any $\alpha\in\mathbb{R}$ the domain of A^α denoted by $\mathrm{V}_\alpha=D(\mathrm{A}^\alpha)$ is a separable Hilbert space when equipped with the scalar product

$$((\boldsymbol{u},\boldsymbol{v}))_\alpha=\sum_{k=1}^{\infty}\lambda_k^{2\alpha}\boldsymbol{u}_k\boldsymbol{v}_k,\text{ for }\boldsymbol{u},\boldsymbol{v}\in\mathrm{V}_\alpha. \tag{2.2}$$

The norm associated to this scalar product will be denoted by $\|\boldsymbol{u}\|_\alpha$, $\boldsymbol{u}\in\mathrm{V}_\alpha$. In what follows we set $\mathrm{V}:=D(\mathrm{A}^{\frac{1}{2}})$.

Next, we consider a nonlinear map $\mathrm{B}(\cdot,\cdot):\mathrm{V}\times\mathrm{V}\to\mathrm{V}^*$ satisfying the following set of assumptions, where hereafter V^* denotes the dual of the Banach space V.

(**B1**) There exists a constant $C_0>0$ such that for any $\theta\in[0,\frac{1}{2})$ and $\gamma\in(0,\frac{1}{2})$ satisfying $\theta+\gamma\in(0,\frac{1}{2}]$, we have

$$\|\mathrm{B}(\boldsymbol{u},\boldsymbol{v})-\mathrm{B}(\boldsymbol{x},\boldsymbol{y})\|_{-\theta}\leq\begin{cases}C_0\|\boldsymbol{u}-\boldsymbol{x}\|_{\frac{1}{2}-(\theta+\gamma)}(\|\boldsymbol{v}\|_\gamma+\|\boldsymbol{y}\|_\gamma)+\|\boldsymbol{v}-\boldsymbol{y}\|_\gamma(\|\boldsymbol{u}\|_{\frac{1}{2}-(\theta+\gamma)}+\|\boldsymbol{x}\|_{\frac{1}{2}-(\theta+\gamma)})\\ \text{for any }\boldsymbol{u},\boldsymbol{x}\in\mathrm{V}_{\frac{1}{2}-(\theta+\gamma)}\text{ and }\boldsymbol{v},\boldsymbol{y}\in\mathrm{V}_\gamma,\\ C_0(\|\boldsymbol{u}\|_\gamma+\|\boldsymbol{x}\|_\gamma)\|\boldsymbol{v}-\boldsymbol{y}\|_{\frac{1}{2}-(\theta+\gamma)}+\|\boldsymbol{u}-\boldsymbol{x}\|_\gamma(\|\boldsymbol{v}\|_{\frac{1}{2}-(\theta+\gamma)}+\|\boldsymbol{y}\|_{\frac{1}{2}-(\theta+\gamma)})\\ \text{for any }\boldsymbol{v},\boldsymbol{y}\in\mathrm{V}_{\frac{1}{2}-(\theta+\gamma)}\text{ and }\boldsymbol{u},\boldsymbol{x}\in\mathrm{V}_\gamma.\end{cases} \tag{2.3}$$

Due to the continuous embedding $\mathrm{V}_{-\theta}\subset\mathrm{V}_{-\frac{1}{2}}$, $\theta\in[0,\frac{1}{2})$, (2.3) holds with θ and $\frac{1}{2}-(\theta+\gamma)$ respectively replaced by $\frac{1}{2}$ and $\frac{1}{2}-\gamma$ where $\gamma>0$ is arbitrary.

In addition to the above, we assume that for any $\varepsilon>0$ there exists a constant $C>0$ such that

$$|\mathrm{B}(\boldsymbol{u},\boldsymbol{v})|\leq C|\boldsymbol{u}|\|\boldsymbol{v}\|_{\frac{1}{2}+\varepsilon},\text{ for any }\boldsymbol{u}\in\mathrm{H},\boldsymbol{v}\in\mathrm{V}_{\frac{1}{2}+\varepsilon}. \tag{2.4}$$

(**B2**) We assume that for any $\boldsymbol{u},\boldsymbol{v}\in \mathrm{V}$

$$\langle \mathrm{A}\boldsymbol{v}+\boldsymbol{b}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{v}\rangle \geq \|\boldsymbol{v}\|^2_{\frac{1}{2}}. \tag{2.5}$$

(**B3**) We also assume that for any $\boldsymbol{u}\in \mathrm{H}$ we have

$$\mathrm{B}(0,\boldsymbol{u})=\mathrm{B}(\boldsymbol{u},0)=0. \tag{2.6}$$

Note that Assumptions (**B1**) and (**B3**) imply

(**B1**)$'$ There exists a constant $C_0>0$ such that for any numbers $\theta\in[0,\frac{1}{2})$ and $\gamma\in(0,\frac{1}{2})$ satisfying $\theta+\gamma\in(0,\frac{1}{2}]$, we have

$$\|\mathrm{B}(\boldsymbol{u},\boldsymbol{v})\|_{-\theta}\leq C_0\begin{cases}\|\boldsymbol{u}\|_{\frac{1}{2}-(\theta+\gamma)}\|\boldsymbol{v}\|_{\gamma} & \text{for any } \boldsymbol{u}\in \mathrm{V}_{\frac{1}{2}-(\theta+\gamma)} \text{ and } \boldsymbol{v}\in \mathrm{V}_{\gamma},\\ \|\boldsymbol{u}\|_{\gamma}\|\boldsymbol{v}\|_{\frac{1}{2}-(\theta+\gamma)} & \text{for any } \boldsymbol{v}\in \mathrm{V}_{\frac{1}{2}-(\theta+\gamma)}, \text{ and } \boldsymbol{u}\in \mathrm{V}_{\gamma}.\end{cases} \tag{2.7}$$

If $\theta=\frac{1}{2}$, then (2.7) holds with $\frac{1}{2}-(\theta+\gamma)$ replaced by $\frac{1}{2}-\gamma$ where $\gamma>0$ is arbitrary.

Let $\{\mathfrak{w}_j;\, j\in\mathbb{N}\}$ be a sequence of mutually independent and identically distributed standard Brownian motions on $\mathfrak{U}$. Let $\mathscr{H}$ be separable Hilbert space and $\mathscr{L}_1(\mathscr{H})$ be the space of all trace class operators on $\mathscr{H}$. Recall that if $Q\in\mathscr{L}_1(\mathscr{H})$ is a symmetric, positive operator and $\{\varphi_j;j\in\mathbb{N}\}$ is an orthonormal basis of $\mathscr{H}$ consisting of eigenvectors of Q, then the series

$$W(t)=\sum_{j=1}^{\infty}\sqrt{q_j}\mathfrak{w}_j(t)\varphi_j,\quad t\in[0,T],$$

where $\{q_j;\, j\in\mathbb{N}\}$ are the eigenvalues of Q, converges in $L^2(\Omega;C([0,T];\mathscr{H}))$ and it defines an $\mathscr{H}$-valued Wiener process with covariance operator Q. Furthermore, for any positive integer $\ell>0$ there exists a constant $C_\ell>0$ such that

$$\mathbb{E}\|W(t)-W(s)\|^{2\ell}_{\mathscr{H}}\leq C_\ell|t-s|^{\ell}(\mathrm{Tr}Q)^{\ell}, \tag{2.8}$$

for any $t,s\geq 0$ with $t\neq 0$. Before proceeding further we recall few facts about stochastic integral. Let K be a separable Hilbert space, $\mathscr{L}(\mathscr{H},\mathrm{K})$ be the space of all bounded linear

K-valued operators defined on $\mathscr{H}$, $\mathscr{M}_T^2(\mathrm{K})$ be the space of all equivalence classes of $\mathbb{F}$-progressively measurable processes $\Psi:\Omega\times[0,T]\to\mathrm{K}$ satisfying

$$\mathbb{E}\int_0^T \|\Psi(s)\|_{\mathrm{K}}^2 ds < \infty.$$

If $Q\in\mathscr{L}_1(\mathscr{H})$ is a symmetric, positive and trace class operator then $Q^{\frac{1}{2}}\in\mathscr{L}_2(\mathscr{H})$ and for any $\Psi\in\mathscr{L}(\mathscr{H},\mathrm{K})$ we have $\Psi\circ Q^{\frac{1}{2}}\in\mathscr{L}_2(\mathscr{H},\mathrm{K})$, where $\mathscr{L}_2(\mathscr{H},\mathrm{K})$ (with $\mathscr{L}_2(\mathscr{H}):=\mathscr{L}_2(\mathscr{H},\mathscr{H})$) is the Hilbert space of all operators $\Psi\in\mathscr{L}(\mathscr{H},\mathrm{K})$ satisfying

$$\|\Psi\|_{\mathscr{L}_2(\mathscr{H},\mathrm{K})}^2 = \sum_{j=1}^{\infty}\|\Psi\varphi_j\|_{\mathrm{K}}^2 < \infty.$$

Furthermore, from the theory of stochastic integration on infinite dimensional Hilbert space, see [36], for any $\mathscr{L}(\mathscr{H},\mathrm{K})$-valued process Ψ such that $\Psi\circ Q^{1/2}\in\mathscr{M}_T^2(\mathscr{L}_2(\mathscr{H},\mathrm{K}))$ the process M defined by

$$M(t)=\int_0^t \Psi(s)dW(s), t\in[0,T],$$

is a K-valued martingale. Moreover, we have the following Itô's isometry

$$\mathbb{E}\left(\left\|\int_0^t \Psi(s)dW(s)\right\|_{\mathrm{K}}^2\right)=\mathbb{E}\left(\int_0^t \|\Psi(s)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(\mathscr{H},\mathrm{K})}^2 ds\right),\ \forall t\in[0,T],$$

and the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left(\sup_{0\le s\le t}\left\|\int_0^s \Psi(\tau)dW(\tau)\right\|_{\mathrm{K}}^q\right)\le C_q\mathbb{E}\left(\int_0^t \|\Psi(s)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(\mathscr{H},\mathrm{K})}^2 ds\right)^{\frac{q}{2}},\ \forall t\in[0,T],\ \forall q\in(1,\infty).$$

Now, we impose the following set of conditions on the nonlinear term $G(\cdot)$ and the Wiener process W.

(**N**) Let $\mathscr{H}$ be a separable Hilbert space. We assume that the driving noise W is a $\mathscr{H}$-valued Wiener process with a positive and symmetric covariance operator $Q\in\mathscr{L}_1(\mathscr{H})$.

(**G**) We assume that the nonlinear function $G:\mathrm{H}\to\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac{1}{4}})$ is measurable and that there exists a constant $C_1>0$ such that for any $\boldsymbol{u}\in\mathrm{H}$, $\boldsymbol{v}\in\mathrm{H}$ we have

$$\|G(\boldsymbol{u})-G(\boldsymbol{v})\|_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac{1}{4}})}\le C_1|\boldsymbol{u}-\boldsymbol{v}|.$$

Remark 2.1.

(a) *Note that the above assumption implies that* $G:\mathrm{H}\to\mathscr{L}(\mathscr{H},\mathrm{H})$ *is globally Lipschitz and of at most linear growth, i.e, there exists a constant* $C_2>0$ *such that*

$$\|G(\boldsymbol{u})-G(\boldsymbol{v})\|_{\mathscr{L}(\mathscr{H},\mathrm{H})}\le C_2|\boldsymbol{u}-\boldsymbol{v}|,$$
$$|G(\boldsymbol{u})|\le C_2(1+|\boldsymbol{u}|),$$

for any $\boldsymbol{u},\boldsymbol{v}\in\mathrm{H}$.

(b) *There also exists a number* $C_3>0$ *such that*

$$\|G(\boldsymbol{u})-G(\boldsymbol{v})\|_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac{1}{4}})}\le C_3\|\boldsymbol{u}-\boldsymbol{v}\|_{\frac{1}{4}},$$
$$\|G(\boldsymbol{u})\|_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac{1}{4}})}\le C_3(1+\|\boldsymbol{u}\|_{\frac{1}{4}}),$$

for any $\boldsymbol{u},\boldsymbol{v}\in\mathrm{V}_{\frac{1}{4}}$.

(c) *Owing to item (a) of the present remark, if* $\boldsymbol{u}\in\mathscr{M}^2_T(\mathrm{H})$, *then* $G(\boldsymbol{u})\circ Q^{\frac{1}{2}}\in\mathscr{M}^2_T(\mathscr{L}_2(\mathscr{H},\mathrm{H}))$ *and the stochastic integral* $\int_0^t G(\boldsymbol{u}(s))dW(s)$ *is a well defined* H*-valued martingale.*

To close the current subsection we formulate the following remark.

Remark 2.2. *Our assumptions on our problem do not imply the assumptions in neither [57] nor [13]. To justify this claim assume that the coefficient of the noise* G *of our paper and those of [57] and [13] are both zero. Let us now set*

$$A(t,u)=-\mathrm{A}u-B(u,u),$$

which basically corresponds to the drift in both [57] and [13]. For the sake of simplicity we take $\theta=0$ *and* $\gamma=\frac{1}{4}$ *in our assumption* ***(B1)***. *The spaces* H *and* V *in [57] and [13] are respectively* V_0 *and* $\mathrm{V}_{\frac{1}{2}}$ *in our framework. The map* $A(t,u)$ *defined above satisfies*

$$\langle A(t,u)-A(t,v),u-v\rangle\le-|u-v|^2+C_0|u-v|\|u-v\|_{\frac{1}{4}}\left(\|u\|_{\frac{1}{4}}+\|v\|_{\frac{1}{4}}\right).$$

This implies that our assumptions does not imply neither [57, Assumptions 2.1(i) and (2.2)(1)] nor [13, Assumption (H2)].

2.2.2 Preliminary results

In this subsection we recall and derive some results that will be used in the remaining part of the paper. To this end, we first define the notion of solution of (1.1).

Definition 2.3. *An $\mathbb{F}$-adapted process $\boldsymbol{u}$ is called a weak solution of (1.1) (in the sense of PDEs) if the following conditions are satisfied*

(i) $\boldsymbol{u}\in L^2(0,T;\mathrm{V})\cap C([0,T];\mathrm{H})\quad \mathbb{P}$*-a.s.,*

(ii) for every $t\in[0,T]$ *we have* $\mathbb{P}$*-a.s.*

$$(\boldsymbol{u}(t),\phi)=(\boldsymbol{u}_0,\phi)-\int_0^t(\langle \mathrm{A}\boldsymbol{u}(s)+\boldsymbol{b}(\boldsymbol{u}(s),\boldsymbol{u}(s)),\phi\rangle)\,ds+\int_0^t\langle\phi,G(\boldsymbol{u}(s))d\boldsymbol{W}(s)\rangle, \quad (2.9)$$

for any $\phi\in\mathrm{V}$.

Definition 2.4. *An $\mathbb{F}$-adapted process $\boldsymbol{u}\in C([0,T];\mathrm{H})$ $\mathbb{P}$-a.s. is called a mild solution to (1.1) if for every $t\in[0,T]$,*

$$\boldsymbol{u}(t)=e^{-t\mathrm{A}}\boldsymbol{u}_0+\int_0^t e^{-(t-r)\mathrm{A}}\mathrm{B}(\boldsymbol{u}(r),\boldsymbol{u}(r))dr+\int_0^t e^{-(t-r)\mathrm{A}}G(\boldsymbol{u}(r))d\boldsymbol{W}(r),\ \mathbb{P}\text{-}a.s. \quad (2.10)$$

Remark 2.5. *Observe that if $\boldsymbol{u}\in L^2(0,T;\mathrm{V})\cap C([0,T],\mathrm{H})$ is a mild solution to (1.1), then for any $t>s\geq 0$,*

$$\boldsymbol{u}(t)=e^{-(t-s)\mathrm{A}}\boldsymbol{u}(s)+\int_s^t e^{-(t-r)\mathrm{A}}\mathrm{B}(\boldsymbol{u}(r),\boldsymbol{u}(r))dr+\int_s^t e^{-(t-r)\mathrm{A}}G(\boldsymbol{u}(r))dW(r),\ \mathbb{P}\text{-}a.s.$$

In fact, we have

$$\begin{aligned}\boldsymbol{u}(t)=&e^{-(t-s)\mathrm{A}}\left(e^{-s\mathrm{A}}\boldsymbol{u}_0+\int_0^s e^{-(s-r)\mathrm{A}}\mathrm{B}(\boldsymbol{u}(r),\boldsymbol{u}(r))dr+\int_0^s e^{-(s-r)\mathrm{A}}G(\boldsymbol{u}(r))dW(r)\right)\\&+\int_s^t e^{-(t-r)\mathrm{A}}\mathrm{B}(\boldsymbol{u}(r),\boldsymbol{u}(r))dr+\int_s^t e^{-(t-r)\mathrm{A}}G(\boldsymbol{u}(r))dW(r)\\=&e^{-(t-s)\mathrm{A}}\boldsymbol{u}(s)+\int_s^t e^{-(t-r)\mathrm{A}}\mathrm{B}(\boldsymbol{u}(r),\boldsymbol{u}(r))dr+\int_s^t e^{-(t-r)\mathrm{A}}G(\boldsymbol{u}(r))dW(r),\ \mathbb{P}\text{-}a.s.\end{aligned}$$

This remark is used later to prove a very important lemma for our analysis, see Lemma 2.12.

Next, we state and give a short proof of the following results.

Proposition 2.6. *If the assumptions (B1) to (B3) hold and (G) is satisfied with* $\mathrm{V}_{\frac{1}{4}}$ *replaced by* H *and* $\boldsymbol{u}_0 \in L^2(\Omega,\mathrm{H})$*, then the problem* (1.1) *has a unique global mild, which is also a weak, solution* $\boldsymbol{u}$*. Moreover, if* $\boldsymbol{u}_0 \in L^{2p}(\Omega,\mathrm{H})$ *for any real number* $p\in[2,8]$*, then there exists a constant* $\mathcal{C}>0$ *such that*

$$\mathbb{E}\sup_{t\in[0,T]}|\boldsymbol{u}(t)|^{2p}+\mathbb{E}\int_0^T|\boldsymbol{u}(s)|^{2p-2}|\mathrm{A}^{\frac{1}{2}}\boldsymbol{u}(s)|^2ds\leq\mathcal{C}(1+\mathbb{E}|\boldsymbol{u}_0|^{2p}), \tag{2.11}$$

and

$$\mathbb{E}\left(\int_0^T|\mathrm{A}^{\frac{1}{2}}\boldsymbol{u}(s)|^2ds\right)^p\leq\mathcal{C}(1+\mathbb{E}|\boldsymbol{u}_0|^{2p}). \tag{2.12}$$

If, in addition, Assumption (G) is satisfied and $\boldsymbol{u}_0\in L^p(\Omega,\mathrm{V}_{\frac{1}{4}})$ *with* $p\in[2,8]$*, then there exists a constant* $C>0$ *such that*

$$\mathbb{E}\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|^p_{\frac{1}{4}}+\mathbb{E}\left(\int_0^T\|\boldsymbol{u}(s)\|^2_{\frac{3}{4}}ds\right)^p\leq C(1+\mathbb{E}\|\boldsymbol{u}_0\|^p_{\frac{1}{4}}+(\mathbb{E}|\boldsymbol{u}_0|^{2p})^2). \tag{2.13}$$

Proof. Let us first prove the existence of a local mild solution. For this purpose, we study the properties of B in order to apply a contraction principle as in [24, Theorem 3.15]. Let $\mathrm{B}(\cdot)$ be the mapping defined by $\mathrm{B}(\boldsymbol{x})=\mathrm{B}(\boldsymbol{x},\boldsymbol{x})$ for any $\boldsymbol{x}\in\mathrm{V}_\beta$. Let $\beta\in(0,\frac{1}{2})$. Using Assumptions (B1) with $\theta=\frac{1}{2}-\beta$, $\gamma=\beta$, we derive that

$$\|\mathrm{B}(\boldsymbol{x})-\mathrm{B}(\boldsymbol{y})\|_{\beta-\frac{1}{2}}\leq C_0|\boldsymbol{x}-\boldsymbol{y}|(\|\boldsymbol{x}\|_\beta+\|\boldsymbol{y}\|_\beta)+C\|\boldsymbol{x}-\boldsymbol{y}\|_\beta(|\boldsymbol{x}|+|\boldsymbol{y}|), \tag{2.14}$$

for any $\boldsymbol{x},\boldsymbol{y}\in\mathrm{V}_\beta$. Since, by [92, Theorem 1.18.10, pp 141], V_β coincides with the complex interpolation $[\mathrm{H},D(\mathrm{A}^{\frac{1}{2}})]_{2\beta}$, we infer from the interpolation inequality [92, Theorem 1.9.3, pp 59] and (2.14) that

$$\begin{aligned}\|\mathrm{B}(\boldsymbol{x})-\mathrm{B}(\boldsymbol{y})\|_{\beta-\frac{1}{2}}\leq &C_0|\boldsymbol{x}-\boldsymbol{y}|(|\boldsymbol{x}|^{1-2\beta}\|\boldsymbol{x}\|^{2\beta}_{\frac{1}{2}}+|\boldsymbol{y}|^{1-2\beta}\|\boldsymbol{y}\|^{2\beta}_{\frac{1}{2}})\\&+C\|\boldsymbol{x}-\boldsymbol{y}\|^{2\beta}_{\frac{1}{2}}|\boldsymbol{x}-\boldsymbol{y}|^{1-2\beta}(|\boldsymbol{x}|+|\boldsymbol{y}|),\end{aligned} \tag{2.15}$$

for any $\boldsymbol{x},\boldsymbol{y}\in\mathrm{V}$. Now, we denote by X_T the Banach space $C([0,T];\mathrm{H})\cap L^2(0,T;\mathrm{V})$ endowed with the norm

$$\|\boldsymbol{x}\|_{X_T}=\sup_{t\in[0,T]}|\boldsymbol{x}(t)|+\left(\int_0^T\|\boldsymbol{x}(t)\|^2_{\frac{1}{2}}dt\right)^{\frac{1}{2}}.$$

We recall the following classical result, see [37, Theorem 3, pp 520].

$$\text{The linear map } \Lambda : L^2(0,T;\mathrm{V}^*) \ni f \mapsto \boldsymbol{x}(\cdot) = \int_0^{\cdot} e^{-(\cdot - r)\mathrm{A}} f(r)dr \in X_T \text{ is continuous.} \tag{2.16}$$

Thus, thanks to (2.15), (2.16), and Assumption **(G)** we can apply [24, Theorem 3.15] to infer the existence of a unique local mild solution $\boldsymbol{u}$ with lifespan τ of (1.1) (we refer to [24, Definition 3.1] for the definition of local solution). Let $\{\tau_j; j\in\mathbb{N}\}$ be an increasing sequence of stopping times converging almost surely to the lifespan τ. Using the equivalence lemma in [36, Proposition 6.5] we can easily prove that the local mild solution is also a local weak solution satisfying (2.9) with t replaced by $t\wedge\tau_j$, $j\in\mathbb{N}$. Now, we can prove by arguing as in [25, Appendix A] or [27, Proof of Theorem 4.4] that the local solution $\boldsymbol{u}$ satisfies (2.11) uniformly w.r.t. $j\in\mathbb{N}$. With this observation along with an argument similar to [24, Proof of Theorem 2.10] we conclude that (1.1) admits a global solution (*i.e.*, $\tau = T$ a.s.) $\boldsymbol{u}$ satisfying (2.11) and $\boldsymbol{u}\in X_T$ almost-surely.

As mentioned earlier the proof follows a similar argument as in [25, Appendix A], but for the sake of completeness we sketch the proof of (2.11). We apply Itô's formula first to $|\cdot|^2$ and the process $\boldsymbol{u}(\cdot\wedge\tau_j)$ and then to the map $x\to x^p \quad p\geq 2$ and the process $|\boldsymbol{u}(\cdot\wedge\tau_j)|^2$. Then, using the assumption **(B2)** and **(G)** we infer that there exists a constant $\mathcal{C}>0$ such that for any $j\in\mathbb{N}$

$$\sup_{t\in[0,T]} |\boldsymbol{u}(t\wedge\tau_j))|^{2p} + \int_0^T |\boldsymbol{u}(s)|^{2p-2}|\mathrm{A}^{\frac{1}{2}}\boldsymbol{u}(s)|^2 ds \leq \mathcal{C}\mathbb{E}|\boldsymbol{u}_0|^{2p} + \mathcal{C}\int_0^T |\boldsymbol{u}(s\wedge\tau_j)|^{2p-2}|(1+|\boldsymbol{u}(s\wedge\tau_j)|^2)ds$$
$$+ 2p \sup_{t\in[0,T]} \int_0^{t\wedge\tau_j} |\boldsymbol{u}(s)|^{2p-2}\langle \boldsymbol{u}(s), G(\boldsymbol{u}(s))dW(s)\rangle.$$

Using the Burkholder–Holder–Davis inequality we deduce that

$$\mathbb{E}\sup_{t\in[0,T]} \int_0^{t\wedge\tau_j} |\boldsymbol{u}(s)|^{2p-2}\langle G(\boldsymbol{u}(s)), \boldsymbol{u}(s)\rangle dW(s) \leq \mathbb{E}\left(\int_0^T (|\boldsymbol{u}(s\wedge\tau_j)|^{4p} ds\right)^{1/2}$$
$$+\mathbb{E}\left(\int_0^T (|\boldsymbol{u}(s\wedge\tau_j)|^{4p-2} ds\right)^{1/2}.$$

Using Young's inequality, we infer that for any $\epsilon \in (0,\frac{1}{2})$ there exists a constant $C(\epsilon)>0$ such that

$$\mathbb{E}\left(\int_0^T (|\boldsymbol{u}(s\wedge\tau_j)|^{4p}ds\right)^{1/2} \le \epsilon\mathbb{E}\sup_{t\in[0,T]}|\boldsymbol{u}(t\wedge\tau_j))|^{2p}+C(\epsilon)\mathbb{E}\int_0^T\sup_{s\in[0,t]}|\boldsymbol{u}(s\wedge\tau_j)|^{2p}dt.$$

For the second integral, we need to use Hölder's inequality and then Young's inequality and the previous calculations

$$\mathbb{E}\left(\int_0^T (|\boldsymbol{u}(s\wedge\tau_j)|^{4p-2}ds\right)^{1/2} \le \epsilon\mathbb{E}\sup_{t\in[0,T]}|\boldsymbol{u}(t\wedge\tau_j))|^{2p}+C(\epsilon)\mathbb{E}\int_0^T\sup_{s\in[0,t]}|\boldsymbol{u}(s\wedge\tau_j)|^{2p}dt+T^{\frac{1}{2p}}.$$

Now collecting all the estimates we get that

$$\begin{aligned}(1-2\epsilon)\mathbb{E}\sup_{t\in[0,T]}|\boldsymbol{u}(t\wedge\tau_j))|^{2p}+\int_0^T\mathbb{E}|\boldsymbol{u}(s)|^{2p-2}|\mathrm{A}^{\frac{1}{2}}\boldsymbol{u}(s)|^2ds \le& \mathcal{C}(1+\mathbb{E}|\boldsymbol{u}_0|^{2p})\\ &+\mathcal{C}\mathbb{E}\int_0^T\sup_{s\in[0,t]}|\boldsymbol{u}(s\wedge\tau_j)|^{2p}dt.\end{aligned}$$

Now, choosing $\epsilon=\frac{1}{4}$, applying Gronwall's lemma and passing to the limit as $j\to\infty$ complete the proof of (2.11). The estimate (2.12) easily follows from (2.11), so we omit its proof.

We shall now prove the inequality (2.13). To start with, we will apply Itô's formula to $\varphi(\boldsymbol{u})=\|\boldsymbol{u}\|^2_{\frac{1}{4}}$. Note that thanks to the estimates (2.11) and (2.12), Assumptions **(B1)** and **(G)** we readily check that there exists a constant $C>0$ such that

$$\mathbb{E}\int_0^T\left[\|A\boldsymbol{u}+B(\boldsymbol{u},\boldsymbol{u})\|^2_{-\frac{1}{2}}+\|G(\boldsymbol{u})\|^2_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac{1}{4}})}\right](t)dt\le C.$$

Hence the general Itô's formula in [68, Section 3] is applicable to (1.1) and the functional $\varphi(\boldsymbol{u})(t)=\|\boldsymbol{u}(t)\|^2_{\frac{1}{4}}$. Thus, an application of Itô's formula to the functional $\varphi(\boldsymbol{u})(t\wedge\tau_j)=\|\boldsymbol{u}(t\wedge\tau_j)\|^2_{\frac{1}{4}}$ gives

$$\varphi(\boldsymbol{u}(t\wedge\tau_j))=\varphi(\boldsymbol{u}(0))+\int_0^{t\wedge\tau_j}\varphi\prime(\boldsymbol{u}(s))d\boldsymbol{u}(s)+\frac{1}{2}\int_0^{t\wedge\tau_j}\mathrm{Tr}\left(\varphi''(\boldsymbol{u}(s))G(\boldsymbol{u}(s))Q(G\boldsymbol{u}(s))^*\right)ds,$$

which along with the inequality $\frac{1}{2}\|\varphi''(\boldsymbol{u})\|\le 1$, where the norm is understood as the norm of a bilinear map, implies

$$
\begin{aligned}
\|\boldsymbol{u}(t\wedge\tau_j)\|^2_{\frac14}+2\int_0^{t\wedge\tau_j}&\Big(\|\boldsymbol{u}(s)\|^2_{\frac34}+2\langle \mathrm{A}^{\frac12}\boldsymbol{u}(s),\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))\rangle\Big)ds\\
&\le \|\boldsymbol{u}_0\|^2_{\frac14}+2\int_0^{t\wedge\tau_j}\langle \mathrm{A}^{\frac12}\boldsymbol{u}(s),G(\boldsymbol{u}(s))dW(s)\rangle \\
&\quad +C\,\mathrm{Tr}Q\int_0^{t\wedge\tau_j}\|G(\boldsymbol{u}(s))\|^2_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac14})}ds.
\end{aligned}
\tag{2.17}
$$

Since the embedding $\mathrm{V}_{\frac12+\alpha}\subset \mathrm{V}_{2\alpha}$ is continuous for any $\alpha\in[0,\frac12]$, we can use Assumptions **(B1)**$'$ and the Cauchy inequality to infer that

$$
\begin{aligned}
\left|\int_0^{t\wedge\tau_j}\langle \mathrm{A}^{\frac12}\boldsymbol{u}(s),\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))\rangle ds\right| &\le C\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|_{\frac12}|\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))|ds\\
&\le \frac12\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|^2_{\frac34}ds+C\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|^2_{\frac12-\gamma}\|\boldsymbol{u}(s)\|^2_{\gamma}ds,
\end{aligned}
$$

for some $\gamma\in(0,\frac12)$. From an application of a complex interpolation inequality, see [92, Theorem 1.9.3, pp 59], we infer that

$$
\left|\int_0^{T}\langle \mathrm{A}^{\frac12}\boldsymbol{u}(s),\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))\rangle ds\right| \le \frac12\int_0^{T}\|\boldsymbol{u}(s)\|^2_{\frac34}ds+\int_0^{T}|\boldsymbol{u}(s)|^2\|\boldsymbol{u}(s)\|^2_{\frac12}ds.
$$

Plugging the latter inequality into (2.17), using the assumption on G we obtain

$$
\begin{aligned}
\|\boldsymbol{u}(t\wedge\tau_j)\|^2_{\frac14}+\frac32\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|^2_{\frac34}ds &\le \|\boldsymbol{u}(0)\|^2_{\frac14}+C\sup_{s\in[0,T]}|\boldsymbol{u}(s)|^2\int_0^T\|\boldsymbol{u}(s)\|^2_{\frac12}ds\\
&+CT+C\int_0^T\|\boldsymbol{u}(s)\|^2_{\frac14}ds+2\left|\int_0^{t\wedge\tau_j}\langle \mathrm{A}^{\frac14}\boldsymbol{u}(s),\mathrm{A}^{\frac14}G(\boldsymbol{u}(s))dW(s)\rangle\right|.
\end{aligned}
\tag{2.18}
$$

Taking the supremum over $t\in[0,T]$, then raising both sides of the resulting inequality to the power $p/2$, taking the mathematical expectation, and finally using the Burkholder–Davis–Gundy inequality yield

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\|\boldsymbol{u}(s\wedge\tau_j)\|^p_{\frac14}+2\mathbb{E}\left(\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|^2_{\frac34}ds\right)^{p/2}\\
\leq&\left(C\mathbb{E}\|\boldsymbol{u}(0)\|^p_{\frac14}+CT+C\mathbb{E}\left[\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|^2_{\frac14}ds\right]^{\frac{p}{2}}\right)\\
&+C\left(\mathbb{E}\sup_{s\in[0,T]}|\boldsymbol{u}(s)|^{2p}\right)^{\frac12}\left[\mathbb{E}\left(\int_0^T\|\boldsymbol{u}(s)\|^2_{\frac12}ds\right)^p\right]^{\frac12}\\
&+2C\mathbb{E}\left(\int_0^{t\wedge\tau_j}|\mathrm{A}^{\frac14}\boldsymbol{u}(s)|^2\|G(\boldsymbol{u}(s))\|^2_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac14})}ds\right)^{\frac{p}{4}}.
\end{aligned}
\tag{2.19}
$$

Here we have used the fact that for any integer ℓ and n we can find a constant $C_{\ell,n}$ such that

$$
\sum_{i=1}^n a_i^\ell\leq\left(\sum_{i=1}^n a_i\right)^\ell\leq C_{\ell,n}\sum_{i=1}^n a_i^\ell \tag{2.20}
$$

for a sequence of non-negative numbers $\{a_i; i=1,2,\dots,n\}$.

Using the assumptions on G and Young's inequality we infer that there exists a constant $C>0$ such that for any $j\in\mathbb{N}$

$$
\begin{aligned}
\mathbb{E}\left(\int_0^{t\wedge\tau_j}|\mathrm{A}^{\frac14}\boldsymbol{u}(s)|^2\|G(\boldsymbol{u}(s))\|^2_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac14})}ds\right)^{\frac{p}{4}}\leq CT+\frac14\mathbb{E}\sup_{s\in[0,t]}\|\boldsymbol{u}(s\wedge\tau_j)\|^p_{\frac14}\\
+C\mathbb{E}\left[\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|^2_{\frac14}ds\right]^{\frac{p}{2}},
\end{aligned}
$$

which along with (2.19), (2.11), and (2.12) implies

$$
\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]}\|\boldsymbol{u}(s\wedge\tau_j)\|^p_{\frac14}+2\mathbb{E}\left(\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|^2_{\frac34}ds\right)^{\frac{p}{2}}\leq\mathbb{E}\|\boldsymbol{u}(0)\|^p_{\frac14}+\mathcal{C}^2(1+\mathbb{E}|\boldsymbol{u}_0|^{2p})^2+CT\\
+\mathbb{E}\left[\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|^2_{\frac14}ds\right]^{\frac{p}{2}}.
\end{aligned}
$$

Now, we infer from the interpolation inequality [92, Theorem 1.9.3, pp 59], (2.11) and (2.12) that there exists a constant $C>0$ such that for any $j\in\mathbb{N}$

$$
\mathbb{E}\left[\int_0^{t\wedge\tau_j}\|\boldsymbol{u}(s)\|^2_{\frac14}ds\right]^{\frac{p}{2}}\leq T^{\frac{p}{2}}\mathbb{E}\left(\sup_{s\in[0,T]}|\boldsymbol{u}(s)|^{\frac{p}{2}}\left[\int_0^T\|\boldsymbol{u}(s)\|^2ds\right]^{\frac{p}{4}}\right)\leq CT.
$$

Hence,

$$\mathbb{E}\sup_{s\in[0,t]}\|\boldsymbol{u}(s\wedge\tau_j)\|_{\frac{1}{4}}^p \leq C_T(1+\mathbb{E}\|\boldsymbol{u}(0)\|_{\frac{1}{4}}^p+(\mathbb{E}|\boldsymbol{u}_0|^{2p})^2),$$

from which along with a passage to the limit we readily complete the proof of the proposition.

□

2.2.3 The numerical scheme and the main result

Let N be a positive integer, $\mathrm{H}_N\subset\mathrm{H}$ the linear space spanned by $\{\psi_n; n=1,\dots,N\}$, and $\pi_N:\mathrm{H}\to\mathrm{H}_N$ the orthogonal projection of H onto the finite dimensional subspace H_N. The projection of $\boldsymbol{u}$ by π_N is denoted by

$$\boldsymbol{u}^N:=\pi_N\boldsymbol{u}=\sum_{n=1}^N(\psi_n,\boldsymbol{u})\psi_n, \tag{2.21}$$

for $\boldsymbol{u}\in\mathrm{H}$. The Galerkin approximation of the SPDEs (1.1) reads

$$d\boldsymbol{u}^N=[\pi_N\mathrm{A}\boldsymbol{u}^N+\pi_N\mathrm{B}(\boldsymbol{u}^N,\boldsymbol{u}^N)]dt+\pi_N G(\boldsymbol{u}^N)dW(t),\quad \boldsymbol{u}^N(0)=\pi_N\boldsymbol{u}_0. \tag{2.22}$$

Due to the assumptions **(B1)**-**(B3)** and **(G)**, we can use Proposition 2.6 to prove that (2.22) has a global weak solution.

To derive an approximation of the exact solution $\boldsymbol{u}$ of (1.1) we construct an approximation U^j of the Galerkin solution $\boldsymbol{u}^N$. To this end, let M be a positive integer and $I_M=([t_m,t_{m+1}))_{m=0}^M$ an equidistant grid of mesh-size $k=t_{m+1}-t_m$ covering $[0,T]$. Now, for any $j\in\{0,\dots,M-1\}$ we look for a sequence of $\mathbb{F}$-adapted random variables $\mathrm{U}^j\in\mathrm{H}_N$, $j=0,1,\dots,M$ such that for any $w\in\mathrm{V}$

$$\begin{cases}\mathrm{U}^0=\pi_N\boldsymbol{u}_0,\\ \langle\mathrm{U}^{j+1}-\mathrm{U}^j+k[\pi_N\mathrm{A}\mathrm{U}^{j+1}+\pi_N\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1})],\,w\rangle=\langle w,\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle,\end{cases} \tag{2.23}$$

where $\Delta_{j+1}W:=W(t_{j+1})-W(t_j)$, $j\in\{0,\dots,M-1\}$, is an independent and identically distributed random variables. We will justify in the following proposition that for a given

$U_0 = \pi_N \boldsymbol{u}_0$ the numerical scheme (2.23) admits at least one solution $U^j \in \mathrm{H}_N$, $j \in \{1,\dots,M\}$ and that (2.23) is stable in H and $D(\mathrm{A}^{\frac{1}{4}})$.

Proposition 2.7. *Let the assumptions* ***(B1)-(B3)*** *and* ***(G)*** *hold. Let N and M be two fixed positive integers and $\boldsymbol{u}_0 \in L^{2^p}(\Omega;\mathrm{H})$ for any integer $p \in [2,4]$. Then, for any $j \in \{1,\dots,M\}$ there exists at least a $\mathcal{F}_{t_j}$-measurable random variable $\mathrm{U}^j \in \mathrm{H}_N$ satisfying (2.23). Moreover, there exists a constant $C>0$ (depending only on T and $\mathrm{Tr}Q$) such that*

$$\mathbb{E}\max_{0\leq m\leq M}|\mathrm{U}^m|^2 + \sum_{j=0}^{M-1}|\mathrm{U}^{j+1}-\mathrm{U}^j|^2 + 2k\mathbb{E}\sum_{j=1}^{M}\|\mathrm{U}^j\|_{\frac{1}{2}}^2 \leq C(\mathbb{E}|\boldsymbol{u}_0|^2+1), \tag{2.24}$$

$$\mathbb{E}\left[\max_{1\leq m\leq M}|\mathrm{U}^m|^{2^p} + k\sum_{j=1}^{M}|\mathrm{U}^j|^{2^{p-1}}\|\mathrm{U}^j\|^2_{\frac{1}{2}}\right] \leq C(1+\mathbb{E}|\boldsymbol{u}_0|^{2^{p-1}}), \tag{2.25}$$

and

$$\mathbb{E}\left[k\sum_{j=1}^{M}\|\mathrm{U}^j\|^2_{\frac{1}{2}}\right]^{2^{p-1}} \leq C(1+\mathbb{E}|\boldsymbol{u}_0|^{2^p}). \tag{2.26}$$

Furthermore, if $\boldsymbol{u}_0 \in L^8(\Omega, D(\mathrm{A}^{\frac{1}{4}}))$, then there exists a constant $C>0$ such that

$$\mathbb{E}\max_{1\leq m\leq M}\|\mathrm{U}^m\|_{\frac{1}{4}}^2 + \mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|_{\frac{1}{4}}^2 + k\mathbb{E}\sum_{j=1}^{M}\|\mathrm{U}^j\|_{\frac{3}{4}}^2 \leq C, \tag{2.27}$$

and

$$\mathbb{E}\max_{1\leq m\leq M}\|\mathrm{U}^m\|_{\frac{1}{4}}^4 + \mathbb{E}\left(\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|_{\frac{1}{4}}^2\right)^2 + k^2\mathbb{E}\left(\sum_{j=1}^{M}\|\mathrm{U}^j\|_{\frac{3}{4}}^2\right)^2 \leq C \tag{2.28}$$

Proof. The detailed proofs of the existence, measurability, and the estimates (2.27) and (2.28) will be given in Section 2.3. Thanks to the assumption (**B2**), the proof of the inequalities (2.24)-(2.26) is very similar to the proof of [30], so we omit it. □

We should note that the estimates (2.27) and (2.28) hold even if $\boldsymbol{u}_0 \in L^4(\Omega, D(\mathrm{A}^{\frac{1}{4}}))$, but for the sake of consistency we take $\boldsymbol{u}_0 \in L^8(\Omega, D(\mathrm{A}^{\frac{1}{4}}))$.

Now, we proceed to the statement of the main result of this paper.

Theorem 2.8. *Let the assumptions (B1)-(B3) and (G) hold and assume that* $\boldsymbol{u}_0 \in L^{16}(\Omega;\mathrm{H}) \cap L^8(\Omega;\mathrm{V}_{\frac{1}{4}})$. *Then for any* $\beta \in [0,\frac{1}{4})$, *there exists a constant* $\mathrm{k}_0 > 0$ *such that for any small number* $\varepsilon > 0$ *we have*

$$\begin{aligned} &\max_{1\le j\le M} \mathbb{E}\left(\mathbf{1}_{\Omega_k} \|\boldsymbol{u}(t_j) - \mathrm{U}^j\|_\beta^2\right) \\ &\qquad + 2k\mathbb{E}\left(\mathbb{1}_{\Omega_k} \sum_{j=1}^M \|\boldsymbol{u}(t_j) - \mathrm{U}^j\|^2_{\frac{1}{2}+\beta}\right) < \mathrm{k}_0 k^{-2\varepsilon}[k^{2(\frac{1}{4}-\beta)} + \lambda_N^{-2(\frac{1}{4}-\beta)}], \end{aligned} \tag{2.29}$$

where the set Ω_k *is defined by*

$$\Omega_k = \left\{\omega\in\Omega:\ \sup_{t\in[0,T]} \|\boldsymbol{u}(t,\omega)\|^2_{\frac{1}{4}} < \log k^{-\varepsilon},\ \max_{0\le j\le M} \|\mathrm{U}^j(\omega)\|^2_{\frac{1}{4}} < \log k^{-\varepsilon}\right\}.$$

Proof. The proof of this theorem will be given in Section 2.4. □

Remark 2.9. *Note that owing to* (2.13) *and* (2.28) *and the Markov inequality it is not difficult to prove that the set* Ω_k *satisfies*

$$\lim_{k\searrow 0} \mathbb{P}[\Omega\backslash\Omega_k] = 0.$$

Corollary 2.10. *If all the assumptions of Theorem 2.8 are satisfied, then the solution* $\{\mathrm{U}^j;\, j=1,2,\dots,M\}$ *of the numerical scheme* (2.23) *converges in probability in the Hilbert space* V_β, $\beta\in[0,\frac{1}{4})$. *More precisely, for any small number* $\varepsilon > 0$, *any* $\theta_0 \in \left(0, \frac{1}{4} - \beta - \varepsilon\right)$ *and* $\theta_1 \in (0, \frac{1}{4} - \beta)$ *we have*

$$\lim_{\Theta\nearrow\infty} \lim_{k\searrow 0} \lim_{N\nearrow\infty} \max_{1\le j\le M} \mathbb{P}\left(\|\boldsymbol{u}(t_j) - \mathrm{U}^j\|_\beta + k^{\frac{1}{2}}\left(\sum_{j=1}^M \|\boldsymbol{u}(t_j) - \mathrm{U}^j\|^2_{\frac{1}{2}+\beta}\right)^{\frac{1}{2}} \ge \Theta[k^{\theta_0} + \Lambda_N^{-\theta_1}]\right) = 0. \tag{2.30}$$

Proof. To shorten notation let us set $\mathbf{e}^j := \boldsymbol{u}(t_j) - \mathrm{U}^j$ and

$$\Omega^\Theta_{k,N} = \{\omega\in\Omega;\ \|\mathbf{e}^j\|^2_\beta + k\sum_{j=1}^M \|\mathbf{e}^j\|^2_{\frac{1}{2}+\beta} \ge \Theta[k^{\theta_0} + \Lambda_N^{-\theta_1}]\},$$

for any positive numbers M and k. Let Ω_k be as in the statement of Theorem 2.8. Owing to (2.29), (2.13), (2.28) and the Chebychev-Markov inequality, we can find a constant $\tilde{C}_5 > 0$

such that

$$\begin{aligned}\mathbb{P}\left(\Omega^{\Theta}_{k,N}\right) &= \mathbb{P}(\Omega^{\Theta}_{k,N}\cap\Omega_k)+\mathbb{P}(\Omega^{\Theta}_{k,N}\cap\Omega^{c}_k)\\ &\le \mathbb{P}(\Omega^{\Theta}_{k,N}\cap\Omega_k)+\mathbb{P}(\Omega^{c}_k)\\ &\le \frac{\mathrm{k}_0}{\Theta}k^{2(\frac14-\beta)-2\varepsilon-2\theta_0}+\frac{\mathrm{k}_0}{\Theta}k^{-2\varepsilon}\lambda_N^{-2(\frac14-\beta)+2\theta_1}+\frac{\tilde{C}_5}{\log k^{-\varepsilon}}.\end{aligned}$$

Letting $N\nearrow\infty$, then $k\searrow 0$, and finally $\Theta\nearrow\infty$ in the last line we easily conclude the proof of the corollary. □

To close this section let us make some few remarks. Instead of the scheme (2.23) we could also use a fully-implicit scheme. More precisely, for any $j\in\{0,\dots,M-1\}$ we look for a $\mathcal{F}_{t_j}$-measurable random variable $\mathcal{U}^j\in\mathrm{H}_N$ such that for any $w\in\mathrm{V}$

$$\begin{cases}\mathcal{U}^0=\pi_N\boldsymbol{u}_0,\\ \langle\mathcal{U}^{j+1}-\mathcal{U}^j+k[\pi_N\mathrm{A}\mathcal{U}^{j+1}+\pi_N\mathrm{B}(\mathcal{U}^{j+1},\mathcal{U}^{j+1})],\,w\rangle=\langle w,\pi_N G(\mathcal{U}^j)\Delta_{j+1}W\rangle,\end{cases}\tag{2.31}$$

where $\Delta_{j+1}W:=W(t_{j+1})-W(t_j)$, $j\in\{0,\dots,M-1\}$. We have the following theorem:

Theorem 2.11. *Let the assumptions (B1)-(B3) and (G) hold and assume that* $\boldsymbol{u}_0\in L^{16}(\Omega;\mathrm{H})\cap L^{8}(\Omega;\mathrm{V}_{\frac14})$. *Let N and M be two fixed positive integers. Then,*

(a) for any $j\in\{0,\dots,M-1\}$ there exists a unique $\mathcal{F}_{t_j}$-measurable random variable $\mathcal{U}^j\in\mathrm{H}_N$ satisfying (2.31) *and the estimates* (2.24) *and* (2.28).

(b) For any $\beta\in[0,\frac14)$ there exists a constant $\mathrm{k}_0>0$ such that for any small number $\varepsilon>0$ we have

$$\max_{1\le j\le M}\mathbb{E}\left(\mathbf{1}_{\Omega_k}\|\boldsymbol{u}(t_j)-\mathcal{U}^j\|^2_{\beta}\right)+2k\mathbb{E}\left(\mathbb{1}_{\Omega_k}\sum_{j=1}^{M}\|\boldsymbol{u}(t_j)-\mathcal{U}^j\|^2_{\frac12+\beta}\right)<\mathrm{k}_0k^{-2\varepsilon}[k^{2(\frac14-\beta)}+\lambda_N^{-2(\frac14-\beta)}],$$

where

$$\Omega_k=\left\{\omega:\ \sup_{t\in[0,T]}\|\boldsymbol{u}(t,\omega)\|^2_{\frac14}<\log k^{-\varepsilon},\ \max_{0\le j\le M}\|\mathcal{U}^j(\omega)\|^2_{\frac14}<\log k^{-\varepsilon}\right\}.$$

(c) Moreover, for any small number $\varepsilon>0$, any $\theta_0\in\left(0,\frac{1}{4}-\beta-\varepsilon\right)$ and $\theta_1\in(0,\frac{1}{4}-\beta)$

$$\lim_{\Theta\nearrow\infty}\lim_{k\searrow 0}\lim_{N\nearrow\infty}\max_{1\le j\le M}\mathbb{P}\left(\|\boldsymbol{u}(t_j)-\mathcal{U}^j\|_\beta^2+k^{\frac{1}{2}}\left(\sum_{j=1}^{M}\|\boldsymbol{u}(t_j)-\mathcal{U}^j\|^2_{\frac{1}{2}+\beta}\right)^{\frac{1}{2}}\ge\Theta[k^{\theta_0}+\lambda_N^{-\theta_1}]\right)=0.$$

Proof. The arguments for the proof of this theorem are very similar to those of the proofs of Proposition 2.7, Theorem 2.8, and Corollary 2.10, thus we omit them. □

2.3 Existence and stability analysis of the scheme: Proof of Proposition 2.7

In this section, we will show that for any $j\in\{0,\dots,M-1\}$ the numerical scheme (2.23) admits at least one solution $\mathrm{U}^j\in\mathrm{H}_N$. We will also show that (2.23) is stable in $D(\mathrm{A}^{\frac{1}{4}})$, see Proposition 2.7 for more precision.

Proof of Proposition 2.7. As we mentioned in Subsection 2.2.3 we will only prove the existence, measurability, and the estimates (2.27) and (2.28). The proof of the inequalities (2.24)-(2.26) will be omitted because it is very similar to the proof of [30] (see also [22]).

Proof of the existence. We first establish that for any $j\in\{0,\dots,M-1\}$ there exists $\mathrm{U}^j\in\mathrm{H}_N$ satisfying the numerical scheme (2.23). To this end, let us fix $\omega\in\Omega$ and for a given $\mathrm{U}^j\in\mathrm{H}_N$ consider the map $\Lambda^j_\omega:\mathrm{H}_N\to\mathrm{H}_N$ defined by

$$\langle\Lambda^j_\omega(\boldsymbol{v}),\psi\rangle=\langle\boldsymbol{v}-\mathrm{U}^j(\omega),\psi\rangle+k\langle\mathrm{A}\boldsymbol{v}+\pi_N\mathrm{B}(\mathrm{U}^j(\omega),\boldsymbol{v}),\psi\rangle-\langle\psi,\pi_N G(\mathrm{U}^j(\omega))\Delta_{j+1}W(\omega)\rangle$$

for any $\psi\in\mathrm{H}_N$. Note that since $\mathrm{H}_N\subset D(\mathrm{A})$ the map Λ^j_ω is well-defined. From assumptions **(B1)** and **(G)** and the linearity of A, it is clear that for given U^j the map Λ^j_ω is continuous. Furthermore, using Hölder's inequality, the fact that $\lambda_1|\psi|^2\le\|\psi\|^2_{\frac{1}{2}}$, $\psi\in\mathrm{V}$ and assumptions **(B2)** and **(G)** we derive that

$$\begin{aligned}\langle\Lambda^j_\omega\boldsymbol{v},\boldsymbol{v}\rangle&\ge|\boldsymbol{v}|^2\left(\lambda_1 k+\frac{1}{2}-\frac{k}{2}\right)-\frac{|\mathrm{U}^j(\omega)|^2}{2}\left(1+\|\Delta_{j+1}W(\omega)\|^2_{\mathscr{H}}C_2^2\right)-\frac{1}{2}\|\Delta_{j+1}W(\omega)\|^2_{\mathscr{H}}C_2^2\\&\ge\gamma|\boldsymbol{v}|^2-\Gamma^j_\omega.\end{aligned}$$

Since $k<1$, and by Assumption (**N**), $\|\Delta_{j+1}W\|^2_{\mathscr{H}}<\infty$, the constant γ is positive and $\mu_j=\sqrt{\frac{\Gamma^j_\omega}{\gamma}}<\infty$ whenever $|\mathrm{U}^j|^2<\infty$. Thus, we have $\langle\Lambda^j_\omega\boldsymbol{v},\boldsymbol{v}\rangle\geq 0$ for any $\boldsymbol{v}\in\mathcal{H}^j_N(\omega):=\{\psi\in\mathrm{H}_n;\,|\psi|=R\mu_j\}$ where $R>1$ is an arbitrary constant. Since $\mathrm{U}^0=\pi_N\boldsymbol{u}_0$ is given, we can conclude from the above observations and Brouwer fixed point theorem that there exists at least one $\mathrm{U}^1\in\mathrm{H}_N$ satisfying

$$\Lambda^0_\omega(\mathrm{U}^1)=0 \text{ and } |\mathrm{U}^1|\leq R\mu_0.$$

In a similar way, assuming that $\mathrm{U}^j\in\mathrm{H}_N$, we infer that there exists at least one $\mathrm{U}^{j+1}\in\mathrm{H}_N$ such that

$$\Lambda^j_\omega(\mathrm{U}^{j+1})=0 \text{ and } |\mathrm{U}^{j+1}|\leq R\mu_j.$$

Therefore, we have to prove by induction that given $\mathrm{U}^0\in\mathrm{H}_N$ and a $\mathscr{H}$-valued Wiener process W, for each j, there exists a sequence $\{\mathrm{U}^j;\,j=1,\dots,M\}\subset\mathrm{H}_N$ satisfying the algorithm (2.23).

Proof of the measurability. In order to prove the $\mathcal{F}_{t_j}$-measurability of U^j it is sufficient to show that for each $j\in\{1,\dots,M\}$ one can find a Borel measurable map $\mathscr{E}_j:\mathrm{H}_N\times\mathscr{H}\to\mathrm{H}_N$ such that $\mathrm{U}^j=\mathscr{E}_j(\mathrm{U}^{j-1},\Delta_jW)$. In fact, if such claim is true then by exploiting the $\mathcal{F}_{t_j}$-measurability of Δ_jW one can argue by induction and show that if U^0 is $\mathcal{F}_0$-measurable then $\mathscr{E}_j(\mathrm{U}^{j-1},\Delta_jW)$ is $\mathcal{F}_{t_j}$-measurable, hence U^j is $\mathcal{F}_{t_j}$-measurable. Thus, it remains to prove the existence of $\mathscr{E}_j$. For this purpose we will closely follow [38]. Let $\mathcal{P}(\mathrm{H}_N)$ be the set of subsets of H_N and consider a multivalued map $\mathscr{E}^S_{j+1}:\mathrm{H}_N\times\mathscr{H}\to\mathcal{P}(\mathrm{H}_N)$ such that for each $(\mathrm{U}^j,\eta_{j+1})$, $\mathscr{E}^S_{j+1}(\mathrm{U}^j,\eta_{j+1})$ denotes the set of solutions U^{j+1} of (2.23). From the existence result above we deduce that $\mathscr{E}^S_{j+1}$ maps $\mathrm{H}_N\times\mathscr{H}$ to nonempty closed subsets of H_N. Furthermore, since we are in the finite dimensional space H_N, we can prove, by using the assumptions (**B1**) and (**G**) and the sequential characterization of the closed graph theorem, that the graph of $\mathscr{E}^S_{j+1}$ is closed. From these last two facts and [8, Theorem 3.1] we can find a univocal map $\mathscr{E}_{j+1}:\mathrm{H}_N\times\mathscr{H}\to\mathrm{H}_N$ such that $\mathscr{E}_j(\mathrm{U}^j,\eta_{j+1})\in\mathscr{E}^S_{j+1}(\mathrm{U}^j,\eta_{j+1})$ and $\mathscr{E}_j$ is measurable when $\mathrm{H}_N\times\mathscr{H}$ and H_N are equipped with their respective Borel σ-algebra. This completes the proof of the measurability of the solutions of (2.23).

Proof of (2.24)-(2.26)*.* Thanks to the assumption (**B2**), the proof of the inequalities (2.24)-(2.26) is very similar to the proof of [30], so we omit it and we directly proceed to the proof of the estimates (2.27) and (2.28).

Proof of (2.27). Taking $w=2\mathrm{A}^{\frac{1}{2}}\mathrm{U}^{j+1}$ in (2.23), using the Cauchy-Schwarz inequality and the identity

$$2(a-b)a=|a|^2-|b|^2+|a-b|^2, \tag{3.32}$$

yield

$$\begin{aligned}&\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2-\|\mathrm{U}^j\|_{\frac{1}{4}}^2+\|\mathrm{U}^{j+1}-\mathrm{U}^j\|_{\frac{1}{4}}^2+2k\|\mathrm{U}^{j+1}\|_{\frac{3}{4}}^2\\&\quad\le 2k|\pi_N\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1})|\|\mathrm{U}^{j+1}\|_{\frac{1}{2}}+2\|\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\|_{\frac{1}{4}}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|_{\frac{1}{4}}\\&\qquad+2\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j,\mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle.\end{aligned}$$

Using the fact that $\|\pi_N\|_{\mathscr{L}(\mathrm{H},\mathrm{H}_N)}\le 1$, we obtain

$$\begin{aligned}&\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2-\|\mathrm{U}^j\|_{\frac{1}{4}}^2+\|\mathrm{U}^{j+1}-\mathrm{U}^j\|_{\frac{1}{4}}^2+2k\|\mathrm{U}^{j+1}\|_{\frac{3}{4}}^2\\&\le 2k|\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1})|\|\mathrm{U}^{j+1}\|_{\frac{1}{2}}+2\|G(\mathrm{U}^j)\Delta_{j+1}W\|_{\frac{1}{4}}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|_{\frac{1}{4}}\\&\quad+2\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j,\mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle.\end{aligned} \tag{3.33}$$

Using Assumption **(B1)**$'$, the complex interpolation inequality in [92, Theorem 1.9.3, pp 59], the Young inequality, and the continuous embedding $\mathrm{V}_{\frac{1}{2}}\subset\mathrm{V}_{\frac{1}{4}}$ we obtain

$$\begin{aligned}2|\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1})|\|\mathrm{U}^{j+1}\|_{\frac{1}{2}}&\le C|\mathrm{U}^j|^4\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2+\|\mathrm{U}^{j+1}\|_{\frac{3}{4}}^2\\&\le C|\mathrm{U}^j|^4\|\mathrm{U}^{j+1}\|_{\frac{1}{2}}^2+\|\mathrm{U}^{j+1}\|_{\frac{3}{4}}^2,\end{aligned} \tag{3.34}$$

which implies that

$$\begin{aligned}\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2-\|\mathrm{U}^j\|_{\frac{1}{4}}^2+\frac{1}{2}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|_{\frac{1}{4}}^2+2k\|\mathrm{U}^{j+1}\|_{\frac{3}{4}}^2\le&2Ck|\mathrm{U}^j|^4\|\mathrm{U}^{j+1}\|_{\frac{1}{2}}^2\\&+4\|G(\mathrm{U}^j)\Delta_{j+1}W\|_{\frac{1}{4}}^2\\&+2\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j,\mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle.\end{aligned} \tag{3.35}$$

Since U^j is a constant, adapted and hence progressively measurable process, it is not difficult to prove that

$$2\mathbb{E}\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j,\mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle=0.$$

Using (2.25) and (2.26) with $p=2$ and $p=3$ respectively, we easily prove that there exists a constant $C>0$, depending only on T, such that

$$k\mathbb{E}\left(\sum_{j=0}^{M-1}|\mathrm{U}^j|^4\|\mathrm{U}^{j+1}\|^2_{\frac{1}{2}}\right)\le\left(\mathbb{E}\max_{1\le m\le M}|\mathrm{U}^m|^8\right)^{\frac{1}{2}}\left(\mathbb{E}\left(k\sum_{j=1}^{M}\|\mathrm{U}^j\|^2_{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}$$
$$\le C(1+\mathbb{E}|\boldsymbol{u}_0|^8)^2. \tag{3.36}$$

Now, since U^j is $\mathcal{F}_{t_j}$-measurable and $\Delta_{j+1}W$ is independent of $\mathcal{F}_{t_j}$, we infer that there exists a constant $C>0$ such that for any $j\in\{0,\dots,M-1\}$

$$\mathbb{E}\left(\|G(\mathrm{U}^j)\Delta_{j+1}W\|^2_{\frac{1}{4}}\right)\le\mathbb{E}\left(\mathbb{E}\left(\|G(\mathrm{U}^j)\|^2_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac{1}{4}})}\|\Delta_{j+1}W\|^2_{\mathscr{H}}|\mathcal{F}_{t_j}\right)\right)$$
$$=m\mathbb{E}\left(\|G(\mathrm{U}^j)\|^2_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac{1}{4}})}\mathbb{E}\left(\|\Delta_{j+1}W\|^2_{\mathscr{H}}|\mathcal{F}_{t_j}\right)\right)$$
$$\le Ck\,(\mathrm{tr}Q)^{\frac{1}{2}}\,(1+\mathbb{E}\|\mathrm{U}^j\|^2_{\frac{1}{4}}), \tag{3.37}$$

where (2.8) and Assumption **(G)** along with Remark 2.1-(b) were used to derive the last line of the above chain of inequalities.

Now taking the mathematical expectation in (3.35), summing both sides of the resulting equations from $j=0$ to $m-1$ and using the last three observations imply

$$\max_{1\le m\le M}\mathbb{E}\|\mathrm{U}^m\|^2_{\frac{1}{4}}+\frac{1}{2}\mathbb{E}\left(\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac{1}{4}}\right)+2k\mathbb{E}\sum_{j=1}^{M}\|\mathrm{U}^j\|^2_{\frac{3}{4}}$$
$$\le CT+\mathbb{E}\|\boldsymbol{u}_0\|^2_{\frac{1}{4}}+C\,\mathrm{Tr}Qk\sum_{m=1}^{M}\max_{1\le j\le m}\mathbb{E}\|\mathrm{U}^j\|^2_{\frac{1}{4}},$$

from which along with the discrete Gronwall lemma we infer that there exists a constant $C>0$ such that

$$\max_{1\le m\le M}\mathbb{E}\|\mathrm{U}^m\|^2_{\frac{1}{4}}+\frac{1}{2}\mathbb{E}\left(\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac{1}{4}}\right)$$
$$+2k\mathbb{E}\sum_{j=1}^{M}\|\mathrm{U}^j\|^2_{\frac{3}{4}}\le C(1+\mathbb{E}\|\boldsymbol{u}_0\|^2_{\frac{1}{4}}+[\mathbb{E}|\boldsymbol{u}_0|^8]^2). \tag{3.38}$$

Note that from (3.35) we can derive that there exists a constant $C>0$ such that

$$\mathbb{E}\max_{1\le m\le M}\|\mathrm{U}^m\|^2_{\frac14}\le \mathbb{E}\|\boldsymbol{u}_0\|^2_{\frac14}+Ck\mathbb{E}\sum_{j=0}^{M-1}|\mathrm{U}^j|^4\|\mathrm{U}^{j+1}\|^2_{\frac12}+\mathbb{E}\sum_{j=0}^{M-1}\|G(\mathrm{U}^j)\Delta_{j+1}W\|^2_{\frac14}$$
$$+2\mathbb{E}\max_{1\le m\le M}\sum_{j=0}^{m-1}\langle \mathrm{A}^{\frac14}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W,\mathrm{A}^{\frac14}\mathrm{U}^j\rangle$$
$$=:\sum_{i=1}^4 \mathrm{I}_i.$$

Arguing as in [22, proof of (3.9)] we can establish that

$$\mathrm{I}_4\le\frac12\mathbb{E}\|\boldsymbol{u}_0\|^2_{\frac14}+\frac12\mathbb{E}\max_{1\le m\le M}\|\mathrm{U}^m\|^2_{\frac14}+Ck\sum_{j=0}^{M-1}\mathbb{E}\|\mathrm{U}^j\|^2_{\frac14},$$

which altogether with (3.38) yields that

$$\mathrm{I}_4\le\frac12\mathbb{E}\max_{1\le m\le M}\|\mathrm{U}^m\|^2_{\frac14}+C(1+\mathbb{E}\|\boldsymbol{u}_0\|^2_{\frac14}).$$

Using the same idea as in the proof of (3.37) and using (3.38) we infer that

$$\mathrm{I}_3\le C(1+\mathbb{E}\|\boldsymbol{u}_0\|^2_{\frac14}).$$

Using these two estimates and the inequality (3.36) we derive that there exists a constant $C>0$ such that

$$\mathbb{E}\max_{1\le m\le M}\|\mathrm{U}^m\|^2_{\frac14}\le C(1+\mathbb{E}\|\boldsymbol{u}_0\|^2_{\frac14}+[\mathbb{E}|\boldsymbol{u}_0|^8]^2),$$

which along with (3.38) completes the proof of (2.27).

Now, we continue with the derivation of an estimate of $\max_{1\le m\le M}\mathbb{E}\|\mathrm{U}^m\|^4_{\frac14}$. Multiplying (3.33) by $\|\mathrm{U}^{j+1}\|^2_{\frac14}$ and using identity (3.32) and then summing both sides of the resulting equation from $j=0$ to $m-1$ implies

$$\begin{aligned}\frac{1}{2}\|\mathrm{U}^m\|_{\frac{1}{4}}^4+\frac{1}{2}\sum_{j=0}^{m-1}\Big|\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2-\|\mathrm{U}^j\|_{\frac{1}{4}}^2\Big|^2+\sum_{j=0}^{m-1}\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2\|\mathrm{U}^{j+1}-\mathrm{U}^j\|_{\frac{1}{4}}^2+2k\sum_{j=0}^{m-1}\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2\|\mathrm{U}^{j+1}\|_{\frac{3}{4}}^2\\ \le\frac{1}{2}\|\boldsymbol{u}_0\|_{\frac{1}{4}}^4+Ck\sum_{j=0}^{m-1}|\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1})|^2\|\mathrm{U}^{j+1}\|_{\frac{1}{2}}^2\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2\\ +2\sum_{j=0}^{m-1}\langle \mathrm{A}^{\frac{1}{4}}[\mathrm{U}^{j+1}-\mathrm{U}^j],\mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2\\ +2\sum_{j=0}^{m-1}\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j,\mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2\\ =:\frac{1}{2}\|\boldsymbol{u}_0\|_{\frac{1}{4}}^4+\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3.\end{aligned} \tag{3.39}$$

Thanks to the estimate (3.34) we can estimate J_1 as follows

$$\mathbb{E}\mathrm{J}_1\le CK\mathbb{E}\sum_{j=0}^{M-1}|\mathrm{U}^j|^4\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^4+k\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}\|_{\frac{1}{4}}^2\|\mathrm{U}^{j+1}\|_{\frac{3}{4}}^2=:\mathrm{J}_{1,1}+\mathrm{J}_{1,2}.$$

Since the second term $\mathrm{J}_{1,2}$ can be absorbed in the LHS later on, we will focus on estimating the second term $\mathrm{J}_{1,1}$. We have

$$\begin{aligned}\mathrm{J}_{1,1}\le& Ck\sum_{j=0}^{M-1}|\mathrm{U}^j|^4|\mathrm{U}^{j+1}|^2\|\mathrm{U}^{j+1}\|_{\frac{1}{2}}^2\\ \le& C\Big(\mathbb{E}\max_{0\le j\le M-1}[|\mathrm{U}^j|^8|\mathrm{U}^{j+1}|^4]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[k\sum_{j=1}^{M}\|\mathrm{U}^j\|_{\frac{1}{2}}^2\Big]^2\Big)^{\frac{1}{2}}\\ \le& C\Big(\mathbb{E}[\max_{0\le j\le M-1}|\mathrm{U}^j|^{12}]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[k\sum_{j=1}^{M}\|\mathrm{U}^j\|_{\frac{1}{2}}^2\Big]^2\Big)^{\frac{1}{2}}\\ \le& C(1+\mathbb{E}|\boldsymbol{u}_0|^{16}),\end{aligned}$$

where (2.25) and (2.26) are used to obtain the last line. Hence,

$$\mathbb{E}\mathrm{J}_1\le C(1+\mathbb{E}|\boldsymbol{u}_0|^{16})+\mathbb{E}k\sum_{j=0}^{M-1}\Big(\|\mathrm{U}^{j+1}\|^2-\frac{1}{4}\|\mathrm{U}^{j+1}\|_{\frac{3}{4}}^2\Big).$$

Now we estimate J_2 as follows

$$\begin{aligned}\mathbb{E}\mathrm{J}_2 \leq & C\mathbb{E}\sum_{j=0}^{M-1}\|G(\mathrm{U}^j)\Delta_{j+1}W\|^2_{\frac{1}{4}}\left(\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}-\|\mathrm{U}^j\|^2_{\frac{1}{4}}+\|\mathrm{U}^j\|^2_{\frac{1}{4}}\right)+\frac{1}{2}\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac{1}{4}}\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}\\ \leq & C\mathbb{E}\sum_{j=0}^{M-1}\|G(\mathrm{U}^j)\Delta_{j+1}W\|^4_{\frac{1}{4}}+C\mathbb{E}\sum_{j=0}^{M-1}\|G(\mathrm{U}^j)\Delta_{j+1}W\|^2_{\frac{1}{4}}\|\mathrm{U}^j\|^2_{\frac{1}{4}}+\frac{1}{8}\mathbb{E}\sum_{j=0}^{M-1}\left|\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}-\|\mathrm{U}^j\|^2_{\frac{1}{4}}\right|^2\\ &+\frac{1}{2}\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac{1}{4}}\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}.\end{aligned}$$

As long as J_3 is concerned we have

$$\begin{aligned}\mathbb{E}\mathrm{J}_3 = & 2\mathbb{E}\sum_{j=0}^{m-1}\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j, \mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle\|\mathrm{U}^j\|^2_{\frac{1}{4}}\\ &+2\mathbb{E}\sum_{j=0}^{m-1}\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j, \mathrm{A}^{\frac{1}{4}}G(\mathrm{U}^j)\Delta_{j+1}W\rangle\left(\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}-\|\mathrm{U}^j\|^2_{\frac{1}{4}}\right)\\ = & 2\mathbb{E}\sum_{j=0}^{m-1}\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j, \mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle\left(\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}-\|\mathrm{U}^j\|^2_{\frac{1}{4}}\right)\\ \leq & C\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{A}^{\frac{1}{4}}G(\mathrm{U}^j)\Delta_{j+1}W\|^2_{\frac{1}{4}}\|\mathrm{U}^j\|^2_{\frac{1}{4}}+\frac{1}{8}\mathbb{E}\sum_{j=0}^{M-1}\left|\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}-\|\mathrm{U}^j\|^2_{\frac{1}{4}}\right|^2\end{aligned}$$

because for any j

$$\mathbb{E}\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j, \mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle\|\mathrm{U}^j\|^2_{\frac{1}{4}} = 0.$$

By a similar idea as used to derive (3.37) we can prove that

$$C\mathbb{E}\sum_{j=0}^{M-1}\|G(\mathrm{U}^j)\Delta_{j+1}W\|^4_{\frac{1}{4}}+C\mathbb{E}\sum_{j=0}^{M-1}\|G(\mathrm{U}^j)\Delta_{j+1}W\|^2_{\frac{1}{4}}\|\mathrm{U}^j\|^2_{\frac{1}{4}} \leq C+Ck\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^j\|^4_{\frac{1}{4}}.$$

Thus,

$$\mathbb{E}[\mathrm{J}_2+\mathrm{J}_3] \leq C+Ck\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^j\|^4_{\frac{1}{4}}+\frac{1}{4}\mathbb{E}\sum_{j=0}^{M-1}\left|\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}-\|\mathrm{U}^j\|^2_{\frac{1}{4}}\right|^2+\frac{1}{2}\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac{1}{4}}\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}.$$

Taking the mathematical expectation in (3.39) and by plugging the information about J_i, $i=1,2,3$ in the resulting equation yield

$$\begin{aligned}\max_{1\le m\le M}\frac{1}{2}\mathbb{E}\|\mathrm{U}^m\|^4_{\frac14}+\frac14\mathbb{E}\sum_{j=0}^{M-1}\Big|\|\mathrm{U}^{j+1}\|^2_{\frac14}-\|\mathrm{U}^j\|^2_{\frac14}\Big|^2\\ +\frac12\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}\|^2_{\frac14}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac14}+k\mathbb{E}\sum_{j=1}^{M}\|\mathrm{U}^j\|^2_{\frac14}\|\mathrm{U}^j\|^2_{\frac34}\\ \le C(1+\mathbb{E}|\boldsymbol{u}_0|^{12}+\|\boldsymbol{u}_0\|^4_{\frac14})+Ck\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^j\|^4_{\frac14},\end{aligned}$$

which along with the Gronwall inequality yields

$$\max_{1\le m\le M}\frac12\mathbb{E}\|\mathrm{U}^m\|^4_{\frac14}\le C(1+\mathbb{E}|\boldsymbol{u}_0|^{12}+\|\boldsymbol{u}_0\|^4_{\frac14}).$$

The latter inequality is used in the former one to derive that

$$\begin{aligned}\max_{1\le m\le M}\frac12\mathbb{E}\|\mathrm{U}^m\|^4_{\frac14}+\frac14\mathbb{E}\sum_{j=0}^{M-1}\Big|\|\mathrm{U}^{j+1}\|^2_{\frac14}-\|\mathrm{U}^j\|^2_{\frac14}\Big|^2+\frac12\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}\|^2_{\frac14}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac14}\\ +k\mathbb{E}\sum_{j=1}^{M}\|\mathrm{U}^j\|^2_{\frac14}\|\mathrm{U}^j\|^2_{\frac34}\le C(1+\mathbb{E}|\boldsymbol{u}_0|^{12}+\|\boldsymbol{u}_0\|^4_{\frac14}).\end{aligned}\tag{3.40}$$

Now we continue our analysis with the estimation of $\mathbb{E}\max_{1\le j\le M}\|\mathrm{U}^j\|^4_{\frac14}$. To start with this analysis, we easily derive from (3.39) the following inequality

$$\begin{aligned}\max_{1\le m\le M}\frac12\mathbb{E}\|\mathrm{U}^m\|^4_{\frac14}\le Ck\sum_{j=0}^{M-1}|\mathrm{U}^j|^4\|\mathrm{U}^{j+1}\|^2\|\mathrm{U}^{j+1}\|^2_{\frac12}\\ +C\sum_{j=0}^{M-1}\Big(\|G(\mathrm{U}^j)\Delta_{j+1}\|^4_{\frac14}+\|G(\mathrm{U}^j)\Delta_{j+1}\|^2_{\frac14}\|\mathrm{U}^j\|^2_{\frac14}\Big)\\ +\max_{0\le j\le M-1}\sum_{\ell=0}^{j-1}\langle\mathrm{A}^{\frac14}\mathrm{U}^\ell,\mathrm{A}^{\frac14}\pi_N G(\mathrm{U}^\ell)\Delta_{\ell+1}W\rangle\|\mathrm{U}^\ell\|^2_{\frac14}=:J_1+J_2+J_3.\end{aligned}$$

Arguing as in the proof of (3.37) and using (3.40), the mathematical expectation of J_1+J_2 can be estimated as follows

$$\mathbb{E}(J_1+J_2)\le C\mathbb{E}(1+|\boldsymbol{u}_0|^{16}+\|\boldsymbol{u}_0\|^4_{\frac14}).$$

The same idea as used in the proof of [22, inequality (3.15)] yields

$$\mathbb{E}J_3 \le \frac{1}{4}\mathbb{E}\max_{1\le m\le M}\|\mathrm{U}^m\|^4_{\frac{1}{4}} + C\mathbb{E}\|\boldsymbol{u}_0\|^4_{\frac{1}{4}} + Ck\mathbb{E}\sum_{j=0}^{M-1}\|\mathrm{U}^j\|^4_{\frac{1}{4}},$$

from which altogether with (3.40) we infer that

$$\mathbb{E}J_3 \le C\mathbb{E}(1+|\boldsymbol{u}_0|^{16}+\|\boldsymbol{u}_0\|^4_{\frac{1}{4}}) + \frac{1}{4}\mathbb{E}\max_{1\le m\le M}\|\mathrm{U}^m\|^4_{\frac{1}{4}}.$$

Thus, summing up we have shown that there exists a constant $C>0$ such that

$$\mathbb{E}\max_{1\le m\le M}\|\mathrm{U}^m\|^4_{\frac{1}{4}} \le C\mathbb{E}(1+|\boldsymbol{u}_0|^{16}+\|\boldsymbol{u}_0\|^4_{\frac{1}{4}}). \tag{3.41}$$

Now, we estimate $\mathbb{E}\left(\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac{1}{4}}\right)^2 + \mathbb{E}\left(k\sum_{j=1}^{M}\|\mathrm{U}^j\|^2_{\frac{3}{4}}\right)^2$. To do this we first observe that from (3.35) we infer that

$$\begin{aligned}\left(\frac{1}{2}\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac{1}{4}}\right)^2 + \left(2k\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}\right)^2 &\le C\left(k\sum_{j=0}^{M-1}|\mathrm{U}^j|^4\|\mathrm{U}^{j+1}\|^2_{\frac{1}{2}}\right)^2 \\ +C\left(\sum_{j=0}^{M-1}\|G(\mathrm{U}^j)\Delta_{j+1}W\|^2_{\frac{1}{4}}\right)^2 &+ C\left(\sum_{j=0}^{M-1}\langle \mathrm{A}^{\frac{1}{4}}\mathrm{U}^j, \mathrm{A}^{\frac{1}{4}}\pi_N G(\mathrm{U}^j)\Delta_{j+1}W\rangle\right)^2.\end{aligned} \tag{3.42}$$

Then, using the same strategies to estimate the J_i-s (or J_i), the sum of the three terms in the right hand side of the above inequality can be bounded from above by

$$\left[\mathbb{E}\left(\max_{0\le j\le M}|\mathrm{U}^j|^{16}\right)\right]^{\frac{1}{2}}\left[\mathbb{E}\left(k\sum_{j=1}^{M}\|\mathrm{U}^j\|^2_{\frac{1}{2}}\right)^4\right]^{\frac{1}{2}} + CMk^2\sum_{j=0}^{M}\mathbb{E}\|\mathrm{U}^j\|^4_{\frac{1}{4}} + Ck\sum_{j=0}^{M}\mathbb{E}\|\mathrm{U}^j\|^4_{\frac{1}{4}},$$

which along with the estimate for $\mathbb{E}\max_{1\le m\le M}\|\mathrm{U}^m\|^4_{\frac{1}{4}}$ and the inequalities (2.25) and (2.26) implies that

$$\left(\frac{1}{2}\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_{\frac{1}{4}}\right)^2 + \left(2k\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}\|^2_{\frac{1}{4}}\right)^2 \le \mathbb{E}(1+|\boldsymbol{u}_0|^{16}+\|\boldsymbol{u}_0\|^4_{\frac{1}{4}}). \tag{3.43}$$

The last estimate along with (3.41) completes the proof of (2.28) and hence the whole proposition. □

2.4 Error analysis of the numerical scheme (2.23): Proof of Theorem 2.8

This section is devoted to the analysis of the error $\mathbf{e}_j = \boldsymbol{u}(t_j) - \mathrm{U}^j$ at the time t_j between the exact solution $\boldsymbol{u}$ of (1.1) and the approximate solution given by (2.23). Since the precise statement of the convergence rate is already given in Theorem 2.8, we proceed directly to the promised proof of Theorem 2.8.

Before giving the proof of Theorem 2.8 we state and prove the following important result.

Lemma 2.12. *Let β be as in Theorem 2.8. Then,*

(i) there exists a constant $C_7 > 0$ such that

$$\mathbb{E}\|\boldsymbol{u}(t) - \boldsymbol{u}(s)\|_\beta^2 \le C_7[(t-s)^{2-2\beta} + (t-s)^{2(\frac{1}{4}-\beta)} + (t-s)], \tag{4.44}$$

for any $t, s \ge 0$ and $t \neq s$.

(ii) There also exists a positive constant C_8 such that

$$\mathbb{E}\int_s^t \|\boldsymbol{u}(t) - \boldsymbol{u}(r)\|_{\frac{1}{2}+\beta}^2 dr \le C_8\left((t-s)^{\frac{3}{2}-2\beta} + (t-s)^{2(\frac{1}{4}-\beta)} + (t-s)^{2-2\beta}\right), \tag{4.45}$$

for any $t > s \ge 0$.

Proof of Lemma 2.12. As in the statement of the lemma we divide the proof into two parts.

Proof of item (i). Let $t, s \in [0, T]$ such that $t \neq s$. Without loss of generality we assume that $t > s$. Thanks to (2.10) of Remark 2.5 we have

$$\begin{aligned}\|\boldsymbol{u}(t) - \boldsymbol{u}(s)\|_\beta^2 \le C|\mathrm{A}^{\beta-\frac{1}{4}}(\mathrm{I} - e^{-(t-s)\mathrm{A}})\mathrm{A}^{\frac{1}{4}}\boldsymbol{u}(s)|^2 + C\left|\int_s^t A^\beta e^{-(t-r)\mathrm{A}} B(\boldsymbol{u}(r), \boldsymbol{u}(r)) dr\right|^2 \\ + C\left|\int_s^t \mathrm{A}^\beta e^{-(t-r)\mathrm{A}} G(\boldsymbol{u}(r)) dW(r)\right|^2.\end{aligned}$$

Before proceeding further we recall that there exists a constant $C > 0$ such that for any $\gamma > 0$ and $t \ge 0$, we have

$$\|A^{-\gamma}(\mathrm{I}-e^{-tA})\|_{\mathscr{L}(\mathrm{H})} \le Ct^{\gamma}.$$

Applying this inequality, the Hölder inequality, Assumption **(B1)**$'$, the Itô isometry and Assumption **(G)** imply

$$\begin{aligned}\mathbb{E}\left(\|\boldsymbol{u}(t)-\boldsymbol{u}(s)\|_{\beta}^{2}\right) \le & C(t-s)\mathbb{E}\left(\int_{s}^{t}(t-r)^{-2\beta}\|\boldsymbol{u}(r)\|_{\frac{1}{4}}^{2}\|\boldsymbol{u}(r)\|_{\frac{1}{2}-\frac{1}{4}}^{2}dr\right) \\ & + C(t-s)^{2(\frac{1}{4}-\beta)}\mathbb{E}\|\boldsymbol{u}(s)\|_{\frac{1}{4}}^{2} + \mathbb{E}\int_{s}^{t}|e^{-(t-r)\mathrm{A}}\mathrm{A}^{\beta}G(\boldsymbol{u}(r))|^{2}dr \\ \le & C(t-s)^{2-2\beta}\mathbb{E}\left(\sup_{r\in[s,t]}\|\boldsymbol{u}(r)\|_{\frac{1}{4}}^{2}\sup_{r\in[s,t]}\|\boldsymbol{u}(r)\|_{\frac{1}{2}-\frac{1}{4}}^{2}\right) \\ & + C[(t-s)^{2(\frac{1}{4}-\beta)}+(t-s)]\mathbb{E}\left(\sup_{r\in[s,t]}\|\boldsymbol{u}(r)\|_{\frac{1}{4}}^{4}\right),\end{aligned}$$

from which along with (2.13) we easily infer that

$$\mathbb{E}\left(\|\boldsymbol{u}(t)-\boldsymbol{u}(s)\|_{\beta}^{2}\right) \le C[(t-s)^{2-2\beta}+(t-s)^{2(\frac{1}{4}-\beta)}+(t-s)].$$

Thus, we have just finished the proof of the first part of the lemma.

Proof of item (ii). Let $t > s \ge 0$. Using (2.10) of Remark 2.5, it is not difficult to see that

$$\begin{aligned}\int_{s}^{t}\|\boldsymbol{u}(t)-\boldsymbol{u}(r)\|_{\frac{1}{2}+\beta}^{2}dr \le & C\int_{s}^{t}\left(\int_{r}^{t}|\mathrm{A}^{\frac{1}{2}+\beta}e^{-(t-\tau)\mathrm{A}}\mathrm{B}(\boldsymbol{u}(\tau),\boldsymbol{u}(\tau))|d\tau\right)^{2}dr \\ & + C\int_{s}^{t}\left|\int_{r}^{t}\mathrm{A}^{\frac{1}{4}+\beta}e^{-(t-\tau)\mathrm{A}}[\mathrm{A}^{\frac{1}{4}}G(\boldsymbol{u}(\tau))]dW(\tau)\right|^{2}dr \\ & + C\int_{s}^{t}|\mathrm{A}^{\beta-\frac{1}{4}}(e^{-(t-r)A}-\mathrm{I})\mathrm{A}^{\frac{3}{4}}\boldsymbol{u}(s)|^{2}dr,\end{aligned}$$

from which and the assumption on B we infer that

$$\begin{aligned}\int_{s}^{t}\|\boldsymbol{u}(t)-\boldsymbol{u}(r)\|_{\frac{1}{2}+\beta}^{2}dr \le & C\sup_{0\le\tau\le T}\left(\|\boldsymbol{u}(\tau)\|_{\frac{1}{4}}^{2}\|\boldsymbol{u}(\tau)\|_{\frac{1}{2}-\frac{1}{4}}^{2}\right)\int_{s}^{t}\left(\int_{r}^{t}(t-\tau)^{-\frac{1}{2}-\beta}d\tau\right)^{2}dr \\ & + C\int_{s}^{t}\left|\int_{r}^{t}\mathrm{A}^{\frac{1}{4}+\beta}e^{-(t-\tau)\mathrm{A}}[\mathrm{A}^{\frac{1}{4}}G(\boldsymbol{u}(\tau))]dW(\tau)\right|^{2}dr \\ & + C\int_{s}^{t}(t-r)^{2(\frac{1}{4}-\beta)}\|\boldsymbol{u}(s)\|_{\frac{3}{4}}^{2}dr.\end{aligned}$$

Taking the mathematical expectation and using (2.13) yield

$$\mathbb{E}\left(\mathbb{1}_{\Omega_k}\int_s^t \|\boldsymbol{u}(t)-\boldsymbol{u}(r)\|^2_{\frac{1}{2}+\beta}dr\right) \leq C(t-s)^{2-2\beta}+C(t-s)^{2(\frac{1}{4}-\beta)}\mathbb{E}\int_0^T \|\boldsymbol{u}(r)\|^2_{\frac{1}{2}+\beta}dr$$
$$+\int_s^t \mathbb{E}\left(\left|\int_r^t \mathrm{A}^{\frac{1}{4}+\beta}e^{-(t-\tau)\mathrm{A}}\mathrm{A}^{\frac{1}{4}}G(\boldsymbol{u}(\tau))dW(\tau)\right|^2\right)dr.$$

Owing to the Itô isometry, the assumption (**G**) and (2.13), we obtain

$$\mathbb{E}\left(\int_s^t \|\boldsymbol{u}(t)-\boldsymbol{u}(r)\|^2_{\frac{1}{2}+\beta}dr\right) \leq \mathbb{E}\left(\sup_{0\leq\tau\leq T}(1+\|\boldsymbol{u}(\tau)\|^2_{\frac{1}{4}})\right)\int_s^t\int_r^t (t-\tau)^{-\frac{1}{2}-2\beta}d\tau dr$$
$$+(t-s)^{2-2\beta}+(t-s)^{2(\frac{1}{4}-\beta)},$$

from which altogether with (2.13) we infer that there exists a constant $C>0$ such that

$$\mathbb{E}\left(\int_s^t \|\boldsymbol{u}(t)-\boldsymbol{u}(r)\|^2_{\frac{1}{2}+\beta}dr\right) \leq C(t-s)^{2-2\beta}+C(t-s)^{2(\frac{1}{4}-\beta)}+C(t-s)^{\frac{3}{2}-2\beta},$$

for any $t>s\geq 0$. □

We now give the promised proof of Theorem 2.8.

Proof of Theorem 2.8. Since the embedding $\mathrm{V}_\beta\subset\mathrm{H}$ is continuous for any $\beta\in(0,\frac{1}{4})$, it is sufficient to prove the main theorem for $\beta\in(0,\frac{1}{4})$.

Note that the numerical scheme (2.23) is equivalent to

$$(\mathrm{U}^{j+1},w)+\int_{t_j}^{t_{j+1}}\langle \mathrm{A}\mathrm{U}^{j+1},w\rangle ds$$
$$+\int_{t_j}^{t_{j+1}}\langle\rangle\pi_N\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1}),w\rangle ds=(\mathrm{U}^j,w)+\int_{t_j}^{t_{j+1}}\langle w,\pi_N G(\mathrm{U}^j)dW(s)\rangle \tag{4.46}$$

for any $j\in\{1,\dots,M\}$ and $w\in \mathrm{V}$. Integrating (1.1) and subtracting the resulting equation and the identity (4.46) term by term yield

$$\begin{aligned}&(\mathbf{e}^{j+1}-\mathbf{e}^{j},w)\\&+\int_{t_j}^{t_{j+1}}\langle \mathrm{A}\mathbf{e}^{j+1}+\mathrm{A}(\boldsymbol{u}(s)-\boldsymbol{u}(t_{j+1}))\\&+\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))-\pi_N\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1}),w\rangle ds=\int_{t_j}^{t_{j+1}}\langle w,[G(\boldsymbol{u}(s))-\pi_N G(\mathrm{U}^j)]dW(s)\rangle.\end{aligned}\tag{4.47}$$

Observe that if $\boldsymbol{v}\in D(\mathrm{A}^{\frac{1}{2}+\alpha})$ with $\alpha>\beta$, then $\mathrm{A}^{2\beta}\boldsymbol{v}\in D(\mathrm{A}^{\frac{1}{2}+\alpha-\beta})\subset D(\mathrm{A}^{\frac{1}{2}-\alpha})$, $\mathrm{A}\boldsymbol{v}\in D(\mathrm{A}^{\alpha-\frac{1}{2}})$ and the duality product $\langle \mathrm{A}\boldsymbol{v},\mathrm{A}^{2\beta}\boldsymbol{v}\rangle$ is meaningful. Thus, we are permitted to take $w=2\mathrm{A}^{2\beta}\mathbf{e}^{j+1}$ in (4.47) and derive that

$$\begin{aligned}\|\mathbf{e}^{j+1}\|_\beta^2-\|\mathbf{e}^{j}\|_\beta^2+\|\mathbf{e}^{j+1}-\mathbf{e}^{j}\|_\beta^2+2k\|\mathbf{e}^{j+1}\|^2_{\frac{1}{2}+\beta}-2\int_{t_j}^{t_{j+1}}\|\mathrm{A}^{\frac{1}{2}+\beta}(\boldsymbol{u}(s)-\boldsymbol{u}(t_{j+1}))\|_{\frac{1}{2}+\beta}\|\mathbf{e}^{j+1}\|_{\frac{1}{2}+\beta}ds\\ \leq 2\int_{t_j}^{t_{j+1}}\left|(\mathrm{A}^{\beta-\frac{1}{2}}[\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))-\pi_N\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1})],\mathrm{A}^{\frac{1}{2}+\beta}\mathbf{e}^{j+1})\right|ds\\ +2\int_{t_j}^{t_{j+1}}\langle \mathrm{A}^{2\beta}\mathbf{e}^{j+1},[G(\boldsymbol{u}(s))-\pi_N G(\mathrm{U}^j)]dW(s)\rangle,\end{aligned}$$

where we have used the identity $(\boldsymbol{v}-\boldsymbol{x},2\mathrm{A}^{2\beta}\boldsymbol{v})=\|\boldsymbol{v}\|_\beta^2-\|\boldsymbol{x}\|_\beta^2+\|\boldsymbol{v}-\boldsymbol{x}\|_\beta^2$. Now, by using the identity $\boldsymbol{v}=(\pi_N+[\mathrm{I}-\pi_N])\boldsymbol{v}$, the fact that

$$\begin{aligned}\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))-\pi_N\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1})=\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))-\pi_N\mathrm{B}(\boldsymbol{u}(t_j),\boldsymbol{u}(t_{j+1}))\\+\pi_N\mathrm{B}(\boldsymbol{u}(t_j),\boldsymbol{u}(t_{j+1}))-\mathrm{B}(\mathrm{U}^j,\mathrm{U}^{j+1}),\end{aligned}$$

the Cauchy–Schwarz inequality, the Cauchy inequality $ab\leq \frac{a^2}{4}+b^2$, $a,b>0$ and Assumption **(B1)** we obtain

$$\|\mathbf{e}^{j+1}\|_\beta^2-\|\mathbf{e}^{j}\|_\beta^2+\|\mathbf{e}^{j+1}-\mathbf{e}^{j}\|_\beta^2+k\|\mathbf{e}^{j+1}\|^2_{\frac{1}{2}+\beta}\leq 2\mathcal{L}_j+16C_0^2\sum_{i=1}^{5}\mathscr{N}_{j,i}+2\mathscr{W}_j,\tag{4.48}$$

where for each $j\in\{0,\dots,M-1\}$ the symbols $\mathcal{L}_j$, $\mathcal{N}_{j,i}$, $i=1,\dots,5$, and $\mathcal{W}_j$ are defined by

$$\begin{aligned}
\mathcal{L}_j &:= \int_{t_j}^{t_{j+1}} \|\boldsymbol{u}(s)-\boldsymbol{u}(t_{j+1})\|^2_{\frac{1}{2}+\beta} ds,\\
\mathcal{N}_{j,1} &:= \int_{t_j}^{t_{j+1}} \|\boldsymbol{u}(s)-\boldsymbol{u}(t_{j+1})\|^2_{\beta}(\|\mathrm{U}^j\|^2_\beta+\|\boldsymbol{u}(s)\|^2_\beta) ds,\\
\mathcal{N}_{j,2} &:= \int_{t_j}^{t_{j+1}} \|\mathbf{e}^{j+1}\|^2_{\beta}(\|\mathrm{U}^{j+1}\|^2_\beta+\|\boldsymbol{u}(s)\|^2_\beta) ds,\\
\mathcal{N}_{j,3} &:= \int_{t_j}^{t_{j+1}} \|\boldsymbol{u}(s)-\boldsymbol{u}(t_j)\|^2_{\beta}(|\mathrm{U}^{j+1}|^2+|\boldsymbol{u}(s)|^2) ds,\\
\mathcal{N}_{j,4} &:= \int_{t_j}^{t_{j+1}} \|\mathbf{e}^{j}\|^2_{\beta}(|\mathrm{U}^{j+1}|^2+|\boldsymbol{u}(s)|^2) ds,\\
\mathcal{N}_{j,5} &:= \int_{t_j}^{t_{j+1}} \|(\mathrm{I}-\pi_N)\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))\|^2_{\beta-\frac{1}{2}} ds,\\
\mathcal{W}_j &:= \int_{t_j}^{t_{j+1}} \langle \mathrm{A}^{2\beta}\mathbf{e}^{j+1}, [G(\boldsymbol{u}(s))-\pi_N G(\mathrm{U}^j)]dW(s)\rangle.
\end{aligned}$$

Let $m\in[1,M]$ an arbitrary integer. Summing (4.48) from $j=0$ to $m-1$, multiplying by $\mathbb{1}_{\Omega_k}$, taking the mathematical expectation, and finally taking the maximum over $m\in[1,M]$ imply

$$\begin{aligned}
&\max_{1\le m\le M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^m\|^2_\beta\right]+\sum_{j=0}^{M-1}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^{j+1}-\mathbf{e}^j\|^2_\beta\right]+k\sum_{j=1}^{M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|^2_{\frac{1}{2}+\beta}\right]\\
&\le \mathbb{E}\|\mathbf{e}^0\|^2_\beta+16C_0^2\sum_{j=0}^{M-1}\sum_{i=1}^{5}\mathbb{E}[\mathbb{1}_{\Omega_k}\mathcal{N}_{j,i}]+2\sum_{j=0}^{M-1}\mathbb{E}[\mathbb{1}_{\Omega_k}\mathcal{L}_j]+2\max_{1\le m\le M}\sum_{j=0}^{m-1}\mathbb{E}[\mathbb{1}_{\Omega_k}\mathcal{W}_j].
\end{aligned}$$

Invoking the two items of Lemma 2.12 and the fact that $\|\boldsymbol{u}(s)\|^2_\beta+\max_{0\le j\le M}\|\mathrm{U}^j\|^2_\beta\le f(k)$ on the set Ω_k we infer that

$$\begin{aligned}
&\max_{1\le m\le M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^m\|^2_\beta\right]+\sum_{j=0}^{M-1}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^{j+1}-\mathbf{e}^j\|^2_\beta\right]+k\sum_{j=1}^{M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|^2_{\frac{1}{2}+\beta}\right]\\
&\le \mathbb{E}\|\mathbf{e}^0\|^2_\beta+16C_0^2kf(k)\sum_{j=0}^{M-1}\mathbb{E}\left(\mathbb{1}_{\Omega_k}[\|\mathbf{e}^{j+1}\|^2_\beta+\|\mathbf{e}^j\|^2_\beta]\right)+2C_8f(k)Mk[\Psi(k)+k^{1+\frac{1}{2}-\beta}] \qquad (4.49)\\
&\quad +64C_0^2C_8[f(k)]^2Mk[\Psi(k)+k]+16C_0^2\sum_{j=0}^{M-1}\mathcal{N}_{j,5}+2\max_{1\le m\le M}\sum_{j=0}^{m-1}\mathbb{E}[\mathbb{1}_{\Omega_k}\mathcal{W}_j],
\end{aligned}$$

where $\psi(k):=k^{2-2\beta}+k^{2(\frac{1}{4}-\beta)}$. Now, thanks to Assumption (**B1**)′ we have

$$\begin{aligned}\mathbb{1}_{\Omega_k}\int_{t_j}^{t_{j+1}}\|(\mathrm{I}-\pi_N)\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))\|^2_{\beta-\frac{1}{2}}ds &= \mathbb{1}_{\Omega_k}\int_{t_j}^{t_{j+1}}\sum_{n=N+1}^{\infty}\lambda_n^{2\beta-1}|\mathrm{B}_n(\boldsymbol{u}(s),\boldsymbol{u}(s))|^2ds\\
&\le \lambda_N^{2\beta-1}\int_{t_j}^{t_{j+1}}\mathbb{1}_{\Omega_k}\sum_{n=0}^{\infty}|\mathrm{B}_n(\boldsymbol{u}(s),\boldsymbol{u}(s))|^2ds\\
&\le \lambda_N^{2\beta-1}\int_{t_j}^{t_{j+1}}\mathbb{1}_{\Omega_k}|\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))|^2ds\\
&\le C\lambda_N^{2\beta-1}k\sup_{s\in[0,T]}\|\boldsymbol{u}(s)\|^4_{\frac{1}{4}}.\end{aligned}$$

Hence, owing to (2.13) we find a constant $C>0$ such that

$$\mathbb{E}\mathbb{1}_{\Omega_k}\int_{t_j}^{t_{j+1}}\|(\mathrm{I}-\pi_N)\mathrm{B}(\boldsymbol{u}(s),\boldsymbol{u}(s))\|^2_{\beta-\frac{1}{2}}ds\le C\lambda_N^{2\beta-1}k.$$

Notice also that

$$\begin{aligned}&\sum_{j=0}^{M-1}\|\mathbf{e}^{j+1}\|^2_\beta(\|\mathrm{U}^{j+1}\|^2_\beta+\|\boldsymbol{u}(s)\|^2_\beta)\\
&\quad=\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j+\mathrm{U}^j-\boldsymbol{u}(t_j)+\boldsymbol{u}(t_j)-\boldsymbol{u}(t_{j+1})\|^2_\beta(\|\mathrm{U}^{j+1}\|^2_\beta+\|\boldsymbol{u}(s)\|^2_\beta)\\
&\quad\le 3\sum_{j=0}^{M-1}\left(\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_\beta+\|\mathbf{e}^j\|^2_\beta+\|\boldsymbol{u}(t_j)-\boldsymbol{u}(t_{j+1})\|^2_\beta\right)\left(\max_{0\le j\le M}\|\mathrm{U}^{j+1}\|^2_\beta+\|\boldsymbol{u}(s)\|^2_\beta\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}\left(\mathbb{1}_{\Omega_k}\sum_{j=0}^{M-1}\|\mathbf{e}^{j+1}\|^2_\beta(\|\mathrm{U}^{j+1}\|^2_\beta+\|\boldsymbol{u}(s)\|^2_\beta)\right)-Cf(k)\mathbb{E}\sum_{m=0}^{M-1}\|\mathbf{e}^j\|^2_\beta+f(k)C_7[\psi(k)+k]\\
&\qquad\le C\left(\mathbb{E}\left(\sum_{j=0}^{M-1}\|\mathrm{U}^{j+1}-\mathrm{U}^j\|^2_\beta\right)^2\right)^{\frac{1}{2}}\left(\mathbb{E}\max_{0\le j\le M}\|\mathrm{U}^j\|^4_\beta+\mathbb{E}\sup_{s\in[0,T]}\|\boldsymbol{u}(s)\|^4_\beta\right)^{\frac{1}{2}}.\end{aligned}$$

As long as the initial data is concerned, we have

$$\mathbb{E}\|\mathbf{e}^0\|^2_\beta=\|[\pi_N+(\mathrm{I}-\pi_N)]\boldsymbol{u}_0-\pi_N\boldsymbol{u}_0\|^2_\beta\le\sum_{n=N+1}^{\infty}\lambda_n^{2(\beta-\frac{1}{4})}\lambda_N^{\frac{1}{2}}|\boldsymbol{u}_{0,n}|^2\le\lambda_N^{2(\beta-\frac{1}{4})}\|\boldsymbol{u}_0\|^2.$$

From all the above observations, (4.49), Assumption **(B1)**$'$, (2.24)-(2.26) and (2.28) we infer that there exists a constant $C_9>0$ such that

$$\begin{aligned}\max_{1\le m\le M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^m\|^2_\beta\right]+\sum_{j=0}^{M-1}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^{j+1}-\mathbf{e}^j\|^2_\beta\right]+k\sum_{j=1}^{M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|^2_{\frac12+\beta}\right]\\ \le C_9f(k)[\Psi(k)+k^{1+\frac12-\beta}]+C_9f(k)[\Psi(k)+k]+C_9\left(\lambda_N^{2\beta-1}+\lambda^{2(\beta-\frac14)}\right)+2\max_{1\le m\le M}\sum_{j=0}^{m-1}\mathbb{E}[\mathbb{1}_{\Omega_k}\mathscr{W}_j]\\ +C_9kf(k)\sum_{m=0}^{M-1}\max_{1\le j\le m}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|^\beta\right]+16C_0^2kf(k)\max_{1\le m\le M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^m\|^2_\beta\right].\end{aligned}\tag{4.50}$$

Now we deal with the term containing $\mathscr{W}_j$. After subtracting from $\mathscr{W}_j$ the martingale M_0 with mean zero defined by

$$M_0=\int_{t_j}^{t_{j+1}}\langle \mathrm{A}^\beta\mathbf{e}^{j+1},\mathrm{A}^\beta[G(\boldsymbol{u}(s))-\pi_NG(\mathrm{U}^j)]dW(s)\rangle,$$

then taking the mathematical expectation, using the Young inequality and the Itô isometry give

$$\begin{aligned}\mathbb{E}\mathbb{1}_{\Omega_k}\mathscr{W}_j&\le C\mathbb{E}\mathbb{1}_{\Omega_k}\left\|\int_{t_j}^{t_{j+1}}[G(\boldsymbol{u}(s))-\pi_NG(\mathrm{U}^j)]dW(s)\right\|^2_\beta+\frac14\mathbb{E}\mathbb{1}_{\Omega_k}\|\mathbf{e}^{j+1}-\mathbf{e}^j\|^2_\beta\\ &\le C\int_{t_j}^{t_{j+1}}\mathbb{E}\mathbb{1}_{\Omega_k}\|G(\boldsymbol{u}(s))-\pi_NG(\mathrm{U}^j)\|^2_{\mathscr{L}(\mathscr{H},V_\beta)}ds+\frac14\mathbb{E}\mathbb{1}_{\Omega_k}\|\mathbf{e}^{j+1}-\mathbf{e}^j\|^2_\beta\\ &\le\sum_{i=1}^{3}\mathbb{E}[\mathbb{1}_{\Omega_k}\mathscr{W}_{j,i}]+\frac14\mathbb{E}\mathbb{1}_{\Omega_k}\|\mathbf{e}^{j+1}-\mathbf{e}^j\|^2_\beta,\end{aligned}$$

where the first two symbols $\mathscr{W}_{j,i}$, $i\in\{1,2\}$ satisfy the following equalities and inequalities

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\Omega_k}\mathscr{W}_{j,1}]&=C\int_{t_j}^{t_{j+1}}\mathbb{E}\mathbb{1}_{\Omega_k}\|\pi_NG(\boldsymbol{u}(s))-\pi_NG(\boldsymbol{u}(t_j))\|^2_{\mathscr{L}(\mathscr{H},V_\beta)}ds\\ &\le CC_3^2\int_{t_j}^{t_{j+1}}\mathbb{E}\|\boldsymbol{u}(s)-\boldsymbol{u}(t_j)\|^2_\beta ds\\ &\le CC_3^2C_7^2k[k^{2-2\beta}+k^{2(\frac14-\beta)}+k];\end{aligned}$$

$$\mathbb{E}[\mathbb{1}_{\Omega_k}\mathscr{W}_{j,2}] = C\int_{t_j}^{t_{j+1}} \mathbb{E}\mathbb{1}_{\Omega_k}\|\pi_N G(\boldsymbol{u}(t_j)) - \pi_N G(U^j)\|^2_{\mathscr{L}(\mathscr{H},V_\beta)} ds$$
$$\leq CC_3^2 k\mathbb{E}\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|^2_\beta,$$

where Lemma 2.12 was used to get the last line.

The third term $\mathscr{W}_{j,3}$ satisfies

$$\mathbb{E}[\mathbb{1}_{\Omega_k}\mathscr{W}_{j,3}] = \int_{t_j}^{t_{j+1}} \mathbb{E}\left(\mathbb{1}_{\Omega_k}\|(\mathrm{I}-\pi_N)G(\boldsymbol{u}(s))\|^2_{\mathscr{L}(\mathscr{H},V_\beta)}\right) ds$$
$$= \int_{t_j}^{t_{j+1}} \mathbb{E}\left(\mathbb{1}_{\Omega_k}\sum_{n=N+1}^{\infty}\lambda_n^{2(\beta-\frac{1}{4})}\lambda_n^{\frac{1}{2}}\sup_{h\in\mathscr{H},\|h\|_{\mathscr{H}}\leq 1}|G_n(\boldsymbol{u}(s))h|^2\right) ds$$
$$\leq \lambda_N^{2(\beta-\frac{1}{4})}\int_{t_j}^{t_{j+1}} \mathbb{E}\left(\mathbb{1}_{\Omega_k}\sum_{n=1}^{\infty}\lambda_n^{\frac{1}{2}}\sup_{h\in\mathscr{H},\|h\|_{\mathscr{H}}\leq 1}|G_n(\boldsymbol{u}(s))h|^2\right) ds$$
$$\leq \lambda_N^{2(\beta-\frac{1}{4})} k\mathbb{E}\left(\mathbb{1}_{\Omega_k}\sup_{s\in[0,T]}\|G(\boldsymbol{u}(s))\|^2_{\mathscr{L}(\mathscr{H},\mathrm{V}_{\frac{1}{4}})}\right).$$

Now, using Assumption (**G**) and the estimate (2.13) we infer that

$$\mathbb{E}[\mathbb{1}_{\Omega_k}\mathscr{W}_{j,3}] \leq CC_3^2\lambda_N^{2(\beta-\frac{1}{4})}k,$$

for any $j\in[0,M]$. Thus, summing up we have obtained that

$$2\max_{1\leq m\leq M}\sum_{j=0}^{m-1}\mathbb{E}[\mathbb{1}_{\Omega_k}\mathscr{W}_j] \leq CC_3^2C_7^2T[\psi(k)+k] + CC_3^2T\lambda_N^{2(\beta-\frac{1}{4})}$$
$$+CC_3^2k\sum_{m=0}^{M-1}\max_{1\leq j\leq m}\mathbb{E}[\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|^2_\beta] + \frac{1}{2}\sum_{m=0}^{M-1}\mathbb{E}\left(\mathbb{1}_{\Omega_k}\|\mathbf{e}^{m+1}-\mathbf{e}^m\|^2_\beta\right).$$

By plugging this last estimate into (4.49), we find a constant $C_{10}>0$ such that

$$\max_{1\leq m\leq M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^m\|^2_\beta\right] + \sum_{j=0}^{M-1}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^{j+1}-\mathbf{e}^j\|^2_\beta\right] + 2k\sum_{j=1}^{M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|^2_{\frac{1}{2}+\beta}\right]$$
$$\leq C_{10}f(k)[\Psi(k)+k+k^{1+\frac{1}{2}-\beta}] + C_{10}f(k)[\Psi(k)+k] + C_{10}\lambda_N^{2\beta-1} + C_{10}\lambda_N^{2(\beta-\frac{1}{4})}$$
$$+C_{10}k[f(k)+1]\sum_{m=0}^{M-1}\max_{1\leq j\leq m}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|^\beta\right].$$

Now, an application of the discrete Gronwall lemma yields

$$\begin{aligned}&\max_{1\le m\le M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^m\|_\beta^2\right]+\sum_{j=0}^{M-1}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^{j+1}-\mathbf{e}^j\|_\beta^2\right]+2k\sum_{j=1}^{M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|_{\frac12+\beta}^2\right]\\ &\le\left(C_{10}f(k)[\Psi(k)+k+k^{1+\frac12-\beta}]+C_{10}f(k)[\Psi(k)+k]+C_{10}\lambda_N^{2\beta-1}+C_{10}\lambda_N^{2(\beta-\frac14)}\right)e^{C_{10}T[f(k)+1]}.\end{aligned}$$

Since

$$\min\{k^{2-2\beta},k^{1+\frac12-\beta},k^{2(\frac14-\beta)},k\}=k^{2(\frac14-\beta)}\text{ and }\min\{\lambda_N^{2(\beta-\frac14)},\lambda_N^{2\beta-1}\}=\lambda_N^{2(\beta-\frac14)},$$

for any $\beta\in[0,\frac14)$, and $k^\varepsilon f(k)=k^\varepsilon\log k^{-\varepsilon}\le\frac12$, then for any $k>0$ and $\varepsilon\in\left(0,2(\frac14-\beta)\right)$, we derive that there exists a constant $C>0$ such that

$$\begin{aligned}&\max_{1\le m\le M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^m\|_\beta^2\right]+\sum_{j=0}^{M-1}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^{j+1}-\mathbf{e}^j\|_\beta^2\right]\\ &+2k\sum_{j=1}^{M}\mathbb{E}\left[\mathbb{1}_{\Omega_k}\|\mathbf{e}^j\|_{\frac12+\beta}^2\right]\le Ck^{-2\varepsilon}[k^{2(\frac14-\beta)}+\lambda_N^{-2(\frac14-\beta)}].\end{aligned}\tag{4.51}$$

This estimate completes the proof of the Theorem 2.8. □

2.5 Motivating Examples

In this section, we give two examples of evolution equations to which we can apply our abstract result.

2.5.1 Stochastic GOY and Sabra shell models

The first examples we can take is the GOY and Sabra shell models. To describe this model let us denote by $\mathbb{C}$ the field of complex numbers, $\mathbb{C}^{\mathbb{N}}$ the set of all $\mathbb{C}$-valued sequences, and we set

$$\mathrm{H}=\left\{\boldsymbol{u}=(\boldsymbol{u}_n)_{n\in\mathbb{N}}\subset\mathbb{C};\sum_{n=1}^{\infty}|\boldsymbol{u}_n|^2<\infty\right\}.$$

Let k_0 be a positive number and $\lambda_n = k_0 2^n$ be a sequence of positive numbers. The space H is a separable Hilbert space when endowed with the scalar product defined by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{k=1}^{\infty} \boldsymbol{u}_k \bar{\boldsymbol{v}}_k, \text{ for } \boldsymbol{u}, \boldsymbol{v} \in \mathrm{H},$$

where $\bar{z}$ denotes the conjugate of any complex number z.

We define a linear map A with domain

$$D(\mathrm{A}) = \{\boldsymbol{u} \in \mathrm{H}; \sum_{n=1}^{\infty} \lambda_n^4 |\boldsymbol{u}_n|^2 < \infty\},$$

by setting

$$\mathrm{A}\boldsymbol{u} = (\lambda_n^2 \boldsymbol{u}_n)_{n\in\mathbb{N}}, \text{ for } \boldsymbol{u} \in D(\mathrm{A}).$$

It is not hard to check that A is a self-adjoint and strictly positive operator. Moreover, the embedding $D(\mathrm{A}^\alpha) \subset D(\mathrm{A}^{\alpha+\varepsilon})$ is compact for any $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Thanks to this observation we can and will assume that there exists an orthonormal basis $\{\psi_n; n \in \mathbb{N}\}$ of H such that

$$\mathrm{A}\psi_n = \lambda_n \psi_n.$$

We can characterize the spaces $D(\mathrm{A}^\alpha)$, $\alpha \in \mathbb{R}$ as follow

$$D(\mathrm{A}^\alpha) = \{\boldsymbol{u} = (\boldsymbol{u}_n)_{n\in\mathbb{N}} \subset \mathbb{C}; \sum_{n=1}^{\infty} \lambda_n^{4\alpha} |\boldsymbol{u}_n|^2 < \infty\}.$$

For any $\alpha \in \mathbb{R}$ the space $\mathrm{V}_\alpha = D(\mathrm{A}^\alpha)$ is a separable Hilbert space when equipped with the scalar product

$$((\boldsymbol{u}, \boldsymbol{v}))_\alpha = \sum_{k=1}^{\infty} \lambda_k^{4\alpha} \boldsymbol{u}_k \bar{\boldsymbol{v}}_k, \text{ for } \boldsymbol{u}, \boldsymbol{v} \in \mathrm{V}_\alpha. \tag{5.52}$$

The norm associated to this scalar product will be denoted by $\|\boldsymbol{u}\|_\alpha$, $\boldsymbol{u} \in \mathrm{V}_\alpha$. In what follows we set $\mathrm{V} = D(\mathrm{A}^{\frac{1}{2}})$.

Now, let $\alpha_0 > \frac{1}{2}$ and $\{\mathfrak{w}_j; j \in \mathbb{N}\}$ be a sequence of mutually independent and identically distributed standard Brownian motions on filtered complete probability space $\mathfrak{U} = (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual condition. We set

$$W(t) = \sum_{n=0}^{\infty} \lambda_n^{-\alpha_0} \mathfrak{w}_n(t)\psi_n.$$

The process W defines a H-valued process with covariance $\mathrm{A}^{-2\alpha_0}$ which is of trace class. We also consider a Lipschitz map $g:[0,\infty)\to\mathbb{R}$ such that $|g(0)|<\infty$. We define a map $G:\mathrm{H}\to\mathscr{L}(\mathrm{H},\mathrm{V}_{\frac{1}{4}})$ defined by

$$G(u)h = g(\|u\|_0)h, \text{ for any } u\in\mathrm{H}, h\in\mathrm{H}.$$

This map satisfies Assumption (**G**).

With the above notation, the stochastic evolution equation describing our randomly perturbed GOY and Sabra shell models is given by

$$\begin{cases} d\boldsymbol{u} = [\mathrm{A}\boldsymbol{u} + \mathrm{B}(\boldsymbol{u},\boldsymbol{u})]dt + G(u)dW, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{5.53}$$

where $\mathrm{B}(\cdot,\cdot)$ is a bilinear map defined on $\mathrm{V}\times\mathrm{V}$ taking values in the dual space V^*. More precisely, we assume that the nonlinear term

$$\begin{aligned} \mathrm{B}:\ & \mathbb{C}^{\mathbb{N}}\times\mathbb{C}^{\mathbb{N}}\to\mathbb{C}^{\mathbb{N}}, \\ & (\boldsymbol{u},\boldsymbol{v})\mapsto \mathrm{B}(\boldsymbol{u},\boldsymbol{v}) = (b_1(\boldsymbol{u},\boldsymbol{v}),\dots,b_n(\boldsymbol{u},\boldsymbol{v}),\dots) \end{aligned}$$

for the GOY shell model (see [55]) is defined by

$$\begin{aligned} b_n(\boldsymbol{u},\boldsymbol{v}) &:= (\mathrm{B}(\boldsymbol{u},\boldsymbol{v}))_n \\ &:= i\lambda_n\left(\frac{1}{4}\overline{v}_{n-1}\overline{u}_{n+1} - \frac{1}{2}(\overline{u}_{n+1}\overline{v}_{n+2} + \overline{v}_{n+1}\overline{u}_{n+2}) + \frac{1}{8}\overline{u}_{n-1}\overline{v}_{n-2}\right), \end{aligned}$$

and for the Sabra shell model, it is defined by

$$\begin{aligned} b_n(\boldsymbol{u},\boldsymbol{v}) := (\mathrm{B}(\boldsymbol{u},\boldsymbol{v}))_n &:= \frac{i}{3}\lambda_{n+1}\left[\overline{v}_{n+1}u_{n+2} + 2\overline{u}_{n+1}v_{n+2}\right] \\ &+ \frac{i}{3}\lambda_n\left[\overline{u}_{n-1}v_{n+1} - \overline{v}_{n-1}u_{n+1}\right] \\ &+ \frac{i}{3}\lambda_{n-1}\left[2u_{n-1}v_{n-2} + u_{n-2}v_{n-1}\right], \end{aligned}$$

for any $\boldsymbol{u}=(u_1,\dots,u_n,\dots)\in\mathbb{C}^{\mathbb{N}}$ and $\boldsymbol{v}=(v_1,\dots,v_n,\dots)\in\mathbb{C}^{\mathbb{N}}$.

Lemma 2.13. *(a) For any non-negative numbers α and β such that $\alpha+\beta\in(0,\frac{1}{2}]$, there exists a constant $c_0>0$ such that*

$$\|\mathrm{B}(\boldsymbol{u},\boldsymbol{v})\|_{-\alpha}\le c_0\begin{cases}\|\boldsymbol{u}\|_{\frac{1}{2}-(\alpha+\beta)}\|\boldsymbol{v}\|_{\beta} & \text{for any } \boldsymbol{u}\in \mathrm{V}_{\frac{1}{2}-(\alpha+\beta)}, \boldsymbol{v}\in\mathrm{V}_\beta\\ \|\boldsymbol{u}\|_{\beta}\|\boldsymbol{v}\|_{\frac{1}{2}-(\alpha+\beta)} & \text{for any } \boldsymbol{v}\in \mathrm{V}_{\frac{1}{2}-(\alpha+\beta)}, \boldsymbol{u}\in\mathrm{V}_\beta.\end{cases}\tag{5.54}$$

(b) For any $\boldsymbol{u}\in\mathrm{H},\boldsymbol{v}\in\mathrm{V}$

$$\langle\boldsymbol{b}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{v}\rangle=0.\tag{5.55}$$

Proof. The item (b) was proved in [33, Proposition 1], thus we omit its proof.

Item (a) can be viewed as a generalization of [33, Proposition 1]. We will just prove the latter item for the Sabra shell model since the proofs for the two models are very similar. Let $\boldsymbol{u}\in\mathrm{V}_{\frac{1}{2}-(\alpha+\beta)}$, $\boldsymbol{v}\in\mathrm{V}_\beta$, and $\boldsymbol{w}\in\mathrm{V}_\alpha$ such that $\|\boldsymbol{w}\|_\alpha\le1$. We have

$$\begin{aligned}|\langle\mathrm{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\rangle|=&|\sum_{n=1}^\infty b_n(\boldsymbol{u},\boldsymbol{v})\bar{\boldsymbol{w}}_n|\le\sum_{n=1}^\infty|b_n(\boldsymbol{u},\boldsymbol{v})||\boldsymbol{w}_n|\\ \le&\frac{1}{3}\sum_{n=1}^\infty\lambda_{n+1}\left(|\boldsymbol{u}_{n+1}|\cdot|\boldsymbol{v}_{n+2}|+|\boldsymbol{u}_{n+2}|\cdot|\boldsymbol{v}_{n+1}|\right)|\boldsymbol{w}_n|\\ &+\frac{1}{3}\sum_{n=1}^\infty\lambda_{n}\left(|\boldsymbol{u}_{n-1}|\cdot|\boldsymbol{v}_{n+1}|+|\boldsymbol{u}_{n+1}|\cdot|\boldsymbol{v}_{n-1}|\right)|\boldsymbol{w}_n|\\ &+\frac{1}{3}\sum_{n=1}^\infty\lambda_{n-1}\left(|\boldsymbol{u}_{n-1}|\cdot|\boldsymbol{v}_{n-2}|+|\boldsymbol{u}_{n-2}|\cdot|\boldsymbol{v}_{n-1}|\right)|\boldsymbol{w}_n|\\ \le&I_1+I_2+I_3.\end{aligned}$$

For the term I_1 we have

$$\begin{aligned}I_1\le&\frac{1}{3}\sum_{n=1}^\infty\lambda_{n+1}|\boldsymbol{u}_{n+1}|\cdot|\boldsymbol{v}_{n+2}||\boldsymbol{w}_n|+\frac{1}{3}\sum_{n=1}^\infty\lambda_{n+1}|\boldsymbol{u}_{n+2}|\cdot|\boldsymbol{v}_{n+1}||\boldsymbol{w}_n|\\ \le&I_{1,1}+I_{1,2}.\end{aligned}$$

We will treat the term $I_{1,1}$. By Hölder's inequality we have

$$\begin{aligned}
I_{1,1} &\leq \frac{1}{3}\sum_{n=1}^{\infty} k_0 2\lambda^{1-2\alpha}|\boldsymbol{u}_{n+1}|\cdot|\boldsymbol{v}_{n+2}|\lambda_n^{2\alpha}|\boldsymbol{w}_n| \\
&\leq \frac{2}{3}k_0\left(\sum_{n=1}^{\infty} k_0 2\lambda_n^{2-4(\alpha+\beta)}|\boldsymbol{u}_{n+1}|^2\lambda_n^{4\beta}|\boldsymbol{v}_{n+2}|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\lambda_n^{4\alpha}|\boldsymbol{w}_n|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\|\boldsymbol{w}\|_\alpha \leq 1$ and $\lambda_{n+p} = k_0^p 2^p \lambda_n$ we can find a constant $C>0$ depending only on α,β and k_0 such that

$$\begin{aligned}
I_{1,1} &\leq C\left(\max_{k\in\mathbb{N}}\lambda_{n+1}^{2-4(\alpha+\beta)}|\boldsymbol{u}_{n+1}|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\lambda_{n+2}^{4\beta}|\boldsymbol{v}_n|^2\right)^{\frac{1}{2}} \\
&\leq C\left(\sum_{n=1}^{\frac{1}{2}}\lambda_{n+1}^{4[\frac{1}{2}-(\alpha+\beta)]}|\boldsymbol{u}_{n+1}|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\lambda_{n+2}^{4\beta}|\boldsymbol{v}_n|^2\right)^{\frac{1}{2}},
\end{aligned}$$

from which we easily derive that

$$I_{1,1} \leq C\|\boldsymbol{u}\|_{\frac{1}{2}-(\alpha+\beta)}\|\boldsymbol{v}\|_\beta.$$

One can use an analogous argument to show that

$$I_{1,2} \leq C\|\boldsymbol{u}\|_{\frac{1}{2}-(\alpha+\beta)}\|\boldsymbol{v}\|_\beta.$$

Hence,

$$I_1 \leq C\|\boldsymbol{u}\|_{\frac{1}{2}-(\alpha+\beta)}\|\boldsymbol{v}\|_\beta.$$

Using a similar argument we can also prove that for any non-negative numbers α and β satisfying $\alpha+\beta\in(0,\frac{1}{2}]$ there exists a constant $C>0$ such that

$$I_2 + I_3 \leq C\|\boldsymbol{u}\|_{\frac{1}{2}-(\alpha+\beta)}\|\boldsymbol{v}\|_\beta,$$

for any $\boldsymbol{u}\in\mathrm{V}_{\frac{1}{2}-(\alpha+\beta)}$ and $\boldsymbol{v}\in\mathrm{V}_\beta$. Therefore, for any non-negative numbers α and β satisfying $\alpha+\beta\in(0,\frac{1}{2}]$ we can find a constant $C>0$ such that

$$\|\mathrm{B}(\boldsymbol{u},\boldsymbol{v})\|_{-\alpha} \leq C\|\boldsymbol{u}\|_{\frac{1}{2}-(\alpha+\beta)}\|\boldsymbol{v}\|_\beta,$$

for any $\boldsymbol{u}\in \mathrm{V}_{\frac{1}{2}-(\alpha+\beta)}$ and $\boldsymbol{v}\in \mathrm{V}_{\beta}$. Interchanging the role of $\boldsymbol{u}$ and $\boldsymbol{v}$ we obtain that for any two numbers α and β as above there exists a positive constant C such that

$$\|\mathrm{B}(\boldsymbol{u},\boldsymbol{v})\|_{-\alpha}\leq C\|\boldsymbol{v}\|_{\frac{1}{2}-(\alpha+\beta)}\|\boldsymbol{u}\|_{\beta},$$

for any $\boldsymbol{v}\in \mathrm{V}_{\frac{1}{2}-(\alpha+\beta)}$ and $\boldsymbol{u}\in \mathrm{V}_{\beta}$. Thus, we have just completed the proof of the lemma for the Sabra shell model. As we mentioned earlier, the case of the GOY model can be dealt with a similar argument. □

For more mathematical results related to shell models we refer to [6], [11], [12], and references therein.

2.5.2 Stochastic nonlinear heat equation

Let $\mathscr{O}$ be a bounded domain of $\mathbb{R}^d$, $d=1,2$. We assume that its boundary $\partial\mathscr{O}$ is of class $\mathcal{C}^\infty$. Throughout this section we will denote by $\mathrm{H}^\theta(\mathscr{O})$, $\theta\in\mathbb{R}$, the (fractional) Sobolev spaces as defined in [92] and $\mathrm{H}^1_0(\mathscr{O})$ be the space of functions $\boldsymbol{u}\in\mathrm{H}^1$ such that $\boldsymbol{u}_{|_\mathscr{O}}=0$. In particular, we set $\mathrm{H}=L^2(\mathscr{O})$ and we denote its scalar product by $(\cdot,\cdot)$.

We define a continuous bilinear map $\mathfrak{a}:\mathrm{H}^1_0(\mathscr{O})\times\mathrm{H}^1_0(\mathscr{O})\to\mathbb{R}$ by setting

$$\mathfrak{a}(\boldsymbol{u},\boldsymbol{v})=(\nabla\boldsymbol{u},\nabla\boldsymbol{v}),$$

for any $\boldsymbol{u},\boldsymbol{v}\in\mathrm{H}^1_0(\mathscr{O})$. Thanks to the Riesz representation there exists a densely linear map A with domain $D(\mathrm{A})\subset\mathrm{H}$ such that

$$\langle\mathrm{A}\boldsymbol{v},\boldsymbol{u}\rangle=\mathfrak{a}(\boldsymbol{v},\boldsymbol{u}),$$

for any $\boldsymbol{u},\boldsymbol{v}\in\mathrm{H}^1_0(\mathscr{O})$. It is well known that A is a self-adjoint and definite positive and its eigenfunctions $\{\psi_n;n\in\mathbb{N}\}\subset\mathcal{C}^\infty(\mathscr{O})$ form an orthonormal basis of H. The family of eigenvalues associated to $\{\psi_n;n\in\mathbb{N}\}$ is denoted by $\{\lambda_n;n\in\mathbb{N}\}$. Observe that the asymptotic behaviour of the eigenvalues is given by $\lambda_n\sim\lambda_1 n^{\frac{2}{d}}$. For any $\alpha\in\mathbb{R}$ we set $\mathrm{V}_\alpha=D(\mathrm{A}^\alpha)$, in particular we put $\mathrm{V} := D(\mathrm{A}^{\frac{1}{2}})$. We always understand that the norm in V_α is denoted by $\|\cdot\|_0$.

Now, let $\alpha_0 > \frac{d+1}{4}$ and $\{\mathfrak{w}_j; j \in \mathbb{N}\}$ be a sequence of mutually independent and identically distributed standard Brownian motions on filtered complete probability space $\mathfrak{U} = (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual condition. We set

$$W(t) = \sum_{n=0}^{\infty} \lambda_n^{-\alpha_0} \mathfrak{w}_n(t) \psi_n.$$

The process W defines a H-valued with covariance $\mathrm{A}^{-2\alpha_0}$ which is of trace class. We also consider a Lipschitz map $g : [0, \infty) \to \mathbb{R}$ such that $|g(0)| < \infty$. We define a map $G : \mathrm{H} \to \mathscr{L}(\mathrm{H}, \mathrm{V}_{\frac{1}{4}})$ defined by

$$G(u)h = g(\|u\|_0)h, \text{ for any } u \in \mathrm{H}, h \in \mathrm{H}.$$

This map satisfies Assumption (**G**).

The second example we can treat is the stochastic nonlinear heat equation

$$d\boldsymbol{u} - [\Delta \boldsymbol{u} - |\boldsymbol{u}|\boldsymbol{u}]dt = g(\|\boldsymbol{u}\|_0)dW, \tag{5.56a}$$

$$\boldsymbol{u} = 0 \text{ on } \partial\mathscr{O}, \tag{5.56b}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0 \quad x \in \mathscr{O}. \tag{5.56c}$$

This stochastic system can be rewritten as an abstract stochastic evolution equation

$$d\boldsymbol{u} + [\mathrm{A}\boldsymbol{u} + B(\boldsymbol{u}, \boldsymbol{u})]dt = G(\boldsymbol{u})dW, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in \mathrm{H},$$

where A and G are defined as above and the $D(\mathrm{A}^{-\frac{1}{2}})$-valued nonlinear map B is defined on $\mathrm{H} \times D(\mathrm{A}^{\frac{1}{2}})$ or $D(\mathrm{A}^{\frac{1}{2}}) \times \mathrm{H}$ by setting

$$\mathrm{B}(\boldsymbol{u}, \boldsymbol{v}) = |\boldsymbol{u}|\boldsymbol{v},$$

for any $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{H} \times D(\mathrm{A}^{\frac{1}{2}})$ or $(\boldsymbol{u}, \boldsymbol{v}) D(\mathrm{A}^{\frac{1}{2}}) \times \mathrm{H}$. It is clear that

$$\langle \mathrm{A}\boldsymbol{v} + \mathrm{B}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{v} \rangle \geq \|\boldsymbol{v}\|_{\frac{1}{2}}^2, \tag{5.57}$$

for any $\boldsymbol{u}, \boldsymbol{v} \in \mathrm{V}$. Here we should note that thanks to the solution of Kato's square root problem in [3, Theorem 1], see also [1, Section 7], we have $\|\boldsymbol{u}\|_{\frac{1}{2}} \simeq |\nabla \boldsymbol{u}|$ for any $\boldsymbol{u} \in \mathrm{H}_0^1(\mathscr{O})$, i.e, $\mathrm{V} = \mathrm{H}_0^1(\mathscr{O})$.

Now we claim that for any numbers $\alpha \in [0, \frac{1}{2})$ and $\beta \in (0, \frac{1}{2})$ such that $\alpha + \beta \in (0, \frac{1}{2})$, there exists a constant $c_0 > 0$ such that

$$\|\mathrm{B}(\boldsymbol{u}, \boldsymbol{v})\|_{-\alpha} \leq c_0 \begin{cases} \|\boldsymbol{u}\|_{\frac{1}{2}-(\alpha+\beta)} \|\boldsymbol{v}\|_{\beta} & \text{for any } \boldsymbol{u} \in \mathrm{V}_{\frac{1}{2}-(\alpha+\beta)}, \boldsymbol{v} \in \mathrm{V}_{\beta} \\ \|\boldsymbol{u}\|_{\beta} \|\boldsymbol{v}\|_{\frac{1}{2}-(\alpha+\beta)} & \text{for any } \boldsymbol{v} \in \mathrm{V}_{\frac{1}{2}-(\alpha+\beta)}, \boldsymbol{u} \in \mathrm{V}_{\beta}, \end{cases} \tag{5.58}$$

and

$$\|\mathrm{B}(\boldsymbol{u}, \boldsymbol{v})\|_{-\frac{1}{2}} \leq c_0 \|\boldsymbol{u}\|_{\frac{1}{4}} \|\boldsymbol{v}\|_{\frac{1}{4}} \qquad \text{for any } \boldsymbol{v} \in \mathrm{V}_{\frac{1}{4}}, \boldsymbol{u} \in \mathrm{V}_{\frac{1}{4}}. \tag{5.59}$$

To prove these inequalities, let $\beta > 0$ such that $\alpha + \beta < \frac{1}{2}$. Since

$$\left(\frac{1}{2} - \alpha\right) + \left(\frac{1}{2} - 1 + 2(\alpha + \beta)\right) + \left(\frac{1}{2} - \beta\right) = 1,$$

we have

$$|\langle |\boldsymbol{u}| \boldsymbol{v}, \boldsymbol{w} \rangle| \leq C_0 \|\boldsymbol{u}\|_{L^r} \|\boldsymbol{v}\|_{L^s} \|\boldsymbol{w}\|_{L^q}, \tag{5.60}$$

where the constants q, r, s are defined through

$$\frac{1}{q} = \frac{1}{2} - \alpha, \; \frac{1}{s} = \alpha + \beta, \; \frac{1}{r} = \frac{1}{2} - \beta.$$

Recall that $\mathrm{V}_\alpha \subset \mathrm{H}^{2\alpha} \subset L^q$ with $\frac{1}{q} = \frac{1}{2} - \alpha$ if $\alpha \in (0, \frac{1}{2})$ and $q \in [2, \infty)$ arbitrary if $\alpha = \frac{1}{2}$. Then, we derive from (5.60) that the second inequality in (5.58) holds. By interchanging the role of r and s we derive that the first inequality in (5.58) also holds. One can establish (5.59) with the same argument. The estimates (5.58) and (5.59) easily imply (2.3) and (2.7).

Now we need to check that $\mathrm{B}(\cdot, \cdot)$ satisfies (2.4). For this purpose we observe that there exists a constant $C > 0$ such that

$$|\mathrm{B}(\boldsymbol{u}, \boldsymbol{v})| \leq C \|\boldsymbol{u}\|_0 \|\boldsymbol{v}\|_{L^\infty},$$

which with the continuous embedding $\mathrm{V}_{\frac{1}{2}+\varepsilon} \subset L^\infty$ for any $\varepsilon > 0$ implies (2.4).

Chapter 3

Time-discretization scheme of stochastic 2-D Navier–Stokes equations by a penalty-projection method

A time-discretization of the stochastic incompressible Navier–Stokes problem by penalty method is analyzed. The main issue concerns the nonlinear term which in the stochastic framework prevents from using a Gronwall argument. Moreover, the approximate solution is slightly compressible and therefore, the nonlinear term does not satisfy the additional orthogonal property. Usually in two-dimension and with a periodic boundary condition this orthogonal property allows to get some useful estimates. To tackle these issues we use the classical decomposition of the solution into an Ornstein–Uhlenbeck process and a solution of a deterministic Navier–Stokes equation depending on a stochastic process. The first part is stochastic but linear while the second one is nonlinear but deterministic. Both sub problems are still approximated with a numerical scheme based on penalty method. Error estimates for both of them are derived, combined, and eventually arrive at a convergence in probability with order 1/4 of the main algorithm towards the initial problem for the pair of variables velocity and pressure. The strong convergence of the scheme is achieved by means of the Bayes formula.

3.1 Introduction

Let $T>0$ and $\mathfrak{P}:=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a filtered probability space with the filtration $\mathbb{F}:=(\mathscr{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions. We refer to the following system of equations as the stochastic *incompressible Navier–Stokes problem* (SNS),

$$\begin{cases} \boldsymbol{u}_t-\nu\Delta\boldsymbol{u}+[\boldsymbol{u}\cdot\nabla]\boldsymbol{u}+\nabla\mathsf{p}=\dot{\boldsymbol{W}}, \text{ in } \mathbb{R}^2, \\ \qquad\qquad\qquad\qquad \operatorname{div}\boldsymbol{u}=0, \text{ in } \mathbb{R}^2. \end{cases} \tag{1.1}$$

Here $\boldsymbol{u}=\{\boldsymbol{u}(t,\boldsymbol{x}):t\in[0,T]\}$ and $\mathsf{p}=\{\mathsf{p}(t,\boldsymbol{x}):t\in[0,T]\}$ are unknown stochastic processes on $\mathbb{R}^2$, representing respectively the velocity and the pressure of a fluid with kinematic viscosity ν filling the whole space $\mathbb{R}^2$, in each point of $\mathbb{R}^2$.

In $\mathbb{R}^2$, we endow (1.1) with an initial condition,

$$\boldsymbol{u}(0,\boldsymbol{x})=\boldsymbol{u}_0(\boldsymbol{x})$$

and periodic boundary conditions,

$$\boldsymbol{u}(t,\boldsymbol{x}+L\boldsymbol{b}_j)=\boldsymbol{u}(t,\boldsymbol{x}), \qquad j=1,2, \quad t\in[0,T],$$

where $\boldsymbol{u}$ has a vanishing spatial average. Here $(\boldsymbol{b}_1,\boldsymbol{b}_2)$ is the canonical basis of $\mathbb{R}^2$ and $L>0$ is the period in the jth direction; $D=(0,L)\times(0,L)$ is the square of the period. The term $\boldsymbol{W}:=\{\boldsymbol{W}(t):t\in[0,T]\}$ is a $\boldsymbol{\mathcal{K}}$-valued Wiener process where $\boldsymbol{\mathcal{K}}$ is a separable Hilbert space.

An incompressible fluid flow is usually modeled with a deterministic Navier–Stokes equation. The stochastic Navier–Stokes (1.1) is a well known model that captures fluid instabilities under ambient noise [14] or small scales perturbation for homogeneous turbulent flow, see e.g. [9], [15], and [80].

Strong approximation of Stochastic Partial Differential Equations (SPDEs) such as the SNS is mostly the natural approach because of its link with the numerical analysis of deterministic equations. However, this type of approximation is often inaccessible for nonlinear SPDEs. Indeed, when the nonlinearity is neither globally Lipschitz nor monotone, weak convergence or convergence in probability are frequently considered, see e.g. [5], [22], [39], [41], [49], [60], and [76]. Another notion, the *speed of convergence in probability*, was first put forward by Printems in [82] for some parabolic SPDEs. Regardless of the type of convergence, we may also have to consider different approaches according to the characteristic of the equation. In particular for the SNS, we can use e.g. a numerical approximation using an Ornstein–Uhlenbeck as an auxiliary step such as in [49], or using splitting methods such as in [10, 28], or using the Wiener chaos expansion such as in [60], or using the layer method (probabilistic representation) such as in [76]. Carelli and Prohl proved in [30] that a speed of convergence in probability can be derived from some direct numerical approximations of the SNS. Here the convergence concerns only one variable, the velocity field.

The SNS shares the same complexity as its deterministic counter part, when it comes to computations. Velocity and pressure are both coupled by the incompressibility constraint, which often requires a saddle point problem to solve. To break this saddle point character of the system, velocity and pressure are decoupled by perturbing the divergence free condition by a penalty method [72, Chapter 3] and choosing a penalty operator in a similar fashion as in [34]. This consists, for every $\varepsilon > 0$, to solve the penalized version of (1.1), i.e.

$$\begin{cases} \boldsymbol{u}_t^\varepsilon - \nu\Delta\boldsymbol{u}^\varepsilon + [\boldsymbol{u}^\varepsilon\cdot\nabla]\boldsymbol{u}^\varepsilon + \frac{1}{2}(\operatorname{div}\boldsymbol{u}^\varepsilon)\boldsymbol{u}^\varepsilon + \nabla\mathsf{p}^\varepsilon = \dot{\boldsymbol{W}}, \text{ in } \mathbb{R}^2, \\ \qquad\qquad\qquad\qquad \operatorname{div}\boldsymbol{u}^\varepsilon + \varepsilon\mathsf{p}^\varepsilon = 0, \text{ in } \mathbb{R}^2. \end{cases} \tag{1.2}$$

This belongs to a more general class of approximation methods for the Navier–Stokes equation, called projection and quasi-compressible methods. This includes the artificial compressibility method, the pressure stabilization, and the pressure correction method. For a complete survey or review on these methods, the reader is referred for instance to [56] or the monograph [83]. Even though these methods are already very popular and efficient in the deterministic framework, the paper of Carelli, Hausenblas, and Prohl, see [28], is the only work, which treats on projection and quasi-compressible methods for the stochastic Stokes equation by using the pressure stabilization and the pressure correction methods to derive an algorithm based on a time marching strategy. The artificial compressibility method has already been used to prove existence and pathwise uniqueness of global strong solutions of SNS, see [74], or adapted solutions to the backward SNS by a local monotonicity argument, see [93]. Concerning the penalty method, it has been introduced in [89] by Temam for the deterministic Navier–Stokes equations where he established its convergence. Since then, the method has been improved by Shen with the addition of error estimates in a sequence of papers including [86] and [87]. It has been used (with a different penalty operator) in a stochastic framework in [26] as an auxiliary step to prove the existence of a spatially homogeneous solution of a SNS driven by a spatially homogeneous Wiener random field.

In this paper, we study a semi-implicit time-discretization scheme for the full stochastic incompressible 2D Navier–Stokes equation based on the penalized system Equation (1.2). Formally, the scheme consists of solving the following equations:

Given $0<\eta<1/2$, $\alpha>1$, $\boldsymbol{u}_0$, $\phi^0=0$. For $\ell=1,\dots,M$:

- Step 1 (Penalization): Find $\tilde{\boldsymbol{u}}^\ell$ such that

$$\tilde{\boldsymbol{u}}^\ell-\nu k\Delta\tilde{\boldsymbol{u}}^\ell+k\tilde{B}(\tilde{\boldsymbol{u}}^\ell,\tilde{\boldsymbol{u}}^\ell)-k^{1-\eta}\nabla\mathrm{div}\,\tilde{\boldsymbol{u}}^\ell=\Delta_\ell\boldsymbol{W}+\boldsymbol{u}^{\ell-1}-k\nabla\phi^{\ell-1};$$

- Step 2: Find ϕ^ℓ such that

$$\Delta\phi^\ell=\Delta\phi^{\ell-1}+(\alpha k)^{-1}\mathrm{div}\,\tilde{\boldsymbol{u}}^\ell;$$

- Step 3 (Projection): $\boldsymbol{u}^\ell=\boldsymbol{P}_{\mathbb{H}}\tilde{\boldsymbol{u}}^\ell$, i.e.

$$\boldsymbol{u}^\ell=\tilde{\boldsymbol{u}}^\ell-\alpha k\nabla(\phi^\ell-\phi^{\ell-1}),\qquad \mathsf{p}^\ell=\widetilde{\mathsf{p}}^\ell+\phi^\ell+\alpha(\phi^\ell-\phi^{\ell-1}).$$

More details are given in Algorithm 3.4. We focus on the time-discretization, since different technical endeavors may obscure the main difficulty of the time-discretization. A paper which is similar to ours is [30], where the authors show the convergence in probability of a space-time discretization of stochastic incompressible Navier–Stokes in 2D. The numerical schemes they use are implicit/semi-implicit in time and use a divergence-free finite element pairing such as the Scott–Vogelius finite element for the velocity and the pressure. The proof needs also some a priori estimates of the approximate solution in $\mathbb{V}$, the divergence-free space with finite enstrophy. These estimates are obtained by means of the additional orthogonal property of the nonlinear term in 2D and under periodic boundary conditions, i.e. $\langle[\boldsymbol{u}.\nabla]\boldsymbol{u},\Delta\boldsymbol{u}\rangle=0$ for each $\boldsymbol{u}\in\mathbb{V}$. As we see in Equation (1.2), the approximate solution is only slightly compressible, thus $\boldsymbol{u}^{\varepsilon}\notin\mathbb{V}$. Even with the projection step added, the additional orthogonal property required in [30] is inapplicable here. To overcome this issue we use the classical decomposition of the SNS into an Ornstein–Uhlenbeck process and a deterministic SNS. This decomposition has already been used for different purpose, e.g. in [23], [45], [46], [48], and [49]. The algorithm depends on the spatial perturbation parameter $\varepsilon>0$, a stability preserving parameter $\alpha>1$, and the time-step k. If we fix $\varepsilon=k^{\eta}$ with some $0<\eta<1/2$ and with any $\alpha>1$, a speed of convergence in probability of order $1/4$ is obtained for both velocity and pressure. Then, by means of the law of total probability, we deduce strong convergence of the scheme for both variables velocity and pressure. In this context, we respond to the lack of results regarding (speed of) convergence for the pressure iterates from algorithms based on pseudo-compressible and projection method for stochastic (Navier)–Stokes equations addressed by [28].

This paper is organized as follows. In Section 3.2, we introduce the assumptions and notations used and review some of the basic facts of the SNS, which are important for the proof, such as the time regularity of the solution and present a splitting argument that will be used later on. In the Section 3.3, we develop stability of the main algorithm and derive error estimates for some auxiliary algorithms. In Section 3.8, we treat the speed of convergence in probability, then the strong convergence of the main algorithm.

3.2 Preliminaries

In this section, we present the assumptions and notations used in this work. We also prove the time regularity of the pressure. As a preparatory work, before going into the numerical analysis, we formulate (1.1) according to the classical decomposition of the SNS into an Ornstein–Uhlenbeck process and a deterministic Navier–Stokes depending on a stochastic process.

3.2.1 Functional settings and notations

To introduce a spatial variable process, i.e. a vector-valued process to the Brownian motion $\boldsymbol{W}$, we introduce a family of mutually independent and identically distributed real-valued Brownian motions $\{\beta_j(t): t\in[0,T]\}$, $j\in\mathbb{N}$, and a covariance $\boldsymbol{Q}$. If $\boldsymbol{Q}\in\mathscr{L}(\boldsymbol{\mathcal{K}})$ (the space of bounded linear operators from $\boldsymbol{\mathcal{K}}$ to $\boldsymbol{\mathcal{K}}$) is non-negative definite and symmetric with an orthonormal basis $\{\boldsymbol{d}_j : j\in\mathbb{N}\}$ of eigenfunctions with corresponding eigenvalues $q_j\geq 0$ such that $\sum_{j\in\mathbb{N}} q_j<\infty$, then $\boldsymbol{Q}\in\mathscr{L}_1(\boldsymbol{\mathcal{K}})$ (the space of trace-class operator on $\boldsymbol{\mathcal{K}}$) and the series

$$\boldsymbol{W}(t)=\sum_{j=1}^{\infty}\sqrt{q_j}\beta_j(t)\boldsymbol{d}_j, \qquad \forall\, t\in[0,T],$$

converges in $L^2(\Omega;\mathcal{C}([0,T];\boldsymbol{\mathcal{K}}))$ and it defines a $\boldsymbol{\mathcal{K}}$-valued Wiener process with covariance operator $\boldsymbol{Q}$ also called $\boldsymbol{Q}$-*Wiener process*. Furthermore, for any $\ell\in\mathbb{N}$ there exists a constant $C_\ell>0$ such that

$$\mathbb{E}\|\boldsymbol{W}(t)-\boldsymbol{W}(s)\|_{\boldsymbol{\mathcal{K}}}^{2\ell}\leq C_\ell(t-s)^\ell(\mathrm{Tr}\boldsymbol{Q})^\ell, \qquad \forall\, t\in[0,T] \quad \text{and} \quad \forall\, s\in[0,t). \tag{2.3}$$

Let $\boldsymbol{\mathcal{H}}$ be another separable Hilbert space. We define by $\mathscr{L}_2(\boldsymbol{\mathcal{K}}_{\boldsymbol{Q}},\boldsymbol{\mathcal{H}})$ the space of Hilbert–Schmidt operator from $\boldsymbol{\mathcal{K}}_{\boldsymbol{Q}}$ to $\boldsymbol{\mathcal{H}}$, where $\boldsymbol{\mathcal{K}}_{\boldsymbol{Q}}$ is the separable Hilbert space defined by $\boldsymbol{\mathcal{K}}_{\boldsymbol{Q}}:=\boldsymbol{Q}^{1/2}\boldsymbol{\mathcal{K}}$.

We can define the $\boldsymbol{\mathcal{H}}$-valued Itô integral with respect to a $\boldsymbol{Q}$-Wiener process $\boldsymbol{W}$ by

$$\int_0^t \Phi(s)d\boldsymbol{W}(s):=\sum_{j=1}^{\infty}\int_0^t \Phi(s)\sqrt{q_j}\boldsymbol{d}_j d\beta_j(s), \quad \forall\, t\in[0,T]$$

which is also a $\boldsymbol{\mathcal{H}}$-valued martingale satisfying the Burkholder–Davis–Gundy inequality (see [63, Theorem 3.3.28]), given by

$$\mathbb{E}\sup_{0\leq s\leq t}\left\|\int_0^s \Phi(\tau)d\boldsymbol{W}(\tau)\right\|_{\boldsymbol{\mathcal{H}}}^{2r}\leq C_r\left(\int_0^t\|\Phi(\tau)\|^2_{\mathscr{L}_2(\boldsymbol{\mathcal{K}}_Q,\boldsymbol{\mathcal{H}})}d\tau\right)^r, \forall\, t\in[0,T], \forall\, r>0. \tag{2.4}$$

In the case of scalar functions, we denote the usual Sobolev spaces by $W^{m,2}(D)$ $(m=$

$0,1,2,\ldots,\infty$). The corresponding scalar product and the corresponding norm for any non-negative integer m is denoted by

$$(\boldsymbol{u},\boldsymbol{v})_m = \int_D \sum_{\ell=0}^{m} \partial^\ell \boldsymbol{u}\, \partial^\ell \boldsymbol{v}\, d\boldsymbol{x} \quad \text{and} \quad \|\boldsymbol{u}\|_m = \|\boldsymbol{u}\|_{W^{m,2}} = (\boldsymbol{u},\boldsymbol{u})_m^{1/2}.$$

By $W_0^{m,2}(D)$, we denote the closure in $W^{m,2}(D)$ of the space $\mathcal{C}_0^\infty(D)$ of all smooth functions defined on D with compact support. Further, $W^{-m,2}(D)$ is the space that is dual to $W^{m,2}(D)\cap W_0^{1,2}(D)$. Particularly for $m=0$, the space $W^{m,2}(D)$ is usually denoted by $L^2(D)$ and then the scalar product and norm are denoted simply by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We reserve the notation $\langle\cdot,\cdot\rangle$ for the duality bracket. In general, we denote the usual Lebesgue spaces by L^p, $1\le p\le\infty$, which are endowed with the standard norms denoted by $\|\cdot\|_{L^p}$. We denote by L^p_{per} and $W^{m,2}_{\text{per}}$ the Lebesgue and Sobolev spaces of functions that are periodic and have vanishing spatial average, respectively. The spaces of vector-valued functions will be indicated with Blackboard bold letters, for instance $\mathbb{L}^2_{\text{per}} := (L^2_{\text{per}})^2$. In further analyses, we will not distinguish between the notation of inner products and norms in scalar or vector-valued applications.

The two spaces frequently used in the theory of Navier–Stokes equations are

$$\mathbb{H} = \left\{\boldsymbol{v}\in\mathbb{L}^2_{\text{per}}(D) : \operatorname{div}\boldsymbol{v}=0 \text{ in } \mathbb{R}^2\right\} \quad \text{and} \quad \mathbb{V} = \left\{\boldsymbol{v}\in\mathbb{W}^{1,2}_{\text{per}}(D) : \operatorname{div}\boldsymbol{v}=0 \text{ in } \mathbb{R}^2\right\}.$$

The space $\mathbb{V}$ is a Hilbert space with the scalar product $(\cdot,\cdot)_1$ and the Hilbert norm induced by $\mathbb{W}^{1,2}$.

Let $\boldsymbol{P}_{\mathbb{H}}$ denote the $\mathbb{L}^2$-projection on the space $\mathbb{H}$ also known as *Helmholtz–Leray projector*. As an orthogonal projection, it satisfies the following identity

$$\langle \boldsymbol{P}_{\mathbb{H}}\boldsymbol{v}-\boldsymbol{v}, \boldsymbol{P}_{\mathbb{H}}\boldsymbol{v}\rangle = 0, \quad \forall \boldsymbol{v}\in\mathbb{L}^2_{\text{per}}. \tag{2.5}$$

The projection $\mathbb{P}_{\mathbb{H}}$ is continuous from $\mathbb{W}^{1,2}(D)_0$ into $\mathbb{W}^{1,2}(D)$ (cf. [91, Remark 1.6] and [19, Proposition IV.3.7.]) and we can find a positive constant $C=C(D)$ such that

$$\|\boldsymbol{P}_{\mathbb{H}}\boldsymbol{u}\|_1 \le C\|\boldsymbol{u}\|_1, \qquad \forall \boldsymbol{u}\in\mathbb{W}^{1,2}(D). \tag{2.6}$$

Due to the Helmholtz–Hodge–Leray decomposition, any function $\boldsymbol{u}\in\mathbb{L}^2(D)$ can be represented as $\boldsymbol{u}=\boldsymbol{P}_{\mathbb{H}}\boldsymbol{u}+\nabla\mathsf{q}$, where q is a scalar D-periodic function such that $\mathsf{q}\in L^2_{\text{per}}(D)$. It is natural to introduce the notation $\boldsymbol{P}_{\mathbb{H}}^{\perp}\boldsymbol{u}:=\nabla\mathsf{q}$ and hence write

$$\boldsymbol{u}=\boldsymbol{P}_{\mathbb{H}}\boldsymbol{u}+\boldsymbol{P}_{\mathbb{H}}^{\perp}\boldsymbol{u},\quad \text{with}\quad \boldsymbol{P}_{\mathbb{H}}^{\perp}\boldsymbol{u}\in\mathbb{H}^{\perp}=\left\{\boldsymbol{v}:\boldsymbol{v}\in\mathbb{L}^2(D),\boldsymbol{v}=\nabla\mathsf{q}\right\}.$$

With periodic boundary conditions the Stokes operator $\boldsymbol{A}=-\boldsymbol{P}_{\mathbb{H}}\Delta$ coincides with the Laplacian operator $-\Delta$. The operator $\boldsymbol{A}$ can be seen as an unbounded positive linear selfadjoint operator on $\mathbb{H}$ with domain $\mathscr{D}(A)=\mathbb{W}^{2,2}\cap\mathbb{V}$. We can define the powers $\boldsymbol{A}^{\alpha}$, $\alpha\in\mathbb{R}$, with domain $\mathscr{D}(\boldsymbol{A}^{\alpha})$. The norm $\|\boldsymbol{A}^{s/2}\boldsymbol{u}\|$ on $\mathscr{D}(\boldsymbol{A}^{s/2})$ is equivalence to the norm induced by $\mathbb{W}_0^{s,2}(D)$. In addition, we also have the following equivalence of norm:

Proposition 3.1 (Equivalence of norms)**.** *There exist positive numbers c_1 and c_2 such that $\forall\boldsymbol{u}\in\mathbb{H}$:*

(i) $\|\boldsymbol{A}^{-1}\boldsymbol{u}\|_s\le c_1\|\boldsymbol{u}\|_{s-2},\quad s=1,2;$

(ii) $c_2\|\boldsymbol{u}\|_{-1}^2\le(\boldsymbol{A}^{-1}\boldsymbol{u},\boldsymbol{u})\le c_1^2\|\boldsymbol{u}\|_{-1}^2.$

Proof. The reader is referred to [85, Equation (2.1)] or [83, Lemma 2.3] for the proof. It relies on the elliptic regularity of the Stokes operator and the definition of negative Sobolev norms. □

We now introduce some operators usually associated with the Navier–Stokes equations and their approximations. In particular,

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{u},\boldsymbol{v})&=[\boldsymbol{u}\cdot\nabla]\boldsymbol{v},\quad \tilde{\boldsymbol{B}}(\boldsymbol{u},\boldsymbol{v})=\boldsymbol{B}(\boldsymbol{u},\boldsymbol{v})+(\operatorname{div}\boldsymbol{u})\boldsymbol{v}/2,\\ b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})&=\langle\boldsymbol{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\rangle,\quad \tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\langle\tilde{\boldsymbol{B}}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\rangle.\end{aligned}$$

The trilinear forms b and $\tilde{b}$ satisfy the following properties:

Skew-symmetry property

$$\begin{aligned}b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})&=-b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}),\qquad \boldsymbol{u}\in\mathbb{H}\ \text{and}\ \boldsymbol{v},\boldsymbol{w}\in\mathbb{V},\\ \tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})&=-\tilde{b}(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}),\qquad \boldsymbol{u},\boldsymbol{v}\in\mathbb{W}^{1,2}(D)\ \text{and}\ \boldsymbol{w}\in\mathbb{W}^{1,2}_{\text{per}}(D).\end{aligned}\tag{2.7}$$

Orthogonal property

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v})=0, \quad \forall\,\boldsymbol{u}\in\mathbb{H}, \forall\,\boldsymbol{v}\in\mathbb{W}^{1,2}_{\rm per}(D); \quad \tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v})=0, \quad \forall\,\boldsymbol{u},\boldsymbol{v}\in\mathbb{W}^{1,2}_{\rm per}(D). \tag{2.8}$$

The following estimates of the trilinear form $\tilde{b}$ will be used repeatedly in the upcoming sections. Let $\boldsymbol{v}\in\mathbb{W}^{2,2}(D)\cap\mathbb{W}^{1,2}_{\rm per}(D)$ and $\boldsymbol{u},\boldsymbol{w}\in\mathbb{W}^{1,2}_{\rm per}(D)$; a combination of integration by parts and Hölder inequality gives

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\leq\|\boldsymbol{u}\|_{\mathbb{L}^4}\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_{\mathbb{L}^4}. \tag{2.9}$$

From this estimate we can deduce using the Sobolev embedding $\mathbb{W}^{1,2}(D)\subset\mathbb{L}^4(D)$,

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\leq C(L)\|\boldsymbol{u}\|_1\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|_1, \tag{2.10}$$

or using the Ladyzhenskaya's inequality $\|\boldsymbol{u}\|_{\mathbb{L}^4}\leq C(L)\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|_1^{1/2}$,

$$\tilde{b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\leq C(L)\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|_1^{1/2}\|\boldsymbol{v}\|_1\|\boldsymbol{w}\|^{1/2}\|\boldsymbol{w}\|_1^{1/2}. \tag{2.11}$$

To find more about the above properties or additional properties of b or $\tilde{b}$, and other estimates, the reader is referred to [90, Section 2.3].

3.2.2 General assumption and spatial regularity of the solution

In the following we choose $\boldsymbol{\mathcal{H}}=\mathbb{V}$, i.e. a solenoidal noise in SNS. An example of solenoidal noise is given in [28, Section 6]. We summarize the assumptions needed for data $\boldsymbol{W}$, $\boldsymbol{Q}$, and $\boldsymbol{u}_0$:

(S_1) For $\boldsymbol{Q}\in\mathscr{L}(\boldsymbol{\mathcal{K}})$, let $\boldsymbol{W}=\{\boldsymbol{W}(t):t\in[0,T]\}$ be a $\boldsymbol{Q}$-Wiener process with values in a separable Hilbert space $\boldsymbol{\mathcal{K}}$ defined on the stochastic basis $\mathfrak{P}$.

(S_2) $\boldsymbol{u}_0\in\mathbb{V}$.

In addition, we recall the notion of a strong solution to (1.1).

Definition 3.2 (Strong solution)**.** *Let $T>0$ be given and let Assumptions (S_1) and (S_2) be valid, with $\boldsymbol{\mathcal{H}}=\mathbb{V}$. A $\mathbb{V}$-valued process $\boldsymbol{u}=\{\boldsymbol{u}(t,\cdot):t\in[0,T]\}$ on $(\mathscr{F}_t)_{0\le t\le T}$ is a strong solution to (1.1) if*

(i) $\boldsymbol{u}(\cdot,\cdot,\omega)\in\mathcal{C}([0,T];\mathbb{V})\cap L^2(0,T;\mathbb{W}^{2,2}\cap\mathbb{V})$ $\mathbb{P}$*-a.s.,*

(ii) for every $t\in[0,T]$ and every $\boldsymbol{\varphi}\in\mathbb{V}$, there holds $\mathbb{P}$-a.s.

$$(\boldsymbol{u}(t),\boldsymbol{\varphi})+\nu\int_0^t(\nabla\boldsymbol{u}(s),\nabla\boldsymbol{\varphi})+b(\boldsymbol{u}(s),\boldsymbol{u}(s),\boldsymbol{\varphi})ds=(\boldsymbol{u}_0,\boldsymbol{\varphi})+\int_0^t(\boldsymbol{\varphi},d\boldsymbol{W}(s)).$$

If Assumption (S_1) holds and $\boldsymbol{\mathcal{H}}=\mathbb{V}$, we can prove (cf. [47, Appendix 1]) that the solutions $\boldsymbol{u}$ of (1.1) as defined by Definition 3.2 satisfies for $2\le p<\infty$ the estimate

$$\mathbb{E}\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|^p+\nu\mathbb{E}\left[\int_0^T\|\boldsymbol{u}(s)\|^{p-2}\|\nabla\boldsymbol{u}(s)\|^2ds\right]\le C_{T,p},\tag{2.12}$$

where $C_{T,p}=C_{T,p}(\mathrm{Tr}\boldsymbol{Q},\mathbb{E}\|\boldsymbol{u}_0\|^p,\mathbb{E}\|\boldsymbol{u}_0\|_{\mathbb{V}}^p)>0$. In addition to the above estimate, if Assumption (S_2) holds for $2\le p<\infty$, it is proven in [30, Lemma 2.1] that $\boldsymbol{u}$ satisfies also the estimates

$$\sup_{0\le t\le T}\mathbb{E}\|\boldsymbol{u}(t)\|_{\mathbb{V}}^p+\nu\mathbb{E}\left[\int_0^T\|\boldsymbol{u}(s)\|_{\mathbb{V}}^{p-2}\|\boldsymbol{A}\boldsymbol{u}(s)\|^2ds\right]\le C_{T,p},\tag{2.13}$$

$$\text{and}\quad\mathbb{E}\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|_{\mathbb{V}}^p\le C_{T,p}.\tag{2.14}$$

We associate a pressure p to the velocity $\boldsymbol{u}$ by using a generalization of the de Rham theorem to processes, see [70, Theorem 4.1]. In addition, we also have the following estimate for the pressure:

Proposition 3.3. *Under Assumptions (S_1) and (S_2), there exists a constant $C>0$ such that the velocity fields $\boldsymbol{u}$ and pressure fields p satisfy $\mathbb{P}$-a.s.*

$$\|\mathsf{p}(t)\|\le C\|\boldsymbol{A}^{1/2}\boldsymbol{u}(t)\|^2,\qquad\forall t\in[0,T].\tag{2.15}$$

Proof. To show the Proposition 3.3 we project equation (1.1) into $\mathbb{H}^\perp$ using the projection operator $\boldsymbol{P}_{\mathbb{H}}^\perp$. Since $\boldsymbol{P}_{\mathbb{H}}^\perp$ commutes with the Laplacian operator (we work with a periodic boundary condition) and $\operatorname{div}\boldsymbol{u}=0$, then

$$\boldsymbol{P}_{\mathbb{H}}^{\perp}\boldsymbol{u}_t=0 \qquad \text{and} \qquad \boldsymbol{P}_{\mathbb{H}}^{\perp}\Delta\boldsymbol{u}=0.$$

In Assumption (S_1) we suppose that the forcing term is divergence-free, hence, each solenoidale term vanishes after projection with $\boldsymbol{P}_{\mathbb{H}}^{\perp}$. The remaining terms give

$$\nabla \mathsf{p}(t)=-\boldsymbol{P}_{\mathbb{H}}^{\perp}\boldsymbol{B}(\boldsymbol{u}(t),\boldsymbol{u}(t)), \qquad \forall\, t\in[0,T].$$

It follows from [53, Lemma 2.2] for $r=2, n=2, \delta=1/2, \theta=\rho=1/2$ and (2.11) that

$$\begin{aligned}\|\nabla \mathsf{p}(t)\|_{-1}=\|\boldsymbol{P}_{\mathbb{H}}^{\perp}\boldsymbol{B}(\boldsymbol{u}(t),\boldsymbol{u}(t))\|_{-1}&\leq\|\boldsymbol{P}_{\mathbb{H}}\boldsymbol{B}(\boldsymbol{u}(t),\boldsymbol{u}(t))\|_{-1}+\|\boldsymbol{B}(\boldsymbol{u}(t),\boldsymbol{u}(t))\|_{-1}\\ &\leq C\|\boldsymbol{A}^{1/2}\boldsymbol{u}(t)\|^2.\end{aligned}$$

Finally, it follows by the Nečas inequality for functions with vanishing spatial average (cf. [19, Proposition IV.1.2.]), that there exists a constant $C>0$, such that

$$\|\mathsf{p}(t)\|\leq C\|\boldsymbol{A}^{1/2}\boldsymbol{u}(t)\|^2, \qquad \forall\, t\in[0,T]. \tag{2.16}$$

The constant C comes from the Nečas inequality, more precisely from the definition of the norm in $\mathbb{W}^{-1,2}$ by the Fourier transform. Therefore, C depends on the spatial dimension d and the L^p-estimates for the Fourier transform multipliers. Here we have $d=2$ and $p=2$, but a similar estimate can be obtained for $d\geq 2$ and $2\leq p<\infty$, see [31, Corollaries 1 and 2] and [77, Lemma 7.1]. □

3.2.3 Regularity in time of the solution of the SNS

Lemma 3.4. *Suppose that Assumption (S_1) holds, and $\boldsymbol{\mathcal{H}}=\mathbb{V}$. For the solution of (1.1), with $\boldsymbol{u}_0\in\mathbb{V}$, $2\leq p<\infty$, we can find a constant $C=C(T,p,L)>0$, such that for $0\leq s<t\leq T$ we have*

(*i*) $\mathbb{E}\|\boldsymbol{u}(s)-\boldsymbol{u}(t)\|_{\mathbb{L}^4}^p\leq C|s-t|^{\eta p} \quad \forall\, 0<\eta<\frac{1}{2}$,

(ii) $\mathbb{E}\|\boldsymbol{u}(s)-\boldsymbol{u}(t)\|_{\mathbb{V}}^{p} \leq C|s-t|^{\frac{\eta p}{2}} \quad \forall 0<\eta<\frac{1}{2}$,

(iii) $\mathbb{E}\|\mathsf{p}(s)-\mathsf{p}(t)\|^{\frac{p}{2}} \leq C|s-t|^{\frac{\eta p}{4}} \quad \forall 0<\eta<\frac{1}{2}$.

Proof. The assertions (i) and (ii) are direct quotations of [30, Lemma 2.3]. We only prove the assertion (iii). Let $t\in[0,T]$. Applying the projection $\boldsymbol{P}_{\mathbb{H}}^{\perp}$ on (1.1) we get

$$\nabla \mathsf{p}(t) = -\mathbb{P}_{\mathbb{H}}^{\perp}\boldsymbol{B}(\boldsymbol{u}(t),\boldsymbol{u}(t)).$$

The following identity holds for $0\leq s<t$

$$\nabla(\mathsf{p}(s)-\mathsf{p}(t)) = \mathbb{P}_{\mathbb{H}}^{\perp}\boldsymbol{B}(\boldsymbol{u}(t),\boldsymbol{u}(t)-\boldsymbol{u}(s)) + \mathbb{P}_{\mathbb{H}}^{\perp}\boldsymbol{B}(\boldsymbol{u}(t)-\boldsymbol{u}(s),\boldsymbol{u}(s)). \tag{2.17}$$

Using the Nečas inequality for vanishing spatial average and Proposition 3.3, we obtain

$$\begin{aligned}\|\mathsf{p}(s)-\mathsf{p}(t)\| &\leq \|\mathbb{P}_{\mathbb{H}}^{\perp}\boldsymbol{B}(\boldsymbol{u}(t),\boldsymbol{u}(t)-\boldsymbol{u}(s))\|_{-1} + \|\mathbb{P}_{\mathbb{H}}^{\perp}\boldsymbol{B}(\boldsymbol{u}(t)-\boldsymbol{u}(s),\boldsymbol{u}(s))\|_{-1}\\ &\leq C\|\boldsymbol{u}(t)\|_1\|\boldsymbol{u}(t)-\boldsymbol{u}(s)\|_1 + C\|\boldsymbol{u}(t)-\boldsymbol{u}(s)\|_1\|\boldsymbol{u}(s)\|_1.\end{aligned}$$

Taking the $p/2$-moment and using the Hölder inequality we get

$$\mathbb{E}\|\mathsf{p}(s)-\mathsf{p}(t)\|^{p/2} \leq C(L)\left[(\mathbb{E}\|\boldsymbol{u}(t)\|_1^p)^{1/2} + (\mathbb{E}\|\boldsymbol{u}(s)\|_1^p)^{1/2}\right](\mathbb{E}\|\boldsymbol{u}(t)-\boldsymbol{u}(s)\|_1^p)^{1/2}.$$

We deduce from (2.13) and the assertion (i) of the present lemma that

$$\mathbb{E}\|\mathsf{p}(s)-\mathsf{p}(t)\|^{p/2} \leq C_{T,2}(L)|s-t|^{\eta p/4}. \tag{2.18}$$

□

3.2.4 Classical decomposition of the solution

Before going to the next section we introduce a splitting argument which is essential for the rest of the paper. We consider the auxiliary Stokes equation

$$\begin{cases} d\boldsymbol{z}+[-\nu\Delta\boldsymbol{z}+\nabla\pi]\,dt=d\boldsymbol{W}, \text{ in } \mathbb{R}^2, \\ \operatorname{div}\boldsymbol{z}=0, \text{ in } \mathbb{R}^2, \end{cases} \tag{2.19}$$

with $\boldsymbol{z}(0)=0$ and which corresponds the penalized system

$$\begin{cases} d\boldsymbol{z}^\varepsilon+[-\nu\Delta\boldsymbol{z}^\varepsilon+\nabla\pi^\varepsilon]\,dt=d\boldsymbol{W}, \text{ in } \mathbb{R}^2, \\ \operatorname{div}\boldsymbol{z}^\varepsilon+\varepsilon\pi^\varepsilon=0, \text{ in } \mathbb{R}^2, \end{cases} \tag{2.20}$$

with $\boldsymbol{z}^\varepsilon(0)=0$.

As already pointed out by [30], the nonlinear term of the SNS does not allow to use a Gronwall argument. To tackle this issue, we use the classical decomposition of the solution $\boldsymbol{u}$ into two parts: one part, given by the process $\boldsymbol{z}$, will be random, but linear; the other part, denoted by $\boldsymbol{v}$, will be nonlinear, but deterministic. In this way, we write the solution of (1.1) as $\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{z}$, where $\boldsymbol{v}$ solves

$$\begin{cases} \dfrac{d\boldsymbol{v}}{dt}+\tilde{\boldsymbol{B}}(\boldsymbol{v}+\boldsymbol{z},\boldsymbol{v}+\boldsymbol{z})-\nu\Delta\boldsymbol{v}+\nabla\rho=0, \text{ in } \mathbb{R}^2, \\ \operatorname{div}\boldsymbol{v}=0, \text{ in } \mathbb{R}^2, \end{cases} \tag{2.21}$$

with $\boldsymbol{v}(0)=\boldsymbol{u}_0$. The corresponding penalized system

$$\begin{cases} \dfrac{d\boldsymbol{v}^\varepsilon}{dt}+\tilde{\boldsymbol{B}}(\boldsymbol{v}^\varepsilon+\boldsymbol{z}^\varepsilon,\boldsymbol{v}^\varepsilon+\boldsymbol{z}^\varepsilon)-\nu\Delta\boldsymbol{v}^\varepsilon+\nabla\rho^\varepsilon=0, \text{ in } \mathbb{R}^2, \\ \operatorname{div}\boldsymbol{v}^\varepsilon+\varepsilon\rho^\varepsilon=0, \text{ in } \mathbb{R}^2, \end{cases} \tag{2.22}$$

with $\boldsymbol{v}^\varepsilon(0)=\boldsymbol{u}_0$.

The system (2.21) (resp. (2.22)) are interpreted as deterministic equations which solves $\boldsymbol{v}$ (resp. $\boldsymbol{v}^\varepsilon$) for a given random process $\boldsymbol{z}$ (resp. $\boldsymbol{z}^\varepsilon$).

3.3 Main algorithm and auxiliary results

We consider a time discretization of (1.1) based on the penalized system Equation (1.2). For that purpose we fix $M \in \mathbb{N}$ and introduce an equidistant partition $I_k := \{t_\ell : 1 \leq \ell \leq M\}$ covering $[0,T]$ with mesh-size $k = T/M > 0$, $t_0 = 0$, and $t_M = T$. Here the increment $\Delta_\ell \boldsymbol{W} := \boldsymbol{W}(t_\ell) - \boldsymbol{W}(t_{\ell-1}) \sim \mathcal{N}(0, kQ)$ and we choose an uniform mesh size $k := t_{\ell+1} - t_\ell$. For every $t \in [t_{\ell-1}, t_\ell]$ and all $\boldsymbol{\varphi} \in \mathbb{W}^{1,2}_{\text{per}}$, there hold $\mathbb{P}$-a.s.,

$$(\boldsymbol{u}(t_\ell) - \boldsymbol{u}(t_{\ell-1}), \boldsymbol{\varphi}) + \nu \int_{t_{\ell-1}}^{t_\ell} (\nabla \boldsymbol{u}(s), \nabla \boldsymbol{\varphi}) ds$$
$$+ \int_{t_{\ell-1}}^{t_\ell} \tilde{b}(\boldsymbol{u}(s), \boldsymbol{u}(s), \boldsymbol{\varphi}) ds + \int_{t_{\ell-1}}^{t_\ell} (\nabla \mathsf{p}(s), \boldsymbol{\varphi}) ds = \int_{t_{\ell-1}}^{t_\ell} (\boldsymbol{\varphi}, d\boldsymbol{W}(s)), \tag{3.23}$$
$$(\operatorname{div} \boldsymbol{u}(t_\ell), \chi) = 0. \tag{3.24}$$

Note that instead of b we use $\tilde{b}$. We can switch between both without any confusion since for each $s \in [0,T]$, $\boldsymbol{u}(s) \in \mathbb{H}$.

Now we discretize the penalized system Equation (1.2) instead of the original equation and project the result into $\mathbb{H}$. We derive the following algorithm:

Algorithm 3.4 (Main algorithm)**.** *Assume $\boldsymbol{u}^{\varepsilon,0} := \boldsymbol{u}_0$ with $\|\boldsymbol{u}_0\| \leq C$. Find for every $\ell \in \{1, \ldots, M\}$ a pair of random variables $(\boldsymbol{u}^{\varepsilon,\ell}, \mathsf{p}^{\varepsilon,\ell})$ with values in $\mathbb{W}^{1,2}_{\text{per}} \times L^2_{\text{per}}$, such that we have $\mathbb{P}$-a.s.*

- *Penalization:*

$$(\tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \boldsymbol{u}^{\varepsilon,\ell-1}, \boldsymbol{\varphi}) + \nu k (\nabla \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \nabla \boldsymbol{\varphi}) + k \tilde{b}(\tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \boldsymbol{\varphi})$$
$$+ k(\nabla \tilde{\mathsf{p}}^{\varepsilon,\ell}, \boldsymbol{\varphi}) + k(\nabla \phi^{\varepsilon,\ell-1}, \boldsymbol{\varphi}) = (\Delta_\ell \boldsymbol{W}, \boldsymbol{\varphi}), \forall \boldsymbol{\varphi} \in \mathbb{W}^{1,2}_{\text{per}}, \tag{4.25}$$
$$(\operatorname{div} \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \chi) + \varepsilon(\tilde{\mathsf{p}}^{\varepsilon,\ell}, \chi) = 0, \forall \chi \in L^2_{\text{per}}; \tag{4.26}$$

• *Projection:*

$$(\boldsymbol{u}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell},\boldsymbol{\varphi})+\alpha k(\nabla(\phi^{\varepsilon,\ell}-\phi^{\varepsilon,\ell-1}),\boldsymbol{\varphi})=0,\,\forall\,\boldsymbol{\varphi}\in\mathbb{W}^{1,2}_{\text{per}},\tag{4.27}$$

$$(\operatorname{div}\boldsymbol{u}^{\varepsilon,\ell},\chi)=0,\,\forall\,\chi\in L^2_{\text{per}},\tag{4.28}$$

$$\mathsf{p}^{\varepsilon,\ell}=\tilde{\mathsf{p}}^{\varepsilon,\ell}+\phi^{\varepsilon,\ell}+\alpha(\phi^{\varepsilon,\ell}-\phi^{\varepsilon,\ell-1}).$$

Proposition 3.5. *There exist iterates* $\{\boldsymbol{u}^\ell:1\leq\ell\leq M\}$ *which solve* (4.25) *and* (4.26) *at each time-step. Moreover, for every integer l, with* $1\leq\ell\leq M$*,* $\boldsymbol{u}^\ell$ *is a* $\mathscr{F}_{t_\ell}$*-measurable.*

Proof. Let us fix $\omega\in\Omega$. We use Lax-Milgram fixed-point theorem to show the existence of a $\mathbb{V}$-valued sequence $\{\boldsymbol{u}^{\varepsilon,\ell}:1\leq\ell\leq M\}$.

• Penalization: Since $\boldsymbol{u}^{\varepsilon,0}$ and ϕ^0 are given, and $|\Delta_\ell\boldsymbol{W}(\omega)|_{\boldsymbol{K}}<\infty$ for all $\ell\in\{1,\dots,M\}$, we assume that $\widetilde{\boldsymbol{u}}^{\varepsilon,1}(\omega),\dots,\widetilde{\boldsymbol{u}}^{\varepsilon,\ell-1}(\omega)$ are also given. To find the pair of random variables $(\boldsymbol{u}^{\varepsilon,\ell},\mathsf{p}^{\varepsilon,\ell})$ in Algorithm 3.4 we need first to solve a nonlinear, nonsymmetric variational problem. Therefore, let us denote by $\mathscr{A}$ the nonlinear operator from $\mathbb{V}$ to $\mathbb{V}'$ ($\mathbb{V}'$: dual of $\mathbb{V}$) defined by:

$$\begin{aligned}\langle\mathscr{A}\tilde{\boldsymbol{u}}^{\varepsilon,\ell}(\omega),\boldsymbol{w}(\omega)\rangle=\tilde{\boldsymbol{u}}^{\varepsilon,\ell}(\omega)+\nu(\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}(\omega),\nabla\boldsymbol{w}^\ell(\omega))\\+\tilde{b}(\tilde{\boldsymbol{u}}^{\varepsilon,\ell}(\omega),\tilde{\boldsymbol{u}}^{\varepsilon,\ell}(\omega),\boldsymbol{w}^\ell(\omega)),\qquad\forall\,\boldsymbol{w}^\ell((\omega))\in\mathbb{V}.\end{aligned}\tag{4.29}$$

Because $\tilde{b}$ satisfies the orthogonal property (2.8), then putting $\boldsymbol{w}^\ell=\tilde{\boldsymbol{u}}^{\varepsilon,\ell}$ in (4.29) we thus have

$$\langle\mathscr{A}\tilde{\boldsymbol{u}}^{\varepsilon,\ell}(\omega),\boldsymbol{w}(\omega)\rangle\geq\nu\|\boldsymbol{u}^{\varepsilon,\ell}(\omega)\|^2_{\mathbb{V}}.$$

The operator $\mathscr{A}$ is therefore $\mathbb{V}$-elliptic and the Lax–Milgram theorem allows us to infer the existence of a unique solution of (4.29).

• Projection: If we take $\boldsymbol{\varphi}=\nabla\phi^{\varepsilon,\ell}$ in (4.27) we see that this step is actually a Poisson problem. Since $\boldsymbol{u}^{\varepsilon,\ell}$ is given from the previous step, the existence of a unique solution $\phi^{\varepsilon,\ell}$ is deduced from the ellipticity of the Laplacian operator.

Since $\boldsymbol{u}^{\varepsilon,\ell}=\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\alpha k\nabla(\phi^{\varepsilon,\ell}-\phi^{\varepsilon,\ell-1})$ and $\mathsf{p}^{\varepsilon,\ell}=\tilde{\mathsf{p}}^{\varepsilon,\ell}+\phi^{\varepsilon,\ell}+\alpha(\phi^{\varepsilon,\ell}-\phi^{\varepsilon,\ell-1})$, the existence of a unique $\boldsymbol{u}^{\varepsilon,\ell}$ and $\mathsf{p}^{\varepsilon,\ell}$ follows.

The proof of the $\mathscr{F}_{t_\ell}$-measurability of $\tilde{\boldsymbol{u}}^{\varepsilon,\ell}$ and $\phi^{\varepsilon,\ell}$ can be done in a similar fashion as in [39], see also [5]. Since $\boldsymbol{u}^{\varepsilon,\ell}$ (resp. $\mathsf{p}^{\varepsilon,\ell}$) are obtained from $\tilde{\boldsymbol{u}}^{\varepsilon,\ell}$ (resp. $\phi^{\varepsilon,\ell}$) we also obtain their $\mathscr{F}_{t_\ell}$-measurability. □

3.4.1 Stability

This section is inspired by [22, Lemma 3.1] and [30, Lemma 3.1]. Here we consider a coupled system, the first one is derived from the penalization and the second one is a projection step.

Lemma 3.6. *Let $\phi^{\varepsilon,0}=0$. Suppose that Assumptions (S_1) and (S_2) are valid with $\|\boldsymbol{u}^0\|\leq C$. Then, there exists a positive constant $C=C(L,T,\boldsymbol{u}^0,\nu)$ so that for every $\varepsilon>0$ and $\alpha>1$, the iterates $\{\boldsymbol{u}^{\varepsilon,\ell}:1\leq\ell\leq M\}$ solving Algorithm 3.4 and the intermediary iterates $\{\tilde{\boldsymbol{u}}^{\varepsilon,\ell}:1\leq\ell\leq M\}$, $\{\tilde{\mathsf{p}}^{\varepsilon,\ell}:1\leq\ell\leq M\}$, and $\{\phi^{\varepsilon,\ell}:1\leq\ell\leq M\}$ satisfy for $q=1$ or $q=2$:*

$$(i)\;\; \nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\|\boldsymbol{u}^{\varepsilon,\ell}\|^{2^q-2}\right)\leq C,$$

$$(ii)\;\; k^2\mathbb{E}\max_{1\leq m\leq M}\|\nabla\phi^{\varepsilon,m}\|^2\|\boldsymbol{u}^{\varepsilon,m}\|^{2^q-2}+\varepsilon\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|\|\boldsymbol{u}^{\varepsilon,\ell}\|^{2^q-2}\right)\leq C,$$

$$(iii)\;\; \mathbb{E}\max_{1\leq m\leq M}\|\boldsymbol{u}^{\varepsilon,m}\|^{2^q}+\nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\boldsymbol{u}^{\varepsilon,\ell}\|^2\|\boldsymbol{u}^{\varepsilon,\ell}\|^{2^q-2}\right)\leq C.$$

Proof. The proof consits of three steps. First, we give some preparatory estimates. Then, we handle the case $q=1$, and, finally, we handle the case $q=2$.

Step (I) Preparatory estimate. We take $\boldsymbol{\varphi}=2\tilde{\boldsymbol{u}}^{\varepsilon,\ell}$ in Equation (4.25) and $\chi=\operatorname{div}\tilde{\boldsymbol{u}}^{\varepsilon,\ell}$ in Equation (4.26), and use the orthogonal property (2.8) of $\tilde{b}$, to get

$$\begin{aligned}(\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\boldsymbol{u}^{\varepsilon,\ell-1},\tilde{\boldsymbol{u}}^{\varepsilon,\ell})+2\nu k\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2+2k(\nabla\tilde{\mathsf{p}}^{\varepsilon,\ell},\boldsymbol{u}^{\varepsilon,\ell})&\\ +2k(\nabla\phi^{\varepsilon,\ell-1},\tilde{\boldsymbol{u}}^{\varepsilon,\ell})=2(\Delta_\ell\boldsymbol{W},\tilde{\boldsymbol{u}}^{\varepsilon,\ell}),&\\ (\nabla\tilde{\mathsf{p}}^{\varepsilon,\ell},\tilde{\boldsymbol{u}}^{\varepsilon,\ell})=\frac{1}{\varepsilon}\|\operatorname{div}\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2.&\end{aligned}\tag{4.30}$$

Using the algebraic identity

$$2(a-b)a=a^2-b^2+(a-b)^2\tag{4.31}$$

in (4.30) we obtain

$$\begin{aligned}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2-\|\boldsymbol{u}^{\varepsilon,\ell-1}\|^2+\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2+2\nu k\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2+2\varepsilon k\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2\\+2k(\nabla\phi^{\varepsilon,\ell-1},\tilde{\boldsymbol{u}}^{\varepsilon,\ell})=2(\Delta_\ell\boldsymbol{W},\tilde{\boldsymbol{u}}^{\varepsilon,\ell}).\end{aligned}\tag{4.32}$$

Let $\alpha>0$. We take $\boldsymbol{\varphi}=2\tilde{\boldsymbol{u}}^{\varepsilon,\ell}$ in (4.27) and obtain

$$\frac{\alpha-1}{\alpha}\left(\|\boldsymbol{u}^{\varepsilon,\ell}\|^2-\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2+\|\boldsymbol{u}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\right)=0.\tag{4.33}$$

Then, we take $\boldsymbol{\varphi}=\boldsymbol{u}^{\varepsilon,\ell}+\tilde{\boldsymbol{u}}^{\varepsilon,\ell}$ in (4.27) and obtain

$$\frac{1}{\alpha}\left(\|\boldsymbol{u}^{\varepsilon,\ell}\|^2-\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\right)+\frac{k}{2}(\nabla(\phi^{\varepsilon,\ell}-\phi^{\varepsilon,\ell-1}),\tilde{\boldsymbol{u}}^{\varepsilon,\ell})=0.\tag{4.34}$$

Collecting together (4.32) to (4.34) we obtain

$$\begin{aligned}\|\boldsymbol{u}^{\varepsilon,\ell}\|^2-\|\boldsymbol{u}^{\varepsilon,\ell-1}\|^2+\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2+\frac{\alpha-1}{\alpha}\|\boldsymbol{u}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2+2\nu k\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\\+2\varepsilon k\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2+k(\nabla(\phi^{\varepsilon,\ell-1}+\phi^{\varepsilon,\ell}),\tilde{\boldsymbol{u}}^{\varepsilon,\ell})\leq 2(\Delta_\ell\boldsymbol{W},\tilde{\boldsymbol{u}}^{\varepsilon,\ell}).\end{aligned}\tag{4.35}$$

We take $\boldsymbol{\varphi}=\nabla(\phi^{\varepsilon,\ell}+\phi^{\varepsilon,\ell-1})$ in (4.27) and obtain

$$(\nabla(\phi^{\varepsilon,\ell}+\phi^{\varepsilon,\ell-1}),\tilde{\boldsymbol{u}}^{\varepsilon,\ell})=\alpha k\|\nabla\phi^{\varepsilon,\ell}\|^2-\alpha k\|\nabla\phi^{\varepsilon,\ell-1}\|^2.$$

This implies,

$$\begin{aligned}\|\boldsymbol{u}^{\varepsilon,\ell}\|^2-\|\boldsymbol{u}^{\varepsilon,\ell-1}\|^2+\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2+\frac{\alpha-1}{\alpha}\|\boldsymbol{u}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2+2\nu k\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\\+2\varepsilon k\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2+\alpha k^2\|\nabla\phi^{\varepsilon,\ell}\|^2-\alpha k^2\|\nabla\phi^{\varepsilon,\ell-1}\|^2\leq 2(\Delta_\ell\boldsymbol{W},\tilde{\boldsymbol{u}}^{\varepsilon,\ell}).\end{aligned}\tag{4.36}$$

Step (II) Case $q=1$. Summing (4.36) from $\ell=1$ to $\ell=m$, we get

$$\begin{aligned}\|\boldsymbol{u}^{\varepsilon,m}\|^2+\sum_{\ell=1}^{m}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2+(\tfrac{\alpha-1}{\alpha})\sum_{\ell=1}^{m}\|\boldsymbol{u}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2+2\nu k\sum_{\ell=1}^{M}\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\\+2\varepsilon k\sum_{\ell=1}^{m}\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2+\alpha k^2\|\nabla\phi^{\varepsilon,m}\|^2\leq\|\boldsymbol{u}^0\|^2+2\sum_{\ell=1}^{m}(\Delta_\ell\boldsymbol{W},\tilde{\boldsymbol{u}}^{\varepsilon,\ell}).\end{aligned}\tag{4.37}$$

The last term of the right side can be splitted as follows,

$$\text{Noise}_1^{\varepsilon,m} := 2\sum_{\ell=1}^{m}(\Delta_\ell \boldsymbol{W}, \tilde{\boldsymbol{u}}^{\varepsilon,\ell}) = 2\sum_{\ell=1}^{m}(\Delta_\ell \boldsymbol{W}, \tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}) + 2\sum_{\ell=1}^{m}(\Delta_\ell \boldsymbol{W}, \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}).$$

Let $\delta_1 > 0$ be an arbitrary number. Applying the Young inequality to the first term on the right side, we get

$$\ldots \leq C(\delta_1)\sum_{\ell=1}^{m}\|\Delta_\ell \boldsymbol{W}\|^2 + \delta_1\sum_{\ell=1}^{m}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2 + 2\sum_{\ell=1}^{m}(\Delta_\ell \boldsymbol{W}, \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}). \tag{4.38}$$

Taking first the maximum of (4.38) over $1 \leq m \leq M$, and then the expectation, exactly with this order, give the following estimate

$$\begin{aligned}\mathbb{E}\max_{1\leq m\leq M}\text{Noise}_1^{\varepsilon,m} \leq C(\delta_1)\sum_{\ell=1}^{M}\mathbb{E}\|\Delta_\ell \boldsymbol{W}\|^2 + \delta_1\sum_{\ell=1}^{M}\mathbb{E}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2 \\ + 2\mathbb{E}\max_{1\leq m\leq M}\sum_{\ell=1}^{m}(\Delta_\ell \boldsymbol{W}, \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}).\end{aligned} \tag{4.39}$$

It follows from (2.3), that $\mathbb{E}\|\Delta_\ell \boldsymbol{W}\|^2 = k$. By applying successively the Burkholder–David–Gundy inequality, the Hölder inequality, and the Young inequality to the last term of (4.39), we obtain

$$\begin{aligned}\mathbb{E}\max_{1\leq m\leq M}\text{Noise}_1^{\varepsilon,m} &\leq C(\delta_1,T) + \delta_1\sum_{\ell=1}^{M}\mathbb{E}\|\boldsymbol{u}^{\varepsilon,\ell} - \boldsymbol{u}^{\varepsilon,\ell-1}\|^2 + \mathbb{E}\left(\sum_{\ell=1}^{M}k\|\boldsymbol{u}^{\varepsilon,\ell-1}\|^2\right)^{1/2} \\ &\leq C(\delta_1,T,\boldsymbol{u}^0) + \delta_1\sum_{\ell=1}^{M}\mathbb{E}\|\boldsymbol{u}^{\varepsilon,\ell} - \boldsymbol{u}^{\varepsilon,\ell-1}\|^2 + \delta_1\mathbb{E}\max_{1\leq \ell\leq M}\|\boldsymbol{u}^{\varepsilon,\ell}\|^2.\end{aligned}$$

Now, taking the maximum of (4.37) over $1 \leq m \leq M$, and, then, expectation, give the following estimate,

$$\begin{aligned}&\mathbb{E}\max_{1\leq m\leq M}\left\{\|\boldsymbol{u}^{\varepsilon,m}\|^2 + \alpha k^2\|\nabla\phi^{\varepsilon,m}\|^2\right\} \\ &+ \mathbb{E}\sum_{\ell=1}^{m}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2 + (\tfrac{\alpha-1}{\alpha})\mathbb{E}\sum_{\ell=1}^{m}\|\boldsymbol{u}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2 \\ &+ \nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\right) + \varepsilon\mathbb{E}\left(k\sum_{\ell=1}^{m}\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2\right) \leq \|\boldsymbol{u}^0\|^2 + \mathbb{E}\max_{1\leq m\leq M}\text{Noise}_1^{\varepsilon,m}.\end{aligned} \tag{4.40}$$

The terms with $\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2$ and $\max_{1\le\ell\le M}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2$ of (4.39) are absorbed by the left hand side of (4.40) which leads to

$$\begin{aligned}&(1-\delta_1)\mathbb{E}\max_{1\le m\le M}\|\boldsymbol{u}^{\varepsilon,m}\|^2+\alpha k^2\mathbb{E}\max_{1\le m\le M}\|\nabla\phi^{\varepsilon,m}\|^2\\&+(1-\delta_1)\mathbb{E}\sum_{\ell=1}^{m}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2+(\tfrac{\alpha-1}{\alpha})\mathbb{E}\sum_{\ell=1}^{m}\|\boldsymbol{u}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\\&+\nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\right)+\varepsilon\mathbb{E}\left(k\sum_{\ell=1}^{m}\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2\right)\le C(\delta_1,T,\boldsymbol{u}^0).\end{aligned}\tag{4.41}$$

The parameters α and δ_1 are chosen such that the left hand side stays positive. Thus, we chose $\alpha>1$ and $0<\delta_1<1$.

Step (III) Case $q=2$. We multiply (4.36) by $2\|\boldsymbol{u}^{\varepsilon,\ell}\|^2$ and use again the algebraic identity (4.31) to give

$$\begin{aligned}&\|\boldsymbol{u}^{\varepsilon,\ell}\|^4-\|\boldsymbol{u}^{\varepsilon,\ell-1}\|^4+2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2\|\boldsymbol{u}^{\varepsilon,\ell}\|^2\\&+\tfrac{\alpha-1}{\alpha}\|\boldsymbol{u}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\|\boldsymbol{u}^{\varepsilon,\ell}\|^2+4\nu k\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\|\boldsymbol{u}^{\varepsilon,\ell}\|^2+4\varepsilon k\|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2\|\boldsymbol{u}^{\varepsilon,\ell}\|^2\\&+2\alpha k^2\|\nabla\phi^{\varepsilon,\ell}\|^2\|\boldsymbol{u}^{\varepsilon,\ell}\|^2-2\alpha k^2\|\nabla\phi^{\varepsilon,\ell-1}\|^2\|\boldsymbol{u}^{\varepsilon,\ell}\|^2=2(\Delta_\ell\boldsymbol{W},\tilde{\boldsymbol{u}}^{\varepsilon,\ell})\|\boldsymbol{u}^{\varepsilon,\ell}\|^2.\end{aligned}\tag{4.42}$$

On the left hand side we use the same calculation that we used on the term $\mathrm{Noise}_1^{\varepsilon,m}$. In particular, we compute

$$\begin{aligned}\mathrm{Noise}_2^{\varepsilon,m}:=&\,2\sum_{\ell=1}^{m}(\Delta_\ell\boldsymbol{W},\tilde{\boldsymbol{u}}^{\varepsilon,\ell})\|\boldsymbol{u}^{\varepsilon,\ell}\|^2\\&\le C(\delta_1)\sum_{\ell=1}^{m}\|\Delta_\ell\boldsymbol{W}\|^2+\delta_1\sum_{\ell=1}^{m}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}-\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2\|\boldsymbol{u}^{\varepsilon,\ell}\|^2\\&\qquad+2\sum_{\ell=1}^{m}(\Delta_\ell\boldsymbol{W},\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1})\|\boldsymbol{u}^{\varepsilon,\ell}\|^2.\end{aligned}\tag{4.43}$$

In the next step, we first take the maximum of (4.43) over $1\le m\le M$, and, then, we take the expectation, exactly with this order. Now, applying the Young inequality, and the Hölder inequality, and using (4.41) to bound some terms, we can find a constant

$C = C(\delta_1, \delta_2, L, T, \boldsymbol{u}^0, \nu) > 0$ such that

$$\mathbb{E} \max_{1 \le m \le M} \text{Noise}_2^{\varepsilon,m} \le C + \delta_1 \sum_{\ell=1}^{M} \|\tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 + \delta_2 \mathbb{E} \max_{1 \le \ell \le M} \|\boldsymbol{u}^{\varepsilon,\ell}\|^4. \tag{4.44}$$

Summing up (4.44) for $\ell = 1$ to $\ell = m$, taking the maximum over $1 \le m \le M$, and taking the expectation in (4.42) we have

$$\begin{aligned}
&\mathbb{E} \max_{1 \le m \le M} \left\{ \|\boldsymbol{u}^{\varepsilon,m}\|^4 + \alpha k^2 \|\nabla \phi^{\varepsilon,\ell}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 \right\} + \mathbb{E} \sum_{\ell=1}^{M} \|\tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 \\
&+ \left(\tfrac{\alpha-1}{\alpha}\right) \mathbb{E} \sum_{\ell=1}^{M} \|\boldsymbol{u}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 + \nu \mathbb{E} \left(k \sum_{\ell=1}^{M} \|\nabla \tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 \right) \\
&+ \varepsilon \mathbb{E} \left(k \sum_{\ell=1}^{M} \|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 \right) \\
&\le C(\delta_1, \delta_2, L, T, \boldsymbol{u}^0, \nu) + \delta_1 \sum_{\ell=1}^{M} \|\tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 + \delta_2 \mathbb{E} \max_{1 \le \ell \le M} \|\boldsymbol{u}^{\varepsilon,\ell}\|^4.
\end{aligned} \tag{4.45}$$

The terms with $\|\boldsymbol{u}^{\varepsilon,\ell}\|^4$ and $\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2$ are absorbed by the left side of (4.45). Therefore, we get

$$\begin{aligned}
&\mathbb{E} \max_{1 \le m \le M} \left\{ (1-\delta_2) \|\boldsymbol{u}^{\varepsilon,m}\|^4 + \alpha k^2 \|\nabla \phi^{\varepsilon,m}\|^2 \|\boldsymbol{u}^{\varepsilon,m}\|^2 \right\} \\
&+ (1-\delta_1) \mathbb{E} \sum_{\ell=1}^{M} \|\tilde{\boldsymbol{u}}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 + \left(\tfrac{\alpha-1}{\alpha}\right) \mathbb{E} \sum_{\ell=1}^{M} \|\boldsymbol{u}^{\varepsilon,\ell} - \tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 \\
&+ \nu \mathbb{E} \left(k \sum_{\ell=1}^{M} \|\nabla \tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 \right) + \varepsilon \mathbb{E} \left(k \sum_{\ell=1}^{M} \|\tilde{\mathsf{p}}^{\varepsilon,\ell}\|^2 \|\boldsymbol{u}^{\varepsilon,\ell}\|^2 \right) \le C(\delta_1, \delta_2, L, T, \boldsymbol{u}^0, \nu).
\end{aligned}$$

We conclude by choosing α, δ_1, and δ_2 such that, $(\alpha - 1) > 0$, $(1-\delta_1) > 0$, and $(1-\delta_2) > 0$.

□

In the next lemma we use the LBB inequality (see [4, 21])

$$\|\mathsf{p}\| \le C \sup_{\boldsymbol{\varphi} \in \mathbb{W}^{1,2}} \frac{(\nabla \mathsf{p}, \boldsymbol{\varphi})}{\|\boldsymbol{\varphi}\|_1} \tag{4.46}$$

to transfer the estimate from the velocity fields $\boldsymbol{u}^{\varepsilon,\ell}$ to the pressure fields $\mathsf{p}^{\varepsilon,\ell}$.

We start with a direct discretization of (1.1) which leads to the following algorithm:

Algorithm 3.5. *Assume $\boldsymbol{u}^0 := \boldsymbol{u}_0$ with $\|\boldsymbol{u}^0\| \le C$. Find for every $\ell \in \{1,\dots,M\}$ a pair of random variables $(\boldsymbol{u}^\ell, \mathsf{p}^\ell)$ with values in $\mathbb{V} \times L^2_{\mathrm{per}}$, such that $\mathbb{P}$-a.s.*

$$(\boldsymbol{u}^\ell - \boldsymbol{u}^{\ell-1}, \boldsymbol{\varphi}) + \nu k(\nabla \boldsymbol{u}^\ell, \nabla \boldsymbol{\varphi}) + k\tilde{b}(\boldsymbol{u}^\ell, \boldsymbol{u}^\ell, \boldsymbol{\varphi}) + k(\nabla \mathsf{p}^\ell, \boldsymbol{\varphi}) = (\Delta_\ell \boldsymbol{W}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \mathbb{W}^{1,2}_{\mathrm{per}}, \tag{5.47}$$

$$(\operatorname{div} \boldsymbol{u}^\ell, \chi) = 0, \quad \forall \chi \in L^2_{\mathrm{per}}. \tag{5.48}$$

We define the sequences of errors $\mathrm{E}^\ell = \boldsymbol{u}^\ell - \boldsymbol{u}^{\varepsilon,\ell}, \tilde{\mathrm{E}}^\ell = \boldsymbol{u}^\ell - \tilde{\boldsymbol{u}}^{\varepsilon,\ell}$, and $\mathrm{Q}^\ell = \mathsf{p}^\ell - \mathsf{p}^{\varepsilon,\ell}$. We subtract (4.25) and (4.26) from (5.47) and (5.48), and get

$$(\mathrm{E}^\ell - \mathrm{E}^{\ell-1}, \boldsymbol{\varphi}) + \nu k(\nabla \tilde{\mathrm{E}}^\ell, \nabla \boldsymbol{\varphi}) + k(\nabla \mathrm{Q}^\ell, \boldsymbol{\varphi}) = k\tilde{b}(\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}, \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \boldsymbol{\varphi}) - k\tilde{b}(\boldsymbol{u}^\ell, \boldsymbol{u}^\ell, \boldsymbol{\varphi}), \forall \boldsymbol{\varphi} \in \mathbb{W}^{1,2}_{\mathrm{per}}. \tag{5.49}$$

Lemma 3.7. *Under the assumption of Lemma 3.6, there exists a constant $C = C(L,T,\boldsymbol{u}_0) > 0$ such that for every $\varepsilon > 0$, the iterates $\{\mathsf{p}^{\varepsilon,\ell} : 1 \le \ell \le M\}$ solving Algorithm 3.4 satisfies*

$$\mathbb{E}\left(k\sum_{\ell=1}^{M} \|\mathsf{p}^{\varepsilon,\ell}\|^2\right) \le C.$$

Proof. Since $(\mathrm{E}^\ell - \mathrm{E}^{\ell-1}) \in \mathscr{D}(\boldsymbol{A}^{-1})$, we can take $\boldsymbol{\varphi} = \boldsymbol{A}^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1})$ in (5.49) and use Proposition 3.1. Then we apply the Young inequality, and use estimate (2.11) of $\tilde{b}$. This leads to the following results:

i) $c_2\|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|^2_{-1} \le (\mathrm{E}^\ell - \mathrm{E}^{\ell-1}, \boldsymbol{A}^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1}))$,

ii) $\nu k(\nabla \mathrm{E}^\ell, \nabla \boldsymbol{A}^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1})) \le C(\delta_1)\nu^2 k^2 \|\mathrm{E}^\ell\|_1^2 + \delta_1 \|\mathrm{E}^\ell - \mathrm{E}^{\ell-1}\|^2_{-1}$,

iii) $k(\nabla \mathrm{Q}^\ell, \boldsymbol{A}^{-1}(\mathrm{E}^\ell - \mathrm{E}^{\ell-1})) = 0$, *iv)* $k\tilde{b}(\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1},\tilde{\boldsymbol{u}}^{\varepsilon,\ell},\boldsymbol{A}^{-1}(\mathrm{E}^\ell-\mathrm{E}^{\ell-1}))\leq C(L,\delta_1)\frac{k^2}{2}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|_1^2 \quad +C(L,\delta_1)\frac{k^2}{2}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2+ \delta_1\|\mathrm{E}^\ell-\mathrm{E}^{\ell-1}\|_{-1}^2,$

v) $k\tilde{b}(\boldsymbol{u}^\ell,\boldsymbol{u}^\ell,\boldsymbol{A}^{-1}(\mathrm{E}^\ell-\mathrm{E}^{\ell-1}))\leq C(L,\delta_1)k^2\|\boldsymbol{u}^\ell\|^2\|\boldsymbol{u}^\ell\|_1^2+\delta_1\|\mathrm{E}^\ell-\mathrm{E}^{\ell-1}\|_{-1}^2.$

Fixing $\delta_1>0$ such that $(c_2-3\delta_1)>0$, and collecting *i*), *ii*), *iii*), *iv*), and *v*), we obtain

$$\begin{aligned}(c_2-3\delta_1)\mathbb{E}\sum_{\ell=1}^M\|\mathrm{E}^\ell-\mathrm{E}^{\ell-1}\|_{-1}^2&\leq C(L,\delta_1,\nu)k\mathbb{E}\left(k\sum_{\ell=1}^M\|\boldsymbol{u}^{\varepsilon,\ell}\|_1^2\right)+k\mathbb{E}\left(k\sum_{\ell=1}^M\|\boldsymbol{u}^\ell\|_1^2\right)\\&+k\mathbb{E}\left(k\sum_{\ell=1}^M\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|_1^2\right)+k\mathbb{E}\left(k\sum_{\ell=1}^M\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2\right)+k\mathbb{E}\left(k\sum_{\ell=1}^M\|\boldsymbol{u}^\ell\|^2\|\boldsymbol{u}^\ell\|_1^2\right).\end{aligned}$$

By Lemma 3.6 and [22, Lemma 3.1 (iii)] we obtain

$$\mathbb{E}\sum_{\ell=1}^M\|\mathrm{E}^\ell-\mathrm{E}^{\ell-1}\|_{-1}^2\leq C(T,L,\boldsymbol{u}^0)k. \tag{5.50}$$

Now we rearrange (5.49) and get

$$k(\nabla\mathrm{Q}^\ell,\boldsymbol{\varphi})=-(\mathrm{E}^\ell-\mathrm{E}^{\ell-1},\boldsymbol{\varphi})-\nu k(\nabla\tilde{\mathrm{E}}^\ell,\nabla\boldsymbol{\varphi})+k\tilde{b}(\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1},\tilde{\boldsymbol{u}}^{\varepsilon,\ell},\boldsymbol{\varphi})-k\tilde{b}(\boldsymbol{u}^\ell,\boldsymbol{u}^\ell,\boldsymbol{\varphi}). \tag{5.51}$$

With the skew symmetry property of $\tilde{b}$ (see (2.7)) and the estimate (2.11), identity (5.51) becomes

$$\begin{aligned}\frac{k(\nabla\mathrm{Q}^\ell,\boldsymbol{\varphi})}{\|\boldsymbol{\varphi}\|_1}&\leq\|\mathrm{E}^\ell-\mathrm{E}^{\ell-1}\|_{-1}+\nu k\|\nabla\tilde{\mathrm{E}}^\ell\|+C(L)k\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|_1\\&+C(L)k\|\boldsymbol{u}^{\varepsilon,\ell}\|\|\boldsymbol{u}^{\varepsilon,\ell}\|_1+C(L)k\|\boldsymbol{u}^\ell\|\|\boldsymbol{u}^\ell\|_1.\end{aligned}$$

Using the inequality (4.46), we have

$$\begin{aligned}k^2\|\mathrm{Q}^\ell\|^2&\leq C\|\mathrm{E}^\ell-\mathrm{E}^{\ell-1}\|_{-1}^2+\nu^2k^2\|\nabla\tilde{\mathrm{E}}^\ell\|^2+C(L)k^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|_1^2\\&+C(L)k^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2+C(L)k^2\|\boldsymbol{u}^\ell\|^2\|\boldsymbol{u}^\ell\|_1^2.\end{aligned}$$

Summing up for $\ell=1$ to $\ell=M$, and taking expectation, we obtain

$$k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\mathrm{Q}^{\ell}\|^2\right) \leq C\mathbb{E}\sum_{\ell=1}^{M}\|\mathrm{E}^{\ell}-\mathrm{E}^{\ell-1}\|_{-1}^2+\nu^2 k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\right)$$
$$+\nu^2 k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\boldsymbol{u}^{\ell}\|^2\right)+C(L)k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell-1}\|_1^2\right)$$
$$+C(L)k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2\right)+C(L)k\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\boldsymbol{u}^{\ell}\|^2\|\boldsymbol{u}^{\ell}\|_1^2\right).$$

From Lemma 3.6, [22, Lemma 3.1 (iii)], and estimate (5.50), we obtain

$$\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\mathrm{Q}^{\ell}\|^2\right)\leq C(T,L,\nu,\boldsymbol{u}^0).$$

The Minkowsky inequality and Poincaré inequality imply

$$\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\mathsf{p}^{\varepsilon,\ell}\|^2\right)\leq C(T,L,\nu,\boldsymbol{u}^0)+C(L)\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\mathsf{p}^{\ell}\|^2\right).$$

We finish the proof with using [30, Lemma 3.2 (i)], where the authors proved that

$$\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\mathsf{p}^{\ell}\|^2\right)\leq C(T).$$

□

3.5.1 Auxiliary error estimates

We start with Algorithm 3.6. Let $\boldsymbol{z}=\{\boldsymbol{z}(t,\cdot):t\in[t_{\ell-1},t_\ell]\}$ be the strong solution of (2.19) as defined in Definition 3.2 and $\pi=\{\pi(t,\cdot):t\in[t_{\ell-1},t_\ell]\}$ the associated pressure, i.e. for every

$t \in [t_{\ell-1}, t_\ell]$, all $\boldsymbol{\varphi} \in \mathbb{W}^{1,2}$, and all $\chi \in L^2_{\mathrm{per}}$, we have $\mathbb{P}$-a.s.

$$\begin{aligned}(\boldsymbol{z}(t_\ell) - \boldsymbol{z}(t_{\ell-1}), \boldsymbol{\varphi}) + \nu \int_{t_{\ell-1}}^{t_\ell} (\nabla \boldsymbol{z}(s), \nabla \boldsymbol{\varphi}) ds \\ + \int_{t_{\ell-1}}^{t_\ell} (\nabla \pi(s), \boldsymbol{\varphi}) ds = \int_{t_{\ell-1}}^{t_\ell} (\boldsymbol{\varphi}, d\boldsymbol{W}(s)),\end{aligned} \tag{5.52}$$

$$(\operatorname{div} \boldsymbol{z}(t_\ell), \chi) = 0. \tag{5.53}$$

For (5.52) and (5.53) we have the following algorithm:

Algorithm 3.6 (First auxiliary algorithm)**.** *Let $\boldsymbol{z}^0 := 0$. Find for every $\ell \in \{1, \ldots, M\}$ a pair of random variables $(\boldsymbol{z}^\ell, \pi^\ell)$ with values in $\mathbb{W}^{1,2}_{\mathrm{per}} \times L^2_{\mathrm{per}}$, such that we have $\mathbb{P}$-a.s.*

- *Penalization:*

$$\begin{aligned}(\tilde{\boldsymbol{z}}^{\varepsilon,\ell} - \boldsymbol{z}^{\varepsilon,\ell-1}, \boldsymbol{\varphi}) + \nu k (\nabla \tilde{\boldsymbol{z}}^{\varepsilon,\ell}, \nabla \boldsymbol{\varphi}) + k (\nabla \tilde{\pi}^{\varepsilon,\ell}, \boldsymbol{\varphi}) \\ + k (\nabla \xi^{\varepsilon,\ell-1}, \boldsymbol{\varphi}) = (\Delta_\ell \boldsymbol{W}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \mathbb{W}^{1,2}_{\mathrm{per}},\end{aligned} \tag{6.54}$$

$$(\operatorname{div} \tilde{\boldsymbol{z}}^{\varepsilon,\ell}, \chi) + \varepsilon (\tilde{\pi}^{\varepsilon,\ell}, \chi) = 0, \quad \forall \chi \in L^2_{\mathrm{per}}. \tag{6.55}$$

- *Projection:*

$$(\boldsymbol{z}^{\varepsilon,\ell} - \tilde{\boldsymbol{z}}^{\varepsilon,\ell}, \boldsymbol{\varphi}) + \alpha k (\nabla (\xi^{\varepsilon,\ell} - \xi^{\varepsilon,\ell-1}), \boldsymbol{\varphi}) = 0, \forall \boldsymbol{\varphi} \in \mathbb{W}^{1,2}_{\mathrm{per}}, \tag{6.56}$$

$$(\operatorname{div} \boldsymbol{z}^{\varepsilon,\ell}, \chi) = 0, \forall \chi \in L^2_{\mathrm{per}}, \tag{6.57}$$

$$\pi^{\varepsilon,\ell} = \tilde{\pi}^{\varepsilon,\ell} + \xi^{\varepsilon,\ell} + \alpha (\xi^{\varepsilon,\ell} - \xi^{\varepsilon,\ell-1}).$$

Define the errors $\tilde{\boldsymbol{\epsilon}}^\ell = \boldsymbol{z}(t_\ell) - \tilde{\boldsymbol{z}}^{\varepsilon,\ell}$, $\boldsymbol{\epsilon}^\ell = \boldsymbol{z}(t_\ell) - \boldsymbol{z}^{\varepsilon,\ell}$ and $\varpi^\ell = \pi(t_\ell) - \pi^{\varepsilon,\ell}$. We subtract (6.54) from (5.52) to get

$$\begin{aligned}(\tilde{\boldsymbol{\epsilon}}^\ell - \boldsymbol{\epsilon}^{\ell-1}, \boldsymbol{\varphi}) + \nu \int_{t_{\ell-1}}^{t_\ell} (\nabla (\boldsymbol{z}(s) - \tilde{\boldsymbol{z}}^{\varepsilon,\ell}), \nabla \boldsymbol{\varphi}) ds \\ + \int_{t_{\ell-1}}^{t_\ell} (\nabla (\pi(s) - \tilde{\pi}^{\varepsilon,\ell} - \xi^{\varepsilon,\ell-1}), \boldsymbol{\varphi}) ds = 0,\end{aligned} \tag{6.58}$$

and choose $\chi = \operatorname{div}\boldsymbol{\varphi}$ in (6.55) to get

$$-(\nabla\pi^{\varepsilon,\ell},\boldsymbol{\varphi}) = \frac{1}{\varepsilon}(\nabla\operatorname{div}\tilde{\boldsymbol{z}}^{\varepsilon,\ell},\boldsymbol{\varphi}). \tag{6.59}$$

Thanks to the following identities

$$(\nabla(\boldsymbol{z}(s)-\tilde{\boldsymbol{z}}^{\varepsilon,\ell}),\nabla\boldsymbol{\varphi}) = (\nabla\tilde{\boldsymbol{\epsilon}}^{\ell},\nabla\boldsymbol{\varphi}) + (\nabla(\boldsymbol{z}(s)-\boldsymbol{z}(t_\ell)),\nabla\boldsymbol{\varphi}) \text{ and} \tag{6.60}$$

$$(\nabla(\pi(s)-\pi^{\varepsilon,\ell}-\xi^{\varepsilon,\ell-1}),\boldsymbol{\varphi}) = (\nabla\pi(s),\boldsymbol{\varphi}) + \frac{1}{\varepsilon}(\nabla\operatorname{div}\tilde{\boldsymbol{z}}^{\varepsilon,\ell},\boldsymbol{\varphi}) - (\nabla\xi^{\varepsilon,\ell-1},\boldsymbol{\varphi}), \tag{6.61}$$

the equation (6.58) is reduced to

$$\begin{aligned}(\tilde{\boldsymbol{\epsilon}}^{\ell}-\boldsymbol{\epsilon}^{\ell-1},\boldsymbol{\varphi}) + \nu k(\nabla\tilde{\boldsymbol{\epsilon}}^{\ell},\nabla\boldsymbol{\varphi}) - \frac{k}{\varepsilon}(\nabla\operatorname{div}\tilde{\boldsymbol{z}}^{\varepsilon,\ell},\boldsymbol{\varphi})&\\ -k(\nabla\xi^{\varepsilon,\ell-1},\boldsymbol{\varphi}) = R_\ell^{\boldsymbol{z}}(\boldsymbol{\varphi}) - \int_{t_{\ell-1}}^{t_\ell}(\nabla\pi(s),\boldsymbol{\varphi})ds,&\end{aligned} \tag{6.62}$$

where

$$R_\ell^{\boldsymbol{z}}(\boldsymbol{\varphi}) = \nu\int_{t_{\ell-1}}^{t_\ell}(\nabla(\boldsymbol{z}(t_\ell)-\boldsymbol{z}(s)),\nabla\boldsymbol{\varphi})\,ds.$$

To (6.62) we associate the following projection equation

$$\begin{cases}(\boldsymbol{\epsilon}^{\ell}-\tilde{\boldsymbol{\epsilon}}^{\ell},\boldsymbol{\varphi}) = k\alpha(\nabla(\xi^{\varepsilon,\ell}-\xi^{\varepsilon,\ell-1}),\boldsymbol{\varphi}),\\ \qquad\operatorname{div}\boldsymbol{\epsilon}^{\ell} = 0.\end{cases} \tag{6.63}$$

Lemma 3.8. *Let $\alpha > 1$ and $0 < \eta < 1/2$. For every $\varepsilon > 0$, there exists a constant $C = C(T,\nu,\eta) > 0$ such that*

$$\mathbb{E}\max_{1<m\le M}\|\boldsymbol{\epsilon}^m\|^2 + \nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\boldsymbol{\epsilon}^{\ell}\|^2\right) \le C(k^\eta+\varepsilon). \tag{6.64}$$

Proof. We take $\boldsymbol{\varphi} = 2\tilde{\boldsymbol{\epsilon}}^{\ell}$ in (6.62). Then we use the algebraic identity (4.31) and the fact that $\operatorname{div}\boldsymbol{z}(t_\ell) = 0$ to get

$$\begin{aligned}\|\tilde{\boldsymbol{\epsilon}}^{\ell}\|^2 - \|\boldsymbol{\epsilon}^{\ell-1}\|^2 + \|\tilde{\boldsymbol{\epsilon}}^{\ell}-\boldsymbol{\epsilon}^{\ell-1}\|^2 + 2\nu k\|\nabla\tilde{\boldsymbol{\epsilon}}^{\ell}\|^2 + \frac{2k}{\varepsilon}\|\operatorname{div}\tilde{\boldsymbol{\epsilon}}^{\ell}\|^2&\\ = 2k(\nabla\xi^{\varepsilon,\ell-1},\tilde{\boldsymbol{\epsilon}}^{\ell}) + R_\ell^{\boldsymbol{z}}(2\tilde{\boldsymbol{\epsilon}}^{\ell}) + \int_{t_{\ell-1}}^{t_\ell}(\operatorname{div}\tilde{\boldsymbol{\epsilon}}^{\ell},\pi(s))ds.&\end{aligned} \tag{6.65}$$

Let us take $\boldsymbol{\varphi} = \tilde{\boldsymbol{\epsilon}}^\ell + \boldsymbol{\epsilon}^\ell$ in (6.63) to get

$$\frac{\alpha-1}{\alpha}\left(\|\boldsymbol{\epsilon}^\ell\|^2 - \|\tilde{\boldsymbol{\epsilon}}^\ell\|^2 + \|\tilde{\boldsymbol{\epsilon}}^\ell - \boldsymbol{\epsilon}^\ell\|^2\right) = 0, \tag{6.66}$$

$$\frac{1}{\alpha}\left(\|\boldsymbol{\epsilon}^\ell\|^2 - \|\tilde{\boldsymbol{\epsilon}}^\ell\|^2\right) = \frac{k}{2}(\nabla(\xi^{\varepsilon,\ell} - \xi^{\varepsilon,\ell-1}), \tilde{\boldsymbol{\epsilon}}^\ell). \tag{6.67}$$

Collecting (6.65) to (6.67) together, we arrive at

$$\begin{aligned}&\|\boldsymbol{\epsilon}^\ell\|^2 - \|\boldsymbol{\epsilon}^{\ell-1}\|^2 + \|\tilde{\boldsymbol{\epsilon}}^\ell - \boldsymbol{\epsilon}^{\ell-1}\|^2 + (\tfrac{\alpha-1}{\alpha})\|\tilde{\boldsymbol{\epsilon}}^\ell - \boldsymbol{\epsilon}^\ell\|^2 + 2\nu k\|\nabla\tilde{\boldsymbol{\epsilon}}^\ell\|^2 \\ &+ \frac{2k}{\varepsilon}\|\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^\ell\|^2 \leq 2k(\nabla(\xi^{\varepsilon,\ell} + \xi^{\varepsilon,\ell-1}), \tilde{\boldsymbol{\epsilon}}^\ell) + R_\ell^{\boldsymbol{z}}(2\tilde{\boldsymbol{\epsilon}}^\ell) + \int_{t_{\ell-1}}^{t_\ell}(\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^\ell, \pi(s))ds.\end{aligned} \tag{6.68}$$

First, notice that from (6.63) it follows

$$\tilde{\boldsymbol{\epsilon}}^\ell = \boldsymbol{\epsilon}^\ell - k\alpha\nabla(\xi^{\varepsilon,\ell} - \xi^{\varepsilon,\ell-1}).$$

Therefore, we have

$$2k(\nabla(\xi^{\varepsilon,\ell} + \xi^{\varepsilon,\ell-1}), \tilde{\boldsymbol{\epsilon}}^\ell) = 2\alpha k^2\|\xi^{\varepsilon,\ell-1}\|^2 - 2\alpha k^2\|\xi^{\varepsilon,\ell}\|^2. \tag{6.69}$$

Secondly, applying the Young inequality to $R_\ell^z(2\tilde{\boldsymbol{\epsilon}}^\ell)$, we get

$$R_\ell^z(2\tilde{\boldsymbol{\epsilon}}^\ell) \leq C_{\delta_1}\nu\int_{t_{\ell-1}}^{t_\ell}\|\nabla(\boldsymbol{z}(t_\ell) - \boldsymbol{z}(s))\|^2 ds + \delta_1 k\|\nabla\tilde{\boldsymbol{\epsilon}}^\ell\|^2, \tag{6.70}$$

$$\int_{t_{\ell-1}}^{t_\ell}(\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^\ell, \pi(s))ds \leq \frac{k}{\varepsilon}\|\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^\ell\|^2 + \varepsilon\int_{t_{\ell-1}}^{t_\ell}\|\pi(s)\|^2 ds. \tag{6.71}$$

We add (6.69) to (6.71) with (6.68). Summing up for $\ell = 1$ to $\ell = m$,

$$\begin{aligned}&\|\boldsymbol{\epsilon}^m\|^2 + 2\alpha k^2\|\xi^{\varepsilon,m}\|^2 + (\tfrac{\alpha-1}{\alpha})\sum_{\ell=1}^m\|\tilde{\boldsymbol{\epsilon}}^\ell - \boldsymbol{\epsilon}^\ell\|^2 \\ &+ (2-\delta_1)\nu\left(k\sum_{\ell=1}^m\|\nabla\tilde{\boldsymbol{\epsilon}}^\ell\|^2\right) + \frac{1}{\varepsilon}\mathbb{E}\left(k\sum_{\ell=1}^m\|\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^\ell\|^2\right) \\ &\leq C_{\delta_1}\nu\sum_{\ell=1}^m\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\|\nabla(\boldsymbol{z}(t_\ell) - \boldsymbol{z}(s))\|^2 + \varepsilon\sum_{\ell=1}^m\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\|\pi(s)\|^2 ds.\end{aligned}$$

Using the identity (2.5), we have

$$\frac{1}{\alpha}\|\boldsymbol{\epsilon}^m\|^2+\frac{\alpha-1}{\alpha}\|\tilde{\boldsymbol{\epsilon}}^m\|^2+2\alpha k^2\|\xi^{\varepsilon,m}\|^2+\frac{\alpha-1}{\alpha}\sum_{\ell=1}^{m-1}\|\tilde{\boldsymbol{\epsilon}}^\ell-\boldsymbol{\epsilon}^\ell\|^2$$
$$+(2-\delta_1)\nu\left(k\sum_{\ell=1}^{m}\|\nabla\tilde{\boldsymbol{\epsilon}}^\ell\|^2\right)+\frac{1}{\varepsilon}\mathbb{E}\left(k\sum_{\ell=1}^{m}\|\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^\ell\|^2\right)$$
$$\leq C_{\delta_1}\nu\sum_{\ell=1}^{m}\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\|\nabla(\boldsymbol{z}(t_\ell)-\boldsymbol{z}(s))\|^2+\varepsilon\sum_{\ell=1}^{m}\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\|\pi(s)\|^2ds.$$

Now taking the maximum for $1<m\leq M$, and expectation, we arrive at

$$\begin{aligned}&\mathbb{E}\max_{1\leq m\leq M}\left\{\frac{1}{\alpha}\|\boldsymbol{\epsilon}^m\|^2+\frac{\alpha-1}{\alpha}\|\tilde{\boldsymbol{\epsilon}}^m\|^2+2\alpha k^2\|\xi^{\varepsilon,m}\|^2\right\}\\&+\frac{\alpha-1}{\alpha}\mathbb{E}\sum_{\ell=1}^{M-1}\|\tilde{\boldsymbol{\epsilon}}^\ell-\boldsymbol{\epsilon}^\ell\|^2\\&+(2-\delta_1)\nu\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{\boldsymbol{\epsilon}}^\ell\|^2\right)+\frac{1}{\varepsilon}\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^\ell\|^2\right)\\&\leq C_{\delta_1}\nu\sum_{\ell=1}^{M}\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\|\nabla(\boldsymbol{z}(t_\ell)-\boldsymbol{z}(s))\|^2+\varepsilon\sum_{\ell=1}^{M}\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\|\pi(s)\|^2ds.\end{aligned}\tag{6.72}$$

Finally, we choose $\delta_1>0$ so that $(2-\delta_1)$ stays positive and conclude the proof with Lemma 3.4, Proposition 3.3 and (2.14), and the stability of $\boldsymbol{P}_{\mathbb{H}}$ in $\mathbb{W}^{1,2}$.

□

Lemma 3.9. *Let* $\alpha>1$ *and* $0<\eta<1/2$. *For every* $\varepsilon>0$, *there exists a constant* $C=C(T,\nu,\eta)>0$ *such that we have*

$$\mathbb{E}\left(k\sum_{\ell=1}^{M}\|\varpi^\ell\|^2\right)\leq C(k^\eta+\varepsilon).\tag{6.73}$$

Proof. We substitute (6.63) to (6.62) and arrange the result such that we obtain

$$\begin{aligned}&(\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1},\boldsymbol{\varphi})+\nu k(\nabla\tilde{\boldsymbol{\epsilon}}^\ell,\nabla\boldsymbol{\varphi})\\&\qquad+k(\nabla\varpi^\ell,\boldsymbol{\varphi})=R_\ell^{\boldsymbol{z}}(\boldsymbol{\varphi})+\int_{t_{\ell-1}}^{t_\ell}(\nabla(\pi(t_\ell)-\pi(s)),\boldsymbol{\varphi})ds.\end{aligned}\tag{6.74}$$

Using identity (6.59), we get

$$k(\nabla\varpi^\ell,\boldsymbol{\varphi})=(\boldsymbol{\epsilon}^{\ell-1}-\boldsymbol{\epsilon}^\ell,\boldsymbol{\varphi})+\int_{t_{\ell-1}}^{t_\ell}(\nabla(\pi(t_\ell)-\pi(s)),\boldsymbol{\varphi})ds-k\nu(\nabla\tilde{\boldsymbol{\epsilon}}^\ell,\nabla\boldsymbol{\varphi})+R_\ell^{\boldsymbol{z}}(\boldsymbol{\varphi}).$$

Using inequality (4.46), we derive that

$$\begin{aligned}k^2\|\varpi^{\varepsilon,\ell}\|^2\leq C\|R_\ell^{\boldsymbol{z}}\|_{-1}^2+C\|\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1}\|_{-1}^2+(\nu k)^2\|\nabla\boldsymbol{\epsilon}^\ell\|^2\\+Ck\int_{t_{\ell-1}}^{t_\ell}\|\pi(t_\ell)-\pi(s)\|^2ds.\end{aligned}\tag{6.75}$$

For brevity let us introduce the numbering

$$\begin{aligned}&\mathtt{I}+\mathtt{II}+\mathtt{III}+\mathtt{IV}\\&\qquad:=C\|R_\ell^{\boldsymbol{z}}\|_{-1}^2+C\|\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1}\|_{-1}^2+Ck\int_{t_{\ell-1}}^{t_\ell}\|\pi(t_\ell)-\pi(s)\|^2ds+(\nu k)^2\|\nabla\boldsymbol{\epsilon}^\ell\|^2.\end{aligned}$$

First, we have for $\mathtt{I}$

$$\begin{aligned}\mathtt{I}=\sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}}\frac{(R_\ell^{\boldsymbol{z}}(\boldsymbol{\varphi}))^2}{\|\boldsymbol{\varphi}\|_1^2}&=\left(\int_{t_{\ell-1}}^{t_\ell}\sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}}\nu\frac{(\nabla(\boldsymbol{z}(t_\ell)-\boldsymbol{z}(s)),\nabla\boldsymbol{\varphi})}{\|\boldsymbol{\varphi}\|_1}ds\right)^2\\&\leq C(\nu,L)k\int_{t_{\ell-1}}^{t_\ell}\|\nabla(\boldsymbol{z}(t_\ell)-\boldsymbol{z}(s)\|^2ds.\end{aligned}\tag{6.76}$$

Now, we estimate the term II. Since $\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1}\in\mathscr{D}(\boldsymbol{A}^{-1})$, we can take $\boldsymbol{\varphi}=\boldsymbol{A}^{-1}(\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1})$ in identity (6.74). From the orthogonality we get

$$k(\nabla\varpi^\ell,\boldsymbol{A}^{-1}(\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1}))=\int_{t_{\ell-1}}^{t_\ell}(\nabla(\pi(t_\ell)-\pi(s)),\boldsymbol{A}^{-1}(\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1}))ds=0,\tag{6.77}$$

$$k(\nabla\varpi^\ell,\boldsymbol{A}^{-1}(\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1}))=\int_{t_{\ell-1}}^{t_\ell}(\nabla(\pi(t_\ell)-\pi(s)),\boldsymbol{A}^{-1}(\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1}))ds=0,\tag{6.78}$$

and from Proposition 3.1 we get

$$\mathtt{II}\leq C(\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1},\boldsymbol{A}^{-1}(\boldsymbol{\epsilon}^\ell-\boldsymbol{\epsilon}^{\ell-1})).\tag{6.79}$$

Applying the Young inequality we obtain the following results:

$$R_\ell^{\boldsymbol{z}}(\boldsymbol{A}^{-1}(\boldsymbol{\epsilon}^\ell - \boldsymbol{\epsilon}^{\ell-1})) \leq C(\delta_1,\nu)k\int_{t_{\ell-1}}^{t_\ell} \|\nabla(\boldsymbol{z}(t_\ell) - \boldsymbol{z}(s))\|^2 ds + \delta_1 \mathtt{II}. \tag{6.80}$$

Collecting the last four estimates we obtain

$$(1-2\delta_1)\mathtt{II} \leq C(\delta_1)k^2\|\nabla\tilde{\boldsymbol{\epsilon}}^\ell\|^2 + C(\delta_1,\nu)k\int_{t_{\ell-1}}^{t_\ell} \|\nabla(\boldsymbol{z}(t_\ell) - \boldsymbol{z}(s))\|^2 ds. \tag{6.81}$$

After choosing δ_1 so that $(1-2\delta_1) > 0$, we substitute the estimates of $\mathtt{I}$ and $\mathtt{II}$ in (6.75), let the terms II and IV unchanged, and get in this way the following new estimate

$$\begin{aligned} k^2\|\varpi^\ell\|^2 \leq C(\nu,L)k\int_{t_{\ell-1}}^{t_\ell} \|\nabla(\boldsymbol{z}(t_\ell) - \boldsymbol{z}(s)\|^2 ds + C(\nu)k^2\|\nabla\tilde{\boldsymbol{\epsilon}}^\ell\|^2 \\ + Ck\int_{t_{\ell-1}}^{t_\ell} \|\nabla(\boldsymbol{z}(t_\ell) - \boldsymbol{z}(s))\|^2 ds + Ck\int_{t_{\ell-1}}^{t_\ell} \|\pi(t_\ell) - \pi(s)\|^2 ds. \end{aligned} \tag{6.82}$$

By taking the sum for $\ell = 1$ to $\ell = M$ and expectation in (6.82), we get

$$\begin{aligned} k\mathbb{E}\left(k\sum_{\ell=1}^M \|\varpi^\ell\|^2\right) \leq C(\nu,L)k\sum_{\ell=1}^M \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\nabla(\boldsymbol{z}(t_\ell) - \boldsymbol{z}(s)\|^2 ds + + C(\nu)k\mathbb{E}\left(k\sum_{\ell=1}^M \|\nabla\tilde{\boldsymbol{\epsilon}}^\ell\|^2\right) \\ + C(\nu)k\sum_{\ell=1}^M \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\nabla(\boldsymbol{z}(t_\ell) - \boldsymbol{z}(s))\|^2 ds + Ck\sum_{\ell=1}^M \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}\|\pi(t_\ell) - \pi(s)\|^2 ds. \end{aligned}$$

From Lemma 3.4 (iii) and Lemma 3.8 we obtain

$$k\mathbb{E}\left(k\sum_{\ell=1}^M \|\varpi^\ell\|^2\right) \leq C(L,T,\nu)(k^{\eta+1} + k(k^\eta + \varepsilon)).$$

□

Let $\boldsymbol{v} = \{\boldsymbol{v}(t,\cdot) : t \in [t_{\ell-1}, t_\ell]\}$ be the strong solution of (2.21) as defined in Assumption 3.2 and $\rho = \{\rho(t,\cdot) : t \in [t_{\ell-1}, t_\ell]\}$ the associated pressure, i.e. for every $t \in [t_{\ell-1}, t_\ell]$ and all $\boldsymbol{\varphi} \in$

$\mathbb{W}^{1,2}$, $\chi \in L^2_{\text{per}}$, we have $\mathbb{P}$-a.s.

$$(\boldsymbol{v}(t_\ell),\boldsymbol{\varphi})+\nu\int_{t_{\ell-1}}^{t_\ell}(\nabla\boldsymbol{v}(s),\nabla\boldsymbol{\varphi})ds+\int_{t_{\ell-1}}^{t_\ell}\tilde{b}(\boldsymbol{u}(s),\boldsymbol{u}(s),\boldsymbol{\varphi})ds \\ +\int_{t_{\ell-1}}^{t_\ell}(\nabla\rho(s),\boldsymbol{\varphi})ds=\int_{t_{\ell-1}}^{t_\ell}(\boldsymbol{\varphi},d\boldsymbol{W}(s)), \tag{6.83}$$

$$(\operatorname{div}\boldsymbol{v}(t_\ell),\chi)=0. \tag{6.84}$$

To these equations correspond the following algorithm:

Algorithm 3.7 (Second auxiliary algorithm)**.** *Let* $\boldsymbol{v}^0:=\boldsymbol{u}_0$ *be a given* $\mathbb{V}$*-valued random variable. Find for every* $\ell\in\{1,\ldots,M\}$ *a tuple of random variables* $(\boldsymbol{v}^{\varepsilon,\ell},\rho^{\varepsilon,\ell})$ *with values in* $\mathbb{W}^{1,2}_{\text{per}}\times L^2_{\text{per}}$*, such that we have* $\mathbb{P}$*-a.s.*

- *Penalization:*

$$(\tilde{\boldsymbol{v}}^{\varepsilon,\ell}-\boldsymbol{v}^{\varepsilon,\ell-1},\boldsymbol{\varphi})+\nu k(\nabla\tilde{\boldsymbol{v}}^{\varepsilon,\ell},\nabla\boldsymbol{\varphi})+k\tilde{b}(\tilde{\boldsymbol{v}}^{\varepsilon,\ell}+\tilde{\boldsymbol{z}}^{\varepsilon,\ell},\tilde{\boldsymbol{v}}^{\varepsilon,\ell}+\tilde{\boldsymbol{z}}^{\varepsilon,\ell},\boldsymbol{\varphi}) \\ +k(\nabla\tilde{\rho}^{\varepsilon,\ell},\boldsymbol{\varphi})+k(\nabla\psi^{\varepsilon,\ell-1},\boldsymbol{\varphi})=0,\ \forall\boldsymbol{\varphi}\in\mathbb{W}^{1,2}, \tag{7.85}$$

$$(\operatorname{div}\tilde{\boldsymbol{v}}^{\varepsilon,\ell},\chi)+\varepsilon(\tilde{\rho}^{\varepsilon,\ell},\chi)=0,\ \forall\chi\in L^2_{\text{per}}. \tag{7.86}$$

- *Projection:*

$$(\boldsymbol{v}^{\varepsilon,\ell}-\tilde{\boldsymbol{v}}^{\varepsilon,\ell},\boldsymbol{\varphi})+\alpha k(\nabla(\psi^{\varepsilon,\ell}-\psi^{\varepsilon,\ell-1}),\boldsymbol{\varphi})=0,\forall\boldsymbol{\varphi}\in\mathbb{W}^{1,2}_{\text{per}}, \tag{7.87}$$

$$(\operatorname{div}\boldsymbol{v}^{\varepsilon,\ell},\chi)=0,\forall\chi\in L^2_{\text{per}}, \tag{7.88}$$

$$\rho^{\varepsilon,\ell}=\tilde{\rho}^{\varepsilon,\ell}+\psi^{\varepsilon,\ell}+\alpha(\psi^{\varepsilon,\ell}-\psi^{\varepsilon,\ell-1}).$$

Define the errors $\boldsymbol{\sigma}^\ell=\boldsymbol{v}(t_\ell)-\boldsymbol{v}^{\varepsilon,\ell}$, $\tilde{\boldsymbol{\sigma}}^\ell=\boldsymbol{v}(t_\ell)-\tilde{\boldsymbol{v}}^{\varepsilon,\ell}$, $\tilde{\varrho}^\ell=\rho(t_\ell)-\tilde{\rho}^{\varepsilon,\ell}$, and $\varrho^\ell=\rho(t_\ell)-\rho^{\varepsilon,\ell}$. Subtracting (7.85) from (6.83) we get

$$(\tilde{\boldsymbol{\sigma}}^\ell-\boldsymbol{\sigma}^{\ell-1},\boldsymbol{\varphi})+\nu\int_{t_{\ell-1}}^{t_\ell}(\nabla(\boldsymbol{v}(s)-\tilde{\boldsymbol{v}}^{\varepsilon,\ell}),\nabla\boldsymbol{\varphi})ds+\int_{t_{\ell-1}}^{t_\ell}\tilde{b}(\boldsymbol{u}(s),\boldsymbol{u}(s),\boldsymbol{\varphi})ds \\ -\int_{t_{\ell-1}}^{t_\ell}\tilde{b}(\boldsymbol{u}^\ell,\boldsymbol{u}^\ell,\boldsymbol{\varphi})ds+\int_{t_{\ell-1}}^{t_\ell}(\nabla(\rho(s)-\tilde{\rho}^{\varepsilon,\ell}),\boldsymbol{\varphi})ds-k(\nabla\psi^{\varepsilon,\ell-1},\boldsymbol{\varphi})=0. \tag{7.89}$$

Choosing $\chi = \operatorname{div} \tilde{\boldsymbol{v}}^{\varepsilon,\ell}$ in (7.86) we get

$$-(\nabla \tilde{\rho}^{\varepsilon,\ell}, \boldsymbol{\varphi}) = \frac{1}{\varepsilon}(\nabla \operatorname{div} \tilde{\boldsymbol{v}}^{\varepsilon,\ell}, \boldsymbol{\varphi}). \tag{7.90}$$

Thanks to the identities

$$(\nabla(\boldsymbol{v}(s) - \tilde{\boldsymbol{v}}^{\varepsilon,\ell}), \nabla\boldsymbol{\varphi}) = (\nabla\tilde{\boldsymbol{\sigma}}^{\ell}, \nabla\boldsymbol{\varphi}) + (\nabla(\boldsymbol{v}(s) - \boldsymbol{v}(t_\ell)), \nabla\boldsymbol{\varphi}) \text{ and} \tag{7.91}$$

$$(\nabla(\rho(s) - \tilde{\rho}^{\varepsilon,\ell}), \boldsymbol{\varphi}) = (\nabla(\rho(s), \boldsymbol{\varphi}) + \frac{1}{\varepsilon}(\nabla \operatorname{div} \tilde{\boldsymbol{v}}^{\varepsilon,\ell}, \boldsymbol{\varphi}), \tag{7.92}$$

equation (7.89) is reduced to

$$\begin{aligned}(\tilde{\boldsymbol{\sigma}}^{\ell} - \boldsymbol{\sigma}^{\ell-1}, \boldsymbol{\varphi}) + \nu k(\nabla\tilde{\boldsymbol{\sigma}}^{\ell}, \nabla\boldsymbol{\varphi}) + \frac{k}{\varepsilon}(\nabla \operatorname{div} \tilde{\boldsymbol{v}}^{\varepsilon,\ell}, \boldsymbol{\varphi}) \\ - k(\nabla\psi^{\varepsilon,\ell-1}, \boldsymbol{\varphi}) = Q_\ell(\boldsymbol{\varphi}) + R_\ell^{\boldsymbol{v}}(\boldsymbol{\varphi}) + \int_{t_{\ell-1}}^{t_\ell} (\operatorname{div} \tilde{\boldsymbol{\sigma}}^{\ell}, \rho(s))ds,\end{aligned} \tag{7.93}$$

where

$$Q_\ell(\boldsymbol{\varphi}) = \int_{t_{\ell-1}}^{t_\ell} \Big(\tilde{b}(\boldsymbol{u}(s), \boldsymbol{u}(s), \boldsymbol{\varphi}) - \tilde{b}(\tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \boldsymbol{\varphi})\Big) ds,$$

$$R_\ell^{\boldsymbol{v}}(\boldsymbol{\varphi}) = \nu \int_{t_{\ell-1}}^{t_\ell} (\nabla(\boldsymbol{v}(t_\ell) - \boldsymbol{v}(s)), \nabla\boldsymbol{\varphi})\, ds.$$

To (7.93) we associate the following projection equation

$$\begin{cases} (\boldsymbol{\sigma}^{\ell} - \tilde{\boldsymbol{\sigma}}^{\ell}, \boldsymbol{\varphi}) = k\alpha(\nabla(\psi^{\varepsilon,\ell} - \psi^{\varepsilon,\ell-1}), \boldsymbol{\varphi}), \\ \qquad \operatorname{div} \boldsymbol{\sigma}^{\ell} = 0. \end{cases} \tag{7.94}$$

Let $\kappa_1, \kappa_2, \kappa_3 > 0$ some fixed constants, and let us introduce the sample subsets

$$\begin{aligned} \Omega_{\kappa_1} &= \left\{ \omega \in \Omega : \sup_{0 \le t \le T} \|\boldsymbol{u}(t)\|_{\mathbb{V}}^2 + k\sum_{\ell=1}^{M} \|\boldsymbol{u}^{\ell}\|_1^2 \le \kappa_1 \right\}, \\ \Omega_{\kappa_2} &= \left\{ \omega \in \Omega : \max_{1 \le m \le M} \|\boldsymbol{\epsilon}^m\|^2 + \nu k \sum_{\ell=1}^{M} \|\boldsymbol{\epsilon}^{\ell}\|_1^2 + k\sum_{\ell=1}^{M} \|\varpi^{\ell}\|^2 \le \kappa_2 \right\}, \text{ and} \\ \Omega_{\kappa_3} &= \left\{ \omega \in \Omega : \forall\, 0 \le s < t \le T,\ \|\boldsymbol{u}(s) - \boldsymbol{u}(t)\|_{\mathbb{L}^4}^2 \le \kappa_3 |t-s|^{2\eta} \right\}. \end{aligned} \tag{7.95}$$

In the next paragraph we derive some error estimates on the intersection of these subsets of Ω.

Lemma 3.10. *Let $\alpha>1$ and $0<\eta<1/2$. For every $\varepsilon>0$, there exists a constant $C=C(L,T,\nu)>0$ such that on $\Omega_{\kappa_1}\cap\Omega_{\kappa_2}\cap\Omega_{\kappa_3}$ we have*

$$\max_{1<m\leq M}\|\boldsymbol{\sigma}^m\|^2+\nu k\sum_{\ell=1}^{M}\|\nabla\boldsymbol{\sigma}^\ell\|^2\leq C(\kappa_1\kappa_3 k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^\eta+\varepsilon)\exp(\kappa_1). \tag{7.96}$$

Proof. We take $\boldsymbol{\varphi}=2\tilde{\boldsymbol{\sigma}}^\ell$ in (7.93) and proceed exactly like in the proof of Lemma 3.8 until (6.72). Doing so we arrive at

$$\begin{aligned}\max_{1\leq m\leq M}\big\{(\tfrac{\alpha+1}{2\alpha})\|\boldsymbol{\sigma}^m\|^2+(\tfrac{\alpha-1}{2\alpha})\|\tilde{\boldsymbol{\sigma}}^m\|^2\big\}+(\tfrac{\alpha-1}{2\alpha})\sum_{\ell=1}^{M-1}\|\tilde{\boldsymbol{\sigma}}^\ell-\boldsymbol{\sigma}^\ell\|^2\\ +(\nu-\delta_1)k\sum_{\ell=1}^{M}\|\nabla\tilde{\boldsymbol{\sigma}}^\ell\|^2\leq C(\delta_1)\nu\sum_{\ell=1}^{M}\int_{t_{\ell-1}}^{t_\ell}\|\nabla(\boldsymbol{v}(t_\ell)-\boldsymbol{v}(s))\|^2\\ +\varepsilon\sum_{\ell=1}^{M}\int_{t_{\ell-1}}^{t_\ell}\|\rho(s)\|^2ds+2\max_{1\leq m\leq M}\sum_{\ell=1}^{m}Q_\ell(\tilde{\boldsymbol{\sigma}}^\ell),\end{aligned} \tag{7.97}$$

where

$$Q_\ell(\tilde{\boldsymbol{\sigma}}^\ell)=\int_{t_{\ell-1}}^{t_\ell}\tilde{b}(\boldsymbol{u}(s),\boldsymbol{u}(s),\tilde{\boldsymbol{\sigma}}^\ell)-\tilde{b}(\tilde{\boldsymbol{u}}^{\varepsilon,\ell},\tilde{\boldsymbol{u}}^{\varepsilon,\ell},\tilde{\boldsymbol{\sigma}}^\ell)ds.$$

We aligned the term Q_ℓ into four terms as follows

$$\begin{aligned}Q_\ell(\tilde{\boldsymbol{\sigma}}^\ell)\leq&\int_{t_{\ell-1}}^{t_\ell}\Big(\tilde{b}(\boldsymbol{u}(s),\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell),\tilde{\boldsymbol{\sigma}}^\ell)+\tilde{b}(\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell),\boldsymbol{u}(t_\ell),\tilde{\boldsymbol{\sigma}}^\ell)\\ &+\tilde{b}(\boldsymbol{u}(t_\ell),\boldsymbol{u}(t_\ell)-\tilde{\boldsymbol{u}}^{\varepsilon,\ell},\tilde{\boldsymbol{\sigma}}^\ell)+\tilde{b}(\boldsymbol{u}(t_\ell)-\tilde{\boldsymbol{u}}^{\varepsilon,\ell},\tilde{\boldsymbol{u}}^{\varepsilon,\ell},\tilde{\boldsymbol{\sigma}}^\ell)\Big)ds\\ \leq&\int_{t_{\ell-1}}^{t_\ell}\big(NLT_1(\tilde{\boldsymbol{\sigma}}^\ell)+NLT_2(\tilde{\boldsymbol{\sigma}}^\ell)+NLT_3(\tilde{\boldsymbol{\sigma}}^\ell)+NLT_4(\tilde{\boldsymbol{\sigma}}^\ell)\big)ds.\end{aligned}$$

In the next lines, we will estimate the terms $NLT_j(\tilde{\boldsymbol{\sigma}}^\ell)$, $j=1,\dots,4$, one by one.

- $NLT_1(\tilde{\boldsymbol{\sigma}}^\ell)$: From (2.9), the Sobolev embedding $\mathbb{W}^{1,2}(D)\subset\mathbb{L}^4(D)$, and the Young inequality, we get the estimate

$$NLT_1(\tilde{\boldsymbol{\sigma}}^\ell)\leq|\tilde{b}(\boldsymbol{u}(s),\tilde{\boldsymbol{\sigma}}^\ell,\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell))|\leq C(\delta_1,L)\|\boldsymbol{u}(s)\|_1^2\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2+\delta_1\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.$$

Then, integrating over the time interval $[t_{\ell-1},t_\ell]$ with respect to s, using the Hölder inequality, and since $\omega\in\Omega_{\kappa_1}\cap\Omega_{\kappa_2}\cap\Omega_{\kappa_3}$, we get

$$\begin{aligned}\int_{t_{\ell-1}}^{t_\ell} NLT_1(\tilde{\boldsymbol{\sigma}}^\ell)ds &\le C(\delta_1,L)\int_{t_{\ell-1}}^{t_\ell}\|\boldsymbol{u}(s)\|_1^2\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2ds+\delta_1 k\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2\\ &\le C(\delta_1,L)\sup_{t_{\ell-1}\le s\le t_\ell}\|\boldsymbol{u}(s)\|_1^2\int_{t_{\ell-1}}^{t_\ell}\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2ds+\delta_1 k\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2\\ &\le C(\delta_1,L)\sup_{t_{\ell-1}\le s\le t_\ell}\|\boldsymbol{u}(s)\|_1^2\int_{t_{\ell-1}}^{t_\ell}\kappa_3|s-t_\ell|^{2\eta}ds+\delta_1 k\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2\\ &\le C(\delta_1,L)\kappa_1\kappa_3 k^{2\eta+1}+\delta_1 k\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.\end{aligned}$$

- $NLT_2(\tilde{\boldsymbol{\sigma}}^\ell)$: Again from (2.9) and the Young inequality, we infer

$$NLT_2(\tilde{\boldsymbol{\sigma}}^\ell)\le|\tilde{b}(\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell),\boldsymbol{u}(t_\ell),\tilde{\boldsymbol{\sigma}}^\ell)|\le C(\delta_1)\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2\|\boldsymbol{u}(t_\ell)\|_1^2+\delta_1\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.$$

Again, integrating over the time interval $[t_{\ell-1},t_\ell]$ with respect to s and since $\omega\in\Omega_{\kappa_2}$, we get

$$\begin{aligned}\int_{t_{\ell-1}}^{t_\ell} NLT_2(\tilde{\boldsymbol{\sigma}}^\ell)ds &\le C(\delta_1)\|\boldsymbol{u}(t_\ell)\|_1^2\int_{t_{\ell-1}}^{t_\ell}\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2ds+\delta_1 k\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2\\ &\le C(\delta_1)\|\boldsymbol{u}(t_\ell)\|_1^2\int_{t_{\ell-1}}^{t_\ell}\kappa_3|s-t_\ell|^{2\eta}ds+\delta_1 k\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2\\ &\le C(\delta_1)\kappa_3 k^{2\eta+1}\|\boldsymbol{u}(t_\ell)\|_1^2+\delta_1 k\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.\end{aligned}$$

Summing up from $\ell=1$ to $\ell=M$, using the Hölder inequality, and since $\omega\in\Omega_{\kappa_1}$, we get

$$\begin{aligned}\sum_{\ell=1}^M\int_{t_{\ell-1}}^{t_\ell} NLT_2(\tilde{\boldsymbol{\sigma}}^\ell)ds &\le C(\delta_1)\kappa_3 k^{2\eta+1}\sum_{\ell=1}^M\|\boldsymbol{u}(t_\ell)\|_1^2+\delta_1 k\sum_{\ell=1}^M\|\boldsymbol{\sigma}^\ell\|_1^2\\ &\le C(\delta_1,T)\kappa_1\kappa_3 k^{2\eta}+\delta_1 k\sum_{\ell=1}^M\|\boldsymbol{\sigma}^\ell\|_1^2.\end{aligned}$$

- $NLT_3(\tilde{\boldsymbol{\sigma}}^\ell)$: Since $\boldsymbol{u}(t_\ell)-\tilde{\boldsymbol{u}}^{\varepsilon,\ell}=\tilde{\boldsymbol{\epsilon}}^\ell+\tilde{\boldsymbol{\sigma}}^\ell$ and thanks to the orthogonal property of $\tilde{b}$ (see Equation (2.8)), we have

$$NLT_3(\tilde{\boldsymbol{\sigma}}^\ell)=|\tilde{b}(\boldsymbol{u}(t_\ell),\boldsymbol{u}(t_\ell)-\tilde{\boldsymbol{u}}^{\varepsilon,\ell},\tilde{\boldsymbol{\sigma}}^\ell)|=|\tilde{b}(\boldsymbol{u}(t_\ell),\tilde{\boldsymbol{\epsilon}}^\ell+\tilde{\boldsymbol{\sigma}}^\ell,\tilde{\boldsymbol{\sigma}}^\ell)|=|\tilde{b}(\boldsymbol{u}(t_\ell),\tilde{\boldsymbol{\epsilon}}^\ell,\tilde{\boldsymbol{\sigma}}^\ell)|.$$

From (2.10) and the Young inequality, we have

$$NLT_3(\tilde{\boldsymbol{\sigma}}^\ell) \leq C(\delta_1, L)\|\boldsymbol{u}(t_\ell)\|_1^2 \|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2 + \delta_1 \|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.$$

As before, integrating over the interval $[t_{\ell-1}, t_\ell]$ with respect to s, we obtain

$$\int_{t_{\ell-1}}^{t_\ell} NLT_3(\tilde{\boldsymbol{\sigma}}^\ell) ds \leq C_{\delta_1}(L) k \|\boldsymbol{u}(t_\ell)\|_1^2 \|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2 + \delta_1 k \|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.$$

Summing up from $\ell = 1$ to $\ell = M$, using the Hölder inequality, and since $\omega \in \Omega_{\kappa_1} \cap \Omega_{\kappa_2}$, we have

$$\begin{aligned}
\sum_{\ell=1}^{M} \int_{t_{\ell-1}}^{t_\ell} NLT_3(\boldsymbol{\sigma}^\ell) ds &\leq C(\delta_1, L) k \sum_{\ell=1}^{M} \|\boldsymbol{u}(t_\ell)\|_1^2 \|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2 + \delta_1 k \sum_{\ell=1}^{M} \|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2 \\
&\leq C(\delta_1, L) \max_{1 \leq \ell \leq M} \|\boldsymbol{u}(t_\ell)\|_1^2 \left(k \sum_{\ell=1}^{M} \|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2 \right) + \delta_1 k \sum_{\ell=1}^{M} \|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2 \\
&\leq C(\delta_1, L) \kappa_1 \kappa_2 + \delta_1 k \sum_{\ell=1}^{M} \|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.
\end{aligned}$$

• $NLT_4(\tilde{\boldsymbol{\sigma}}^\ell)$: By similar computations as before and using the fact that $\boldsymbol{u}(t_\ell) - \tilde{\boldsymbol{u}}^{\varepsilon,\ell} = \tilde{\boldsymbol{\epsilon}}^\ell + \tilde{\boldsymbol{\sigma}}^\ell$, we get

$$NLT_4(\tilde{\boldsymbol{\sigma}}^\ell) = |\tilde{b}(\tilde{\boldsymbol{\epsilon}}^\ell + \tilde{\boldsymbol{\sigma}}^\ell, \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \tilde{\boldsymbol{\sigma}}^\ell)| \leq |\tilde{b}(\tilde{\boldsymbol{\epsilon}}^\ell, \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \tilde{\boldsymbol{\sigma}}^\ell)| + |\tilde{b}(\tilde{\boldsymbol{\sigma}}^\ell, \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \tilde{\boldsymbol{\sigma}}^\ell)|,$$

For simplicity, let us introduce the notation

$$NLT_{4,a}(\tilde{\boldsymbol{\sigma}}^\ell) := |\tilde{b}(\tilde{\boldsymbol{\epsilon}}^\ell, \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \tilde{\boldsymbol{\sigma}}^\ell)| \tag{7.98}$$

and

$$NLT_{4,b}(\tilde{\boldsymbol{\sigma}}^\ell) := |\tilde{b}(\tilde{\boldsymbol{\sigma}}^\ell, \tilde{\boldsymbol{u}}^{\varepsilon,\ell}, \tilde{\boldsymbol{\sigma}}^\ell)|. \tag{7.99}$$

We aligned $NLT_{4,a}$ into two terms by replacing $\tilde{\boldsymbol{u}}^{\varepsilon,\ell}$ by $\tilde{\boldsymbol{u}}(t_\ell)+\tilde{\boldsymbol{\epsilon}}^\ell$. Next, we apply (2.10) and (2.11) respectively. Finally, we use the Young inequality to get

$$NLT_{4,a}(\tilde{\boldsymbol{\sigma}}^\ell) \leq |\tilde{b}(\tilde{\boldsymbol{\epsilon}}^\ell,\boldsymbol{u}(t_\ell),\tilde{\boldsymbol{\sigma}}^\ell)|+|\tilde{b}(\tilde{\boldsymbol{\epsilon}}^\ell,\tilde{\boldsymbol{\epsilon}}^\ell,\tilde{\boldsymbol{\sigma}}^\ell)|$$

$$\leq C(\delta_1,L)\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2\|\boldsymbol{u}(t_\ell)\|_1^2+C(\delta_1,L)\|\tilde{\boldsymbol{\epsilon}}^\ell\|^2\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2+\delta_1\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.$$

The term $NLT_{4,b}(\tilde{\boldsymbol{\sigma}}^\ell)$ satisfies the skew-symmetry property (see (2.7)). Therefore, using the estimate (2.11) and the Young inequality, we get

$$NLT_{4,b}(\tilde{\boldsymbol{\sigma}}^\ell) = |\tilde{b}(\tilde{\boldsymbol{\sigma}}^\ell,\tilde{\boldsymbol{\sigma}}^\ell,\tilde{\boldsymbol{u}}^{\varepsilon,\ell})| \leq C(L)\|\tilde{\boldsymbol{\sigma}}^\ell\|\|\tilde{\boldsymbol{\sigma}}^\ell\|_1\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1$$

$$\leq C(\delta_1,L)\|\tilde{\boldsymbol{\sigma}}^\ell\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2+\delta_1\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.$$

From these estimates, we obtain after an integration over the time interval $[t_{\ell-1},t_\ell]$ with respect to s the estimate

$$\int_{t_{\ell-1}}^{t_\ell} NLT_4(\tilde{\boldsymbol{\sigma}}^\ell)ds \leq C(\delta_1,L)k\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2\|\boldsymbol{u}(t_\ell)\|_1^2+C(\delta_1,L)k\|\tilde{\boldsymbol{\epsilon}}^\ell\|^2\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2$$

$$+C(\delta_1,L)k\|\boldsymbol{\sigma}^\ell\|^2\|\boldsymbol{u}^\ell\|_1^2+\delta_1 k\|\boldsymbol{\sigma}^\ell\|_1^2.$$

Then, summing up,

$$\sum_{\ell=1}^M\int_{t_{\ell-1}}^{t_\ell} NLT_4(\tilde{\boldsymbol{\sigma}}^\ell)ds \leq C(\delta_1,L)k\sum_{\ell=1}^M\left[\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2\|\boldsymbol{u}(t_\ell)\|_1^2+\|\tilde{\boldsymbol{\epsilon}}^\ell\|^2\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2+\|\tilde{\boldsymbol{\sigma}}^\ell\|^2\|\tilde{\boldsymbol{u}}^\ell\|_1^2\right]$$

$$+\delta_1 k\sum_{\ell=1}^M\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2$$

$$\leq C(\delta_1,L)\left[\kappa_1\kappa_2+\kappa_2^2+k\sum_{\ell=1}^M\|\tilde{\boldsymbol{\sigma}}^\ell\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2\right]+\delta_1 k\sum_{\ell=1}^M\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2.$$

Finally, the estimates obtained for $NLT_i(\tilde{\boldsymbol{\sigma}}^\ell), i=1\dots 4$ imply

$$\sum_{\ell=1}^M Q_\ell(\tilde{\boldsymbol{\sigma}}^\ell) \leq C(\delta_1,L,T)\left[k\sum_{\ell=1}^M\|\tilde{\boldsymbol{\sigma}}^\ell\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2+(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2)\right]$$

$$+\delta_1 k\sum_{\ell=1}^M\|\tilde{\boldsymbol{\sigma}}^\ell\|_1^2. \tag{7.100}$$

We plug (7.100) into (7.97). We fix δ_1 so that $0<\delta_1<\nu$. Since $\omega\in\Omega_{\kappa_1}\cap\Omega_{\kappa_2}\cap\Omega_{\kappa_3}$, we can find a constant $C=C(\delta_1,L,T)>0$ such that

$$\max_{1\le m\le M}\left\{\left(\tfrac{\alpha+1}{2\alpha}\right)\|\boldsymbol{\sigma}^m\|^2+\left(\tfrac{\alpha-1}{2\alpha}\right)\|\tilde{\boldsymbol{\sigma}}^m\|^2\right\}$$
$$+\left(\tfrac{\alpha-1}{2\alpha}\right)\sum_{\ell=1}^{M-1}\|\tilde{\boldsymbol{\sigma}}^\ell-\boldsymbol{\sigma}^\ell\|^2+(\nu-\delta_1)\left(k\sum_{\ell=1}^{M}\|\nabla\tilde{\boldsymbol{\sigma}}^\ell\|^2\right)$$
$$+\frac{1}{\varepsilon}\left(k\sum_{\ell=1}^{M}\|\mathrm{div}\,\tilde{\boldsymbol{\sigma}}^\ell\|^2\right)\le C\left[k\sum_{\ell=1}^{M}\|\tilde{\boldsymbol{\sigma}}^\ell\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2+(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^\eta+\varepsilon)\right].$$

Since we choose $0<\delta_1<\nu$, we have $(\nu-\delta_1)>0$. In addition, since $\omega\in\Omega_{\kappa_1}$, we apply the Gronwall's Lemma we conclude that

$$\max_{1\le m\le M}\left\{\left(\tfrac{\alpha+1}{2\alpha}\right)\|\boldsymbol{\sigma}^m\|^2+\left(\tfrac{\alpha-1}{2\alpha}\right)\|\tilde{\boldsymbol{\sigma}}^m\|^2\right\}$$
$$+k\sum_{\ell=1}^{M}\|\nabla\tilde{\boldsymbol{\sigma}}^\ell\|^2\le C(\delta_1,L,T)(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^\eta+\varepsilon)\exp(\kappa_1).$$

Remember that $\boldsymbol{P}_{\mathbb{H}}$ is stable in $\mathbb{W}^{1,2}$, thus $\|\nabla\boldsymbol{\sigma}^\ell\|\le C\|\nabla\tilde{\boldsymbol{\sigma}}^\ell\|$. □

Lemma 3.11. *Under the same assumption as in Lemma 3.10, there exists a constant $C=C(L,T,\nu)>0$ such that on $\Omega_{\kappa_1}\cap\Omega_{\kappa_2}\cap\Omega_{\kappa_3}$ the error iterates $\{\varrho^\ell:1\le\ell\le M\}$ of the pressure term in Algorithm 3.7 satisfies*

$$k\sum_{\ell=1}^{M}\|\varrho^\ell\|^2\le C(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^\eta+\varepsilon)\exp(\kappa_1). \tag{7.101}$$

Proof. We add (7.94) and (7.93) and get

$$\begin{aligned}k(\nabla\varrho^\ell,\boldsymbol{\varphi})=Q_\ell(\boldsymbol{\varphi})+R_\ell^{\boldsymbol{v}}(\boldsymbol{\varphi})+\int_{t_{\ell-1}}^{t_\ell}(\nabla(\rho(t_\ell)-\rho(s)),\boldsymbol{\varphi})ds\\-(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1},\boldsymbol{\varphi})-k(\nabla\tilde{\boldsymbol{\sigma}}^\ell,\nabla\boldsymbol{\varphi}).\end{aligned}\tag{7.102}$$

Using the inequality (4.46), we derive that

$$k^2\sum_{\ell=0}^{M}\|\varrho^\ell\|^2 \le C\sum_{\ell=0}^{M}\sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}}\frac{1}{\|\boldsymbol{\varphi}\|_1^2}[Q_\ell(\boldsymbol{\varphi})+R_\ell(\boldsymbol{\varphi})+\int_{t_{\ell-1}}^{t_\ell}(\nabla(\rho(t_\ell)-\rho(s)),\boldsymbol{\varphi})ds$$
$$-(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1},\boldsymbol{\varphi})-k(\nabla\boldsymbol{\sigma}^\ell,\nabla\boldsymbol{\varphi})]^2.$$

For simplicity let us introduce the following abbreviation

$$\widetilde{\mathtt{I}}+\widetilde{\mathtt{II}}+\widetilde{\mathtt{III}}+\widetilde{\mathtt{IV}}+\widetilde{\mathtt{V}}$$
$$:=\frac{1}{\|\boldsymbol{\varphi}\|_1}Q_\ell(\boldsymbol{\varphi})+R_\ell(\boldsymbol{\varphi})+\int_{t_{\ell-1}}^{t_\ell}(\nabla(\rho(t_\ell)-\rho(s)),\boldsymbol{\varphi})ds-(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1},\boldsymbol{\varphi})-k(\nabla\boldsymbol{\sigma}^\ell,\nabla\boldsymbol{\varphi}).$$

In the following we estimate each term of the right side.

- **Term** $\widetilde{\mathtt{I}}$: Here, we get

$$\widetilde{\mathtt{I}}\le C\int_{t_{\ell-1}}^{t_\ell}\sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}}\frac{1}{\|\boldsymbol{\varphi}\|_1}(NLT_1(\boldsymbol{\varphi})+NLT_2(\boldsymbol{\varphi})+NLT_3(\boldsymbol{\varphi})+NLT_4(\boldsymbol{\varphi}))ds,$$
$$\le C\int_{t_{\ell-1}}^{t_\ell}(\widetilde{NLT}_1+\widetilde{NLT}_2+\widetilde{NLT}_3+\widetilde{NLT}_4)ds,$$

where with (2.9) and (2.10) we arrive at

$$\widetilde{NLT}_1\le C(L)\sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}}\frac{1}{\|\boldsymbol{\varphi}\|_1}\|\boldsymbol{u}(s)\|_1\|\boldsymbol{\varphi}\|_1\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}=\|\boldsymbol{u}(s)\|_1\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4},$$
$$\widetilde{NLT}_2\le \sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}}\frac{1}{\|\boldsymbol{\varphi}\|_1}\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}\|\boldsymbol{u}(t_\ell)\|_1\|\boldsymbol{\varphi}\|_1=\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}\|\boldsymbol{u}(t_\ell)\|_1,$$
$$\widetilde{NLT}_3\le C(L)\sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}}\frac{1}{\|\boldsymbol{\varphi}\|_1}\|\boldsymbol{u}(t_\ell)\|_1\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1\|\boldsymbol{\varphi}\|_1=\|\boldsymbol{u}(t_\ell)\|_1\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1,$$
$$\widetilde{NLT}_4\le C(L)\sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}}\frac{1}{\|\boldsymbol{\varphi}\|_1}\{\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1\|\boldsymbol{u}(t_\ell)\|_1\|\boldsymbol{\varphi}\|_1+\|\tilde{\boldsymbol{\epsilon}}^\ell\|\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1\|\boldsymbol{\varphi}\|_1\},$$
$$=C(L)\{\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1\|\boldsymbol{u}(t_\ell)\|_1+\|\tilde{\boldsymbol{\epsilon}}^\ell\|\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1\}.$$

Integrating gives

$$\int_{t_{\ell-1}}^{t_\ell} \widetilde{NLT}_1 ds \leq C(L) \sup_{t_{\ell-1}\leq s\leq t_\ell} \|\boldsymbol{u}(s)\|_1 \int_{t_{\ell-1}}^{t_\ell} \|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4} ds,$$
$$\int_{t_{\ell-1}}^{t_\ell} \widetilde{NLT}_2 ds \leq \|\boldsymbol{u}(t_\ell)\|_1 \int_{t_{\ell-1}}^{t_\ell} \|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4} ds,$$
$$\int_{t_{\ell-1}}^{t_\ell} \widetilde{NLT}_3 ds \leq k\|\boldsymbol{u}(t_\ell)\|_1 \|\tilde{\boldsymbol{\epsilon}}^\ell\|_1,$$
$$\int_{t_{\ell-1}}^{t_\ell} \widetilde{NLT}_4 ds \leq C(L)k\{\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1 \|\boldsymbol{u}(t_\ell)\|_1 + \|\tilde{\boldsymbol{\epsilon}}^\ell\|\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1\}.$$

From the estimates of $\int_{t_{\ell-1}}^{t_\ell} \widetilde{NLT}_i ds$, for $i=1,\dots,4$, we obtain

$$\widetilde{\mathtt{I}}^2 \leq kC(L) \sup_{t_{\ell-1}\leq s\leq t_\ell} \|\boldsymbol{u}(s)\|_1^2 \int_{t_{\ell-1}}^{t_\ell} \|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2 ds + k\|\boldsymbol{u}(t_\ell)\|_1^2 \int_{t_{\ell-1}}^{t_\ell} \|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2 ds$$
$$+ k^2\|\boldsymbol{u}(t_\ell)\|_1^2 \|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2 + C(L)k^2\{\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2 \|\boldsymbol{u}(t_\ell)\|_1^2 + \|\tilde{\boldsymbol{\epsilon}}^\ell\|^2 \|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2\}.$$

Summing up for $\ell=1$ to $\ell=M$ gives

$$\sum_{\ell=1}^M \widetilde{\mathtt{I}}^2 \leq C(L,T)k(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2^2).$$

• **Term** $\widetilde{\mathtt{II}}$: Here, we have

$$\widetilde{\mathtt{II}}^2 \leq k\int_{t_{\ell-1}}^{t_\ell} \sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}} \nu^2 \frac{\|\nabla(\boldsymbol{v}(t_\ell)-\boldsymbol{v}(s)\|^2 \|\nabla\boldsymbol{\varphi}\|^2}{\|\boldsymbol{\varphi}\|_1^2} ds \leq Ck\int_{t_{\ell-1}}^{t_\ell} \nu^2 \|\nabla(\boldsymbol{v}(t_\ell)-\boldsymbol{v}(s)\|^2 ds.$$

Then, summing up and using Lemma 3.4 gives

$$\sum_{\ell=1}^M \widetilde{\mathtt{II}}^2 \leq C(\nu,T) \leq C(\nu)k^{\eta+1}.$$

• **Term** $\widetilde{\mathtt{III}}$: Here, we have

$$\widetilde{\mathtt{III}}^2 \leq \sup_{\boldsymbol{\varphi}\in\mathbb{W}^{1,2}} \frac{1}{\|\boldsymbol{\varphi}\|_1^2} k\int_{t_{\ell-1}}^{t_\ell} \|\rho(t_\ell)-\rho(s)\|^2 \|\boldsymbol{\varphi}\|_1^2 ds = k\int_{t_{\ell-1}}^{t_\ell} \|\rho(t_\ell)-\rho(s)\|^2 ds.$$

Again summing up and using Lemma 3.4 gives

$$\sum_{\ell=1}^{M} \widetilde{\mathtt{III}}^2 \le k \sum_{\ell=1}^{M} \int_{t_{\ell-1}}^{t_\ell} \|\rho(t_\ell) - \rho(s)\|^2 ds \le C_{T,4} k \sum_{\ell=1}^{M} k^{2\eta+1} = C_{T,4} k^{2\eta+1}.$$

• **Term** $\widetilde{\mathtt{IV}}$: Here, we proceed in two steps. First, we estimate $\widetilde{\mathtt{IV}}$ with a term under a weak norm. Then, we use the Proposition 3.1 to bound this later with terms under H^1 or L^2-norm. Thereby, we have

$$\widetilde{\mathtt{IV}} = \sup_{\boldsymbol{\varphi} \in \mathbb{W}^{1,2}} \frac{1}{\|\boldsymbol{\varphi}\|_1} (\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}, \boldsymbol{\varphi}) \le \sup_{\boldsymbol{\varphi} \in \mathbb{W}^{1,2}} \frac{1}{\|\boldsymbol{\varphi}\|_1} \|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1} \|\boldsymbol{\varphi}\|_1 \le \|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1}.$$

Next, since $\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1} \in \mathscr{D}(\boldsymbol{A}^{-1})$, we can take $\boldsymbol{\varphi} = \boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})$ in (7.102), use Proposition 3.1, and arrive at the following estimates:

i) $\|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1}^2 \le C(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}, \boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}))$,

ii) $k(\nabla \tilde{\boldsymbol{\sigma}}^\ell, \nabla \boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) \le \delta_1 \|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1}^2 + C\delta_1 k^2 \|\nabla \tilde{\boldsymbol{\sigma}}^\ell\|^2$,

iii) $k(\nabla \varrho^\ell, \boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) = \int_{t_{\ell-1}}^{t_\ell} (\nabla(\rho(t_\ell) - \rho(s)), \boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) ds = 0$,

iv) $R_\ell^{\boldsymbol{v}}(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) \le C(\nu, \delta_1) k^{\eta+2} + \delta_1 \|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1}^2$.

We aligned the term $Q_\ell(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}))$ as follows

$$\begin{aligned} Q_\ell(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) \le \int_{t_{\ell-1}}^{t_\ell} & NLT_1(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) + NLT_2(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) \\ & + NLT_3(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) + NLT_4(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) ds, \end{aligned}$$

where each of terms $NLT_j(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}))$ for $j=1,2,3,4$, are estimated as follows:

$$NLT_1(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}))\le C_{\delta_1}(L)k\|\boldsymbol{u}(s)\|_1^2\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2+\frac{\delta_1}{k}\|\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1})\|_1^2,$$

$$NLT_2(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}))\le C_{\delta_1}k\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2\|\boldsymbol{u}(t_\ell)\|_1^2+\frac{\delta_1}{k}\|\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1})\|_1^2,$$

$$NLT_3(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}))\le C(\delta_1,L)k\|\boldsymbol{u}(t_\ell)\|_1^2\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2+\frac{\delta_1}{k}\|\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1})\|_1^2,$$

$$\begin{aligned}NLT_4(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}))\le C(\delta_1,L)k&\left\{\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2\|\boldsymbol{u}(t_\ell)\|_1^2+\|\tilde{\boldsymbol{\epsilon}}^\ell\|^2\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2+\|\tilde{\boldsymbol{\sigma}}^\ell\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2\right\}\\&+\frac{2\delta_1}{k}\|\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1})\|_1^2.\end{aligned}$$

All together, the estimates of $NLT_i(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}))$, for $i=1,\dots,4$, lead to

$$\begin{aligned}Q_\ell(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}))\le{}& C(\delta_1,L)k\int_{t_{\ell-1}}^{t_\ell}\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2\left[\|\boldsymbol{u}(s)\|_1^2+\|\boldsymbol{u}(t_\ell)\|_1^2\right]ds\\&+C(\delta_1,L)k^2\left\{\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2\|\boldsymbol{u}(t_\ell)\|_1^2+\|\tilde{\boldsymbol{\epsilon}}^\ell\|^2\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2+\|\tilde{\boldsymbol{\sigma}}^\ell\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2\right\}\\&+C(\delta_1,L)k^2\|\boldsymbol{u}(t_\ell)\|_1^2\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2+4\delta_1\|\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1})\|_1^2.\end{aligned}$$

In addition on Ω_{κ_3} we have

$$\begin{aligned}Q_\ell(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}))\le{}& C(\delta_1,L)(\sup_{t_{\ell-1}\le s\le t_\ell}\|\boldsymbol{u}(s)\|_1^2+C(\delta_1)\|\boldsymbol{u}(t_\ell)\|_1^2)\kappa_3k^{2\eta+2}\\&+C(\delta_1,L)k^2\left\{\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2\|\boldsymbol{u}(t_\ell)\|_1^2+\|\tilde{\boldsymbol{\epsilon}}^\ell\|^2\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2+\|\tilde{\boldsymbol{\sigma}}^\ell\|^2\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2\right\}\\&+C(\delta_1,L)k^2\|\boldsymbol{u}(t_\ell)\|_1^2\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2+4\delta_1\|\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}\|_{-1}^2.\end{aligned}$$

Now summing for $\ell=1$ to $\ell=M$, we have

$$\begin{aligned}\sum_{\ell=1}^M Q_\ell(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}))\le{}& C(\delta_1,L)(\sup_{0\le s\le T}\|\boldsymbol{u}(s)\|_1^2+\max_{1\le\ell\le M}\|\boldsymbol{u}(t_\ell)\|_1^2)\kappa_3k^{2\eta+2}\\&+C(\delta_1,L)k\max_{1\le\ell\le M}\|\boldsymbol{u}(t_\ell)\|_1^2\left(k\sum_{\ell=1}^M\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2\right)+k\max_{1\le\ell\le M}\|\tilde{\boldsymbol{\epsilon}}^\ell\|^2\left(k\sum_{\ell=1}^M\|\tilde{\boldsymbol{\epsilon}}^\ell\|_1^2\right)\\&+k\max_{1\le\ell\le M}\|\tilde{\boldsymbol{\sigma}}^\ell\|^2\left(k\sum_{\ell=1}^M\|\tilde{\boldsymbol{u}}^{\varepsilon,\ell}\|_1^2\right)+4\delta_1\sum_{\ell=1}^M\|\boldsymbol{\sigma}^\ell-\boldsymbol{\sigma}^{\ell-1}\|_{-1}^2.\end{aligned}$$

Since we have due to the assumptions $\omega \in \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}$ we obtain using (7.96)

$$\sum_{\ell=1}^{M} Q_\ell(\boldsymbol{A}^{-1}(\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1})) \leq C(\delta_1, L)k\left(\kappa_1\kappa_3 k^{2\eta+1} + \kappa_1\kappa_2 + \kappa_2^2\right) + 4\delta_1 \sum_{\ell=1}^{M} \|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1}^2 + C(\delta_1, L, T)k(\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2 + k^\eta + \varepsilon)\exp(\kappa_1).$$

All together we obtain,

$$\sum_{\ell=1}^{M} \|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1}^2 \leq 6\delta_1 \sum_{\ell=1}^{M} \|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1}^2 + C(\delta_1)k^2 \sum_{\ell=1}^{M} \|\nabla \tilde{\boldsymbol{\sigma}}^\ell\|^2 + C(\nu, \delta_1)k^{\eta+1} + C(\delta_1, L, T)k(\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2 + k^\eta + \varepsilon)\exp(\kappa_1).$$

The terms with $\|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1}^2$ are absorbed by the left hand side. Thanks to (7.96),

$$(1 - 6\delta_1) \sum_{\ell=1}^{M} \|\boldsymbol{\sigma}^\ell - \boldsymbol{\sigma}^{\ell-1}\|_{-1}^2 \leq C(\delta_1, L, T)k(\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2 + k^\eta + \varepsilon)\exp(\kappa_1).$$

We can choose δ_1 so that $(1 - 6\delta_1) > 0$. Note that $1 \leq \exp(x)$ for all $x \in \mathbb{R}$. Therefore,

$$\sum_{\ell=1}^{M} \widetilde{\mathtt{IV}}^2 \leq C(L, T)k(\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2 + k^\eta + \varepsilon)\exp(\kappa_1).$$

- **Term** $\widetilde{\mathtt{V}}$: Here, we have

$$\widetilde{\mathtt{V}} = \sup_{\boldsymbol{\varphi}_\ell \in \mathbb{W}^{1,2}} \frac{1}{\|\boldsymbol{\varphi}\|_1} k(\nabla \tilde{\boldsymbol{\sigma}}^\ell, \nabla \boldsymbol{\varphi}) \leq \sup_{\boldsymbol{\varphi} \in \mathbb{W}^{1,2}} \frac{1}{\|\boldsymbol{\varphi}\|_1} k\|\nabla \tilde{\boldsymbol{\sigma}}^\ell\| \|\nabla \boldsymbol{\varphi}\| = Ck\|\nabla \tilde{\boldsymbol{\sigma}}^\ell\|.$$

Summing up and using (7.96) gives

$$\sum_{\ell=1}^{M} \widetilde{\mathtt{V}}^2 \leq Ck^2 \sum_{\ell=1}^{M} \|\nabla \tilde{\boldsymbol{\sigma}}^\ell\|^2 \leq C(\delta_1, L, T)k(\kappa_1\kappa_3 k^{2\eta} + \kappa_1\kappa_2 + \kappa_2^2 + k^\eta + \varepsilon)\exp(\kappa_1).$$

Collecting $\widetilde{\mathtt{I}}, \mathtt{II}, \widetilde{\mathtt{III}}, \widetilde{\mathtt{IV}}$, and $\widetilde{\mathtt{V}}$, we obtain

$$
\begin{aligned}
k^2\sum_{\ell=0}^{M}\|\varrho^\ell\|^2 \leq \sum_{\ell=0}^{M}\{\widetilde{\mathtt{I}}+\widetilde{\mathtt{II}}+\widetilde{\mathtt{III}}+\widetilde{\mathtt{IV}}+\widetilde{\mathtt{V}}\}^2,&\\
\leq C(L,T)k(\kappa_1\kappa_2+\kappa_1\kappa_3+\kappa_2^2)+C(\nu,T)k^{\eta+1}+C_{T,4}k^{2\eta+1}&\\
+C(L,T)k(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^\eta+\varepsilon)\exp(\kappa_1)&\\
+C(L,T)k(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^\eta+\varepsilon)\exp(\kappa_1).&
\end{aligned}
$$

Because $1<\exp(x)$ for all $x\in\mathbb{R}$ and with a limiting order term (k^η), we have

$$
k\sum_{\ell=0}^{M}\|\varrho^\ell\|^2 \leq C(L,T,\nu)(\kappa_1\kappa_3k^{2\eta}+\kappa_1\kappa_2+\kappa_2^2+k^\eta+\varepsilon)\exp(\kappa_1).
$$

□

3.8 Main results

Let us define the errors $\mathbf{e}^\ell=\boldsymbol{u}(t_\ell)-\boldsymbol{u}^{\varepsilon,\ell}$ and $\mathsf{q}^\ell=\mathsf{p}(t_\ell)-\mathsf{p}^{\varepsilon,\ell}$. Here in the final section, we use the estimates of the iterates $\{\boldsymbol{\epsilon}^\ell,\varpi^\ell\}_\ell$ and $\{\boldsymbol{\sigma}^\ell,\varrho^\ell\}_\ell$ to derive an estimate for $\{\mathbf{e}^\ell,\mathsf{q}^\ell\}_\ell$, show convergence in probability of Algorithm 3.4, and deduce from that strong convergence.

We set

$$
\mathcal{E}^M := \max_{1\leq m\leq M}\|\mathbf{e}^m\|^2+\nu k\sum_{\ell=1}^{M}\|\nabla\mathbf{e}^\ell\|^2+k\sum_{\ell=1}^{M}\|\mathsf{q}^\ell\|^2, \tag{8.103}
$$

$$
\widetilde{\mathcal{E}}^M := \max_{1\leq m\leq M}\|\mathbf{e}^m\|^2+\left(\nu k\sum_{\ell=1}^{M}\|\nabla\mathbf{e}^\ell\|^2\right)^{1/2}+\left(k\sum_{\ell=1}^{M}\|\mathsf{q}^\ell\|^2\right)^{1/2}, \tag{8.104}
$$

$$
\mathcal{E}_1^M := \max_{1\leq m\leq M}\|\boldsymbol{\epsilon}^m\|^2+\nu k\sum_{\ell=1}^{M}\|\nabla\boldsymbol{\epsilon}^\ell\|^2+k\sum_{\ell=1}^{M}\|\varpi^\ell\|^2, \tag{8.105}
$$

$$
\mathcal{E}_2^M := \max_{1\leq m\leq M}\|\boldsymbol{\sigma}^m\|^2+\nu k\sum_{\ell=1}^{M}\|\nabla\boldsymbol{\sigma}^\ell\|^2+k\sum_{\ell=1}^{M}\|\varrho^\ell\|^2. \tag{8.106}
$$

Theorem 3.12. *Let $\mathcal{E}^M$ be defined in (8.103). If $\varepsilon = \kappa^\eta$, the Algorithm 3.4 converges in probability with order $0 < r < \eta$. In particular, we have*

$$\lim_{\widetilde{C}\to\infty} \lim_{k\to 0} \mathbb{P}\left[\mathcal{E}^M \geq \widetilde{C} k^{\mathrm{r}}\right] = 0.$$

Proof. Let $\widetilde{C}, r > 0$ be some arbitrary constants which will be fixed at the end of the proof. By the Chebyshev inequality

$$\begin{aligned}
\mathbb{P}\left[\mathcal{E}^M \geq \widetilde{C} k^r\right] &\leq \mathbb{P}(\Omega \setminus \Omega_{\kappa_1}) + \mathbb{P}(\Omega \setminus \Omega_{\kappa_2}) + \mathbb{P}(\Omega \setminus \Omega_{\kappa_3}) + \mathbb{P}\left[\mathcal{E}^M \geq \widetilde{C} k^r \middle| \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}\right] \\
&\leq \frac{1}{\kappa_1} \mathbb{E}\left[\sup_{0\leq s\leq T} \|\boldsymbol{u}(s)\|_{\mathbb{V}}^2 + \nu k \sum_{\ell=1}^{M} \|\boldsymbol{u}^\ell\|_1^2\right] + \frac{1}{\kappa_2} \mathbb{E}\left[\max_{1\leq \ell\leq M} \|\boldsymbol{\epsilon}^\ell\|^2 + \nu k \sum_{\ell=1}^{M} \|\boldsymbol{\epsilon}^\ell\|_1^2 + k \sum_{\ell=1}^{M} \|\varpi^\ell\|^2\right] \\
&\quad + \frac{1}{\kappa_3 |t-s|^{2\eta}} \mathbb{E}\left[\|\boldsymbol{u}(s) - \boldsymbol{u}(t)\|_{\mathbb{L}^4}^2\right] + \frac{\mathbb{E}\left[\mathcal{E}^M \middle| \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}\right]}{\widetilde{C} k^r}.
\end{aligned}$$

Observe, that we can write $\mathbf{e}^\ell = \boldsymbol{\epsilon}^\ell + \boldsymbol{\sigma}^\ell$ and $\mathsf{q}^\ell = \varpi^\ell + \varrho^\ell$. Now, it follows by the definition of Ω_{κ_2} (see (7.95)), by Lemma 3.10, and Lemma 3.11

$$\begin{aligned}
\mathbb{E}\left[\mathcal{E}^M \middle| \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}\right] &\leq \mathbb{E}\left[\mathcal{E}_1^M \middle| \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}\right] + \mathbb{E}\left[\mathcal{E}_2^M \middle| \Omega_{\kappa_1} \cap \Omega_{\kappa_2} \cap \Omega_{\kappa_3}\right] \\
&\leq \kappa_2 + C(\kappa_1 \kappa_3 k^{2\eta} + \kappa_1 \kappa_2 + \kappa_2^2 + k^\eta + \varepsilon) \exp(\kappa_1),
\end{aligned}$$

where $\mathcal{E}_1^M$ and $\mathcal{E}_2^M$ are defined by (8.105) and (8.106) respectively. Moreover, by estimate (2.14), Lemma 3.6, and Lemma 3.8 we obtain

$$\begin{aligned}
\mathbb{P}\left[\mathcal{E}^M \geq \widetilde{C} k^r\right] &\leq \frac{\kappa_2 + C(\kappa_1 \kappa_3 k^{2\eta} + \kappa_1 \kappa_2 + \kappa_2^2 + k^\eta + \varepsilon) \exp(\kappa_1)}{\widetilde{C} k^r} + \frac{C}{\kappa_1} + \frac{C(k^\eta + \varepsilon)}{\kappa_2} + \frac{C}{\kappa_3} \\
&\leq \frac{C(\kappa_2 + \kappa_3 k^{2\eta} + \kappa_2^2 + k^\eta + \varepsilon) \exp(2\kappa_1)}{\widetilde{C} k^r} + \frac{C}{\kappa_1} + \frac{C(k^\eta + \varepsilon)}{\kappa_2} + \frac{C}{\kappa_3}.
\end{aligned}$$

Let $\mu > 0$. We fix $\varepsilon = k^\eta$, $\kappa_1 = \ln k^{-\mu/2}$, $\kappa_2 = k^{\mu + r}$, and $\kappa_3 = k^{-\eta}$. Therefore, we have

$$\mathbb{P}\left[\mathcal{E}^M \geq \widetilde{C} k^r\right] \leq \frac{C(k^r + k^{\eta - \mu})}{\widetilde{C} k^r} - \frac{C}{\ln k^\mu} + C k^{\eta - \mu - r} + C k^\eta.$$

Let us remind, that we fixed the constant r in the beginning, such that $\eta-\mu-r>0$. Now, we are ready to go to the limit:

$$\lim_{\widetilde{C}\to\infty}\lim_{k\to 0}\mathbb{P}\left[\mathcal{E}^M\geq\widetilde{C}k^r\right]\leq\lim_{\widetilde{C}\to\infty}\lim_{k\to 0}\left(\frac{C}{\widetilde{C}}-\frac{C}{\ln k^\mu}+Ck^{\eta-\mu-r}+Ck^\eta\right)=\lim_{\widetilde{C}\to\infty}\frac{C}{\widetilde{C}}=0.$$

This gives the assertion. □

A consequence of this theorem is strong convergence of iterates of the scheme. This will be shown by the following corollary.

Corollary 3.13. *Let $\widetilde{\mathcal{E}}^M$ be defined as in* (8.104)*. Under the assumption of Theorem 3.12 we have*

$$\lim_{M\to\infty}\mathbb{E}\left[\widetilde{\mathcal{E}}^M\right]=0.$$

Proof. Let $\widetilde{C}>0$ an arbitrary constant. We define the sample set

$$\Omega_{\widetilde{C},k}:=\left\{\mathcal{E}^M\geq\widetilde{C}k^{\mathrm{r}}\right\}.$$

From the law of total probability we deduce that

$$\mathbb{E}\left[\widetilde{\mathcal{E}}^M\right]=\mathbb{E}\left[\widetilde{\mathcal{E}}^M\middle|\Omega_{\widetilde{C},k}\right]\mathbb{P}(\Omega_{\widetilde{C},k})+\mathbb{E}\left[\widetilde{\mathcal{E}}^M\middle|\Omega\setminus\Omega_{\widetilde{C},k}\right]\mathbb{P}(\Omega\setminus\Omega_{\widetilde{C},k}).$$

Since $\mathbb{P}(\Omega\setminus\Omega_{\widetilde{C},k})\leq 1$, and by definition of $\Omega_{\widetilde{C},k}$,

$$\mathbb{E}\left[\widetilde{\mathcal{E}}^M\right]\leq\mathbb{E}\left[\widetilde{\mathcal{E}}^M\middle|\Omega_{\widetilde{C},k}\right]\mathbb{P}(\Omega_{\widetilde{C},k})+\widetilde{C}k^{r/2}.$$

Using the definition of conditional expectation and the Cauchy–Schwartz inequality we obtain

$$\mathbb{E}\left[\widetilde{\mathcal{E}}^M\middle|\Omega_{\widetilde{C},k}\right]\mathbb{P}(\Omega_{\widetilde{C},k})\leq\mathbb{E}\left[\left(\widetilde{\mathcal{E}}^M\right)^2\right]\left(\mathbb{P}(\Omega_{\widetilde{C},k})\right)^{1/2}.$$

Remember that $\mathbf{e}^\ell = \boldsymbol{u}(t_\ell) - \boldsymbol{u}^{\varepsilon,\ell}$ and $\mathbf{e}^\ell = \mathsf{p}(t_\ell) - \mathsf{p}^{\varepsilon,\ell}$. Using now Equation (2.12), Lemma 3.6(iii), Equation (2.13), Proposition 3.3, and Lemma 3.7, we arrive at

$$\mathbb{E}\left[\left(\widetilde{\mathcal{E}}^M\right)^2\right] \le \mathbb{E}\left[\max_{1\le m\le M}\|\boldsymbol{u}(t_m)\|^4\right] + \mathbb{E}\left[\max_{1\le m\le M}\|\boldsymbol{u}^{\varepsilon,m}\|^4\right] + \mathbb{E}\left(\nu k\sum_{\ell=1}^{M}\|\nabla\boldsymbol{u}(t_\ell)\|^2\right)$$
$$+\mathbb{E}\left(\nu k\sum_{\ell=1}^{M}\|\nabla\boldsymbol{u}^{\varepsilon,\ell}\|^2\right) + \mathbb{E}\left(k\sum_{\ell=1}^{M}\|\mathsf{p}(t_\ell)\|^2\right) + \mathbb{E}\left(k\sum_{\ell=1}^{M}\|\mathsf{p}^{\varepsilon,\ell}\|^2\right) \le C(T,L,\boldsymbol{u}^0,\nu).$$

Consequently, we get

$$\mathbb{E}\left[\widetilde{\mathcal{E}}^M\right] \le C(T,L,\boldsymbol{u}^0,\nu)\left(\mathbb{P}(\Omega_{\widetilde{C},k})\right)^{1/2} + \widetilde{C}k^{r/2}.$$

Now we fix $\widetilde{C} = k^{-r/4}$ from the beginning and define $\widetilde{\Omega}_M := \Omega_{M^{r/4},M^{-1}}$. To conclude, we take the limit for $M\to\infty$ and apply Theorem 3.12,

$$\lim_{M\to\infty}\mathbb{E}\left[\widetilde{\mathcal{E}}^M\right] \le C(T,L,\boldsymbol{u}^0,\nu)\left(\lim_{M\to\infty}\mathbb{P}(\widetilde{\Omega}_M)\right)^{1/2} + \lim_{M\to\infty}\frac{1}{M^{r/4}} = 0.$$

This gives the assertion. □

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