Chair of Mathematics, Statistics and Geometry

Doctoral Thesis

Tilings related to Number Systems and Substitutions

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AFFIDAVIT

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Abstract

The present doctoral thesis contains results that can be grouped into two areas. The first one corresponds to digit systems and tilings. Lagarias and Wang [52] considered *integral self-affine tiles* $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ related to an expanding matrix $A \in \mathbb{Z}^{n\times n}$ and a digit set $\mathcal{D}$ satisfying $|\mathcal{D}| = |\det A|$; these tiles are defined as attractors of an iterated function system, and in general, have intricate shapes and a fractal boundary. They proved, under the assumption that $\mathcal{F}$ has positive measure, the existence of a tiling and a multiple tiling of $\mathbb{R}^n$. Steiner and Thuswaldner [78] introduced *rational self-affine tiles* associated with expanding algebraic numbers or, equivalently, expanding rational matrices $A \in \mathbb{Q}^{n\times n}$ with irreducible characteristic polynomials. They considered only a particular type of digit set, where $\mathcal{D}$ is obtained as a complete residue set modulo the base A: this is a strong assumption because it guarantees the existence of a tile of positive measure. The challenge of this theory is that rational self-affine tiles are not subsets of $\mathbb{R}^n$, and instead are defined in a representation space $\mathbb{K}_A$ which is a subring of a certain Adèle ring (in the case $n = 1$, a base $\frac{a}{b}$ induces a space of the form $\mathbb{R} \times \mathbb{Q}_b$). A central part of our work consisted in proving analogous results to the ones by Lagarias and Wang, in the setting of Steiner and Thuswaldner: we considered $A \in \mathbb{Q}^{n\times n}$ (without the irreducibility restriction) and a set of digits $\mathcal{D}$ (not necessarily a residue set) which induced a rational self-affine tile $\mathcal{F}(A, \mathcal{D})$, and proved topological properties and the existence of a tiling and a lattice multiple tiling. The representation space in this more general case is defined in terms of a projective limit. On top of expanding the existing theory with new results, we made a thorough survey of the mathematical tools necessary to understand it, computed many specific examples in detail, and present multiple illustrations.

The second area covered in this thesis is word combinatorics, in particular, we considered a family of infinite words over a two-letter alphabet that we called *N-continued fraction sequences*, obtained in terms of symbolic substitutions. They are related to N-continued fraction expansions of real numbers and constitute a generalization of Sturmian sequences. We prove combinatorial results of these sequences: in particular, we give balance constants and compute their complexity function, as well as other dynamical results.

As a bridge between the two areas covered, at the end of the thesis we relate N-continued fraction sequences to Rauzy fractals, which are also self-affine sets. For non-unimodular substitutions, the representation space for the corresponding Rauzy fractals has a factor defined in terms of a projective limit which is analogous to the one used for rational self-affine tiles.

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Contents

Chapter 1

Introduction

1.1 Number systems

Number systems are, roughly speaking, a way of representing the elements of a space using strings of digits in relation to a base. They are a way to symbolically depict numbers. They are also called numeral systems, numeration systems, or digit systems. The most commonly used are α-ary number systems where the base is an integer α with $|\alpha| > 1$. Every real number has an α-ary expansion

$$\pm(d_m d_{m-1} \ldots d_0.d_{-1}d_{-2}\ldots)_\alpha := \pm \sum_{j=-\infty}^{m} d_j \alpha^j$$

where every d_j is a *digit* taken from the set $\{0, 1, \ldots, |\alpha| - 1\}$ (the sign "$\pm$" is required only for positive α). A desirable property of a number system is that almost all numbers (in a measure-theoretic sense) can be expanded in a unique way. For example, in the decimal expansion, the number 4 can also be written as $3.9999\ldots$. However, the set of numbers admitting more than one decimal expansion is very small in the sense that it has Lebesgue measure zero. This fact is reflected in the following *tiling property*. Consider the set of *fractional parts* in the decimal number system, that is, the set of numbers that can be expanded using only negative powers of 10: this set is equal to the unit interval $[0, 1]$. On the other hand, the set of numbers whose expansion uses only non-negative powers of 10, that is, numbers with an *integer expansion*, is equal to $\mathbb{Z}$ (if we permit the use of the minus sign). Consider the collection

$$\{[0, 1] + z \mid z \in \mathbb{Z}\}. \tag{1.1}$$

This collection covers $\mathbb{R}$, and the only overlaps occur at the boundary points of the intervals. We thus say that $\{[0, 1] + z \mid z \in \mathbb{Z}\}$ forms a *tiling* of the real line. Here $[0, 1]$ is the *central tile* and $\mathbb{Z}$ is the *translation set*. This tiling property is a geometric interpretation of the fact that almost all real numbers admit a unique expansion in the decimal system, and it works the same for any other α-ary number systems.

Over time, many generalizations of the classical number systems came to the fore. To cite some examples, back in 1885 Grünwald [36] studied number systems with negative integers as bases. In 1936 Kempner [42] and later also Rényi [71] proposed expansions with respect to non-integral real bases (see also [76]). Knuth [46] introduced complex bases and, as we illustrate below, dealt with their relations to fractal sets. Gordon [34] discussed the relevance of number systems with varying digit sets in cryptography, and in [55] binary and hexadecimal expansions were used in this context. The β-expansion introduced by Parry [66] is defined in terms of a base $\beta > 1$ using the greedy algorithm. Kovács [48] considered expansions in base $A \in \mathbb{Z}^{n\times n}$ with digits $\mathcal{D} \subset \mathbb{Z}^n$ satisfying $|\mathcal{D}| = |\det(A)|$, and looked at ways of expanding a vector $x \in \mathbb{R}^n$ in the form

$$x = \sum_{j=-\infty}^{k} A^j d_j, \qquad d_j \in \mathcal{D}. \tag{1.2}$$

A nice example of a representation not obtained using a base's powers is the Fibonacci coding of a positive integer. It is a binary digit expansion that is based on Zeckendorf's theorem, which states that every positive integer can be expressed uniquely as the sum of distinct Fibonacci numbers, provided that the sum does not include any two consecutive Fibonacci numbers. The Fibonacci coding is a sequence of zeroes and ones that ends with "11" and contains no other instances of "11" before the end. It is related to the golden ratio base expansion, which is an example of an irrational base number system, and in particular, is an example of an Ostrowski numeration.

Akiyama et al. [3] studied representations of numbers in a rational base $\frac{p}{q}$ where $p > q \geq 2$ are coprime integers. These representations are obtained through an algorithm that produces less significant digits first and is different from the greedy algorithm. In particular, they obtain unique expansion for non-negative integers. They are related to the rational base number systems that we consider in Chapter 2.

A large portion of this thesis is devoted to the study of the relation between number systems, fractal sets, and tilings. Perhaps one of the first examples of this relation that people with a mathematical background encounter is the ternary Cantor set: it corresponds to the points of the unit interval that can be written in base 3 without using the digit 1. In other ways, the ternary Cantor set corresponds to the set of *fractional parts* of the ternary number system with digit set $\{0, 2\}$. A lot is known about this set: it is compact, as is shown in almost any introductory topology course; it is self-affine: a ternary Cantor set is made up of two smaller ternary Cantor sets, and it has measure zero. The fact that it has measure zero reveals a crucial fact about the underlying number system: when using base 3, the digits 0 and 2 alone are not enough to represent all real numbers. Full intervals of numbers cannot be written with these two digits only. This shows how the geometric properties of a set can shed light on the underlying algebraic properties. Measure zero sets cannot tile the real line: there is no way to cover $\mathbb{R}$ with copies of the ternary Cantor set translated via a discrete set. Moreover, a number system could also "fail" in the other direction: it could have too many digits, and one same number may have (infinitely) many ways to be represented; redundancy is also not desirable. This would geometrically manifest as a set that is the union of smaller copies of itself, but these smaller sets have big overlaps. We will see an example of this phenomenon soon.

This raises the question: given a fixed integer base, for which digit sets $\mathcal{D}$ are (almost) all real numbers expressed uniquely? For prime numbers, the answer is simple: this can only be achieved when $\mathcal{D}$ is a complete set of residues modulo the base (and note that we may also need a minus sign at the beginning). The problem of characterizing these sets $\mathcal{D}$ is difficult even for integer bases and has been studied by Ka-Sing Lau and Rao Hui [57].

Number systems with integer bases can be generalized pretty well by using algebraic integers. Recall that an *algebraic integer* is a complex number that is a zero of a monic polynomial with integer coefficients. Consider, for example, $\alpha = -1 + i$, which is a zero of the monic polynomial $P(X) = X^2 + 2X + 2$. Every complex number can be expressed in this base using the digits 0 and 1 because they constitute a residue set of $\mathbb{Z}[\alpha] \mod \alpha$. Associated with this number system is a famous fractal set known as *Knuth's Twin Dragon*, depicted in Figure 1.1 (see [47, p. 206]). It is defined as the set of fractional parts of this number system, that is, the set of points $x = \sum_{j=1}^{\infty}(-1+i)^{-j}d_{-j}$ with $d_{-j} \in \{0, 1\}$. This set plays the same role as the interval $[0, 1]$ in the decimal system.

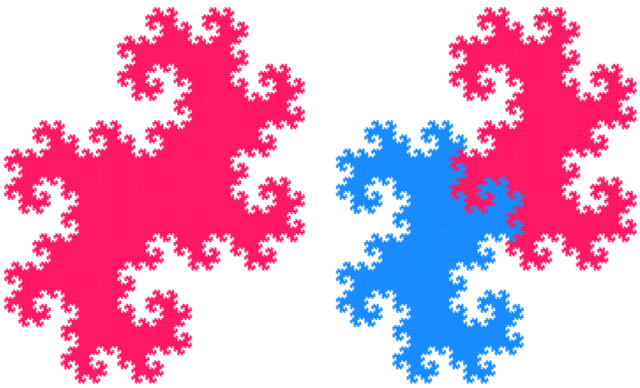


FIGURE 1.1: Knuth's Twin Dragon, a set related to a number system with base $-1 + i$.

1.2 Self-affine tiles

Given a complete metric space K, a family of contraction mappings $f_i : K \to K,\ i = 1, \ldots, N$ is called and *iterated function system* (IFS for short). The Hutchinson operator is a map H defined on the compact subsets of K as

$$H(S) := \bigcup_{i=1}^{N} f_i(S)$$

Hutchinson [39] showed that there is a unique non-empty compact set $S \subset K$ that satisfies $H(S) = S$, that is, S is a fixed point of the operator. The set S is called the *attractor* of the IFS, and the existence and uniqueness are a consequence of the contraction mapping principle. It can be obtained by starting from a compact set S_0 and letting $S_{n+1} := H(S_n)$ for $n \geq 0$. Then $S = \lim_{n\to\infty} S_n$. We will use this result to define self-affine tiles. We follow [82].

Definition 1.2.1 (Digit system)**.** *Let $A \in \mathbb{R}^{n\times n}$ be an expanding matrix, that is, one such that every eigenvalue λ satisfies $|\lambda| > 1$. Suppose that its determinant is an integer $|\det(A)| = m \in \mathbb{Z}$ with $m > 1$. Consider a set of vectors $\mathcal{D} = \{d_1, \ldots, d_m\} \subset \mathbb{R}^n$. We call $(A, \mathcal{D})$ a digit system.*

We will mostly use the term *digit system* when the base is a matrix, and *number system* when the base is a number.

Definition 1.2.2 (Self-affine tile)**.** *Let $(A, \mathcal{D})$ be a digit system. A self-affine tile $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ is a compact subset of $\mathbb{R}^n$ with non-empty interior satisfying the set equation*

$$A\mathcal{F} = \bigcup_{d\in\mathcal{D}} (\mathcal{F} + d). \tag{1.3}$$

This is a particular instance of an *iterated function system*, because the maps $x \mapsto A^{-1}(x + d)$ for $d \in \mathcal{D}$ are contractions in $\mathbb{R}^n$, therefore the existence and uniqueness of the attractor $\mathcal{F}$ is guaranteed. The set $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ is also known as the *central tile* of the digit system. It can be interpreted as the set of fractional parts (see [51]) because it is given explicitly by

$$\mathcal{F}(A, \mathcal{D}) = \Big\{ \sum_{j=1}^{\infty} A^{-j} d_{-j}, \qquad d_{-j} \in \mathcal{D} \Big\},$$

hence the use of the letter $\mathcal{F}$. Note that the assumption that A is expanding is necessary and sufficient for this series to converge. For most digit systems $(A, \mathcal{D})$, the set of fractional parts $\mathcal{F}(A, \mathcal{D})$ has Lebesgue measure zero, so $\mathcal{F}$ is not called a self-affine tile in this case. A necessary condition for $\mathcal{F}$ to have positive measure is that $|\mathcal{D}| = |\det(A)|$, and that is why we enforce this condition; however the converse is not true. If $\mathcal{F}$ has positive measure, it is easy to see that the union on the right of the set equation 1.3 is pairwise essentially disjoint, that is, disjoint outside of a measure zero set. For that reason, the dilated set $A\mathcal{F}$ can be seen as a union of copies of $\mathcal{F}$ and that is why it is called self-affine. Equivalently, $\mathcal{F}$ can be expressed as an essentially disjoint union of contracted copies of itself.

Definition 1.2.3 (Tiling)**.** *Consider a compact set $T \subset \mathbb{R}^n$ of positive measure, known as a tile. Let $\mathcal{S} \subset \mathbb{R}^n$ and consider the collection $\mathcal{C} = \{T + s \mid s \in \mathcal{S}\}$. Then $\mathcal{C}$ is said to be a tiling of $\mathbb{R}^n$ if $\mathbb{R}^n = \bigcup_{s\in\mathcal{S}} T + s$, and, for any $s \neq s'$ in $\mathcal{S}$, the sets $T + s$ and $T + s'$ have disjoint interiors. The set $\mathcal{S}$ is called a translation set. We say that T tiles $\mathbb{R}^n$.*

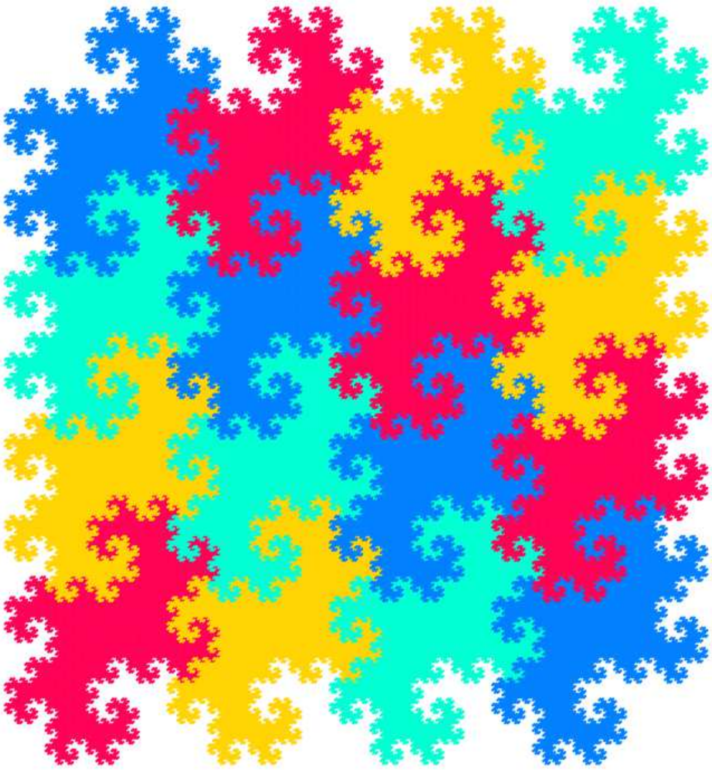


FIGURE 1.2: A tiling of the plane by the Twin Dragon.

When $\mathcal{S}$ is a lattice, we call it a *lattice tiling*. Lattice tilings are particularly interesting because the translation set is a group and hence has an easy structure. The following theorem can be found in [53, Theorem 1.2]:

Theorem 1.2.4. *Let $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ be a self-affine tile for some digit system $(A, \mathcal{D})$ in $\mathbb{R}^n$. Then $\mathcal{F}$ tiles $\mathbb{R}^n$.*

The translation set for the tiling given by $\mathcal{F}(A, \mathcal{D})$ is related (but not always exactly equal to) the set of integer expansions of the digit system.

Definition 1.2.5 (Integer expansions)**.** *Given a digit system $(A, \mathcal{D})$, we define the set of integer expansions as*

$$\mathcal{D}[A] := \Big\{ \sum_{j=0}^{k} A^j d_j, \qquad k \geq 0,\, d_j \in \mathcal{D} \Big\}.$$

Note that, even though the definitions so far are set in $\mathbb{R}^n$, this scenario also includes complex bases, because multiplication by a complex number can be seen as the action of a matrix in $\mathbb{R}^2$. In particular, the twin dragon from Figure 1.1 corresponds to the self-affine tile $\mathcal{F}(A, \mathcal{D})$ where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

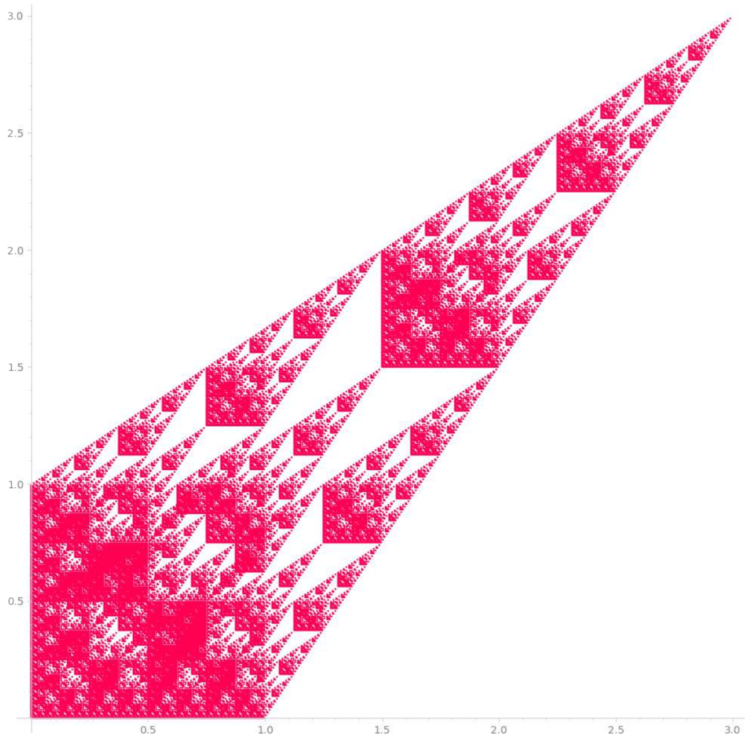


FIGURE 1.3: A self-affine tile in $\mathbb{R}^2$

Note that $\det(A) = 2$, and $-1+i$ is an eigenvalue of A whose complex norm is given by $N(-1+i) = 2$. In this way, self-affine tiles can be defined for complex numbers α whose norm is an integer, by taking the companion matrix of their minimal polynomial. This is the case for algebraic integers. A tiling of $\mathbb{R}^2$ (or $\mathbb{C}$) given by the twin dragon is illustrated in Figure 1.2. The translation set for this tiling corresponds to $\mathbb{Z}^2$ (or $\mathbb{Z}[i]$).

We mention that we refer to some sets, such as the Twin Dragon, as fractals, even though this is not strictly correct. There are two features that we have in mind when informally talking about a fractal set: its self-affine structure and the intricate local shape of its boundary. In the most commonly used definition, attributed to Mandelbrot, a set is said to be a fractal if its Hausdorff dimension is not an integer. Self-affine tiles give tilings of $\mathbb{R}^n$, which means that their Hausdorff dimension is n. However, in many cases, the boundary of a self-affine tile is a fractal set. For example, Wang shows in [82] that the boundary of the Twin Dragon has dimension $2\log_2 \lambda_0$ where λ_0 is the largest root of $X^3 - X^2 - 2$ (which is irrational). In this way, the twin dragon inherits from its boundary the property of having a complex structure at arbitrarily small scales. However, this is not always true: the unit square, for example, is a self-affine tile whose boundary is not a fractal.

In some definitions, tiles are required to be connected or even to be topological disks. We will also consider tilings given by disconnected tiles. A nice example in $\mathbb{R}^2$ of $\mathcal{F}(A, \mathcal{D})$ is shown in Figure 1.3, where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\}.$$

FIGURE 1.4: Illustration of the self-affinity of $\mathcal{F}(A, \mathcal{D})$

The set equation is illustrated in Figure 1.4, where the union of 4 copies of the tile adds up to an enlarged version of it. It is also an example of a set that has a fractal structure even though its dimension is 2.

1.3 Standard vs non-standard

Consider an expanding real matrix $A \in \mathbb{R}^{n\times n}$. So far, the only requirement for a digit set $\mathcal{D} \subset \mathbb{R}^n$ is that $|\mathcal{D}| = |\det(A)|$. We have mentioned that most of the choices of digits do not induce a self-affine tile as in Definition 1.2.2 (that is, $\mathcal{F}(A, \mathcal{D})$ has zero Lebesgue measure for most of the choices of $\mathcal{D}$). It is a central question to determine when the measure of $\mathcal{F}$ is positive.

Definition 1.3.1. *We say that a digit system $(A, \mathcal{D})$ is a standard digit system if $\mathcal{D}$ is a complete residue set of $\mathbb{Z}^n[A] \mod A$.*

Standard digit systems give rise to self-affine tiles (see [53, Corollary 1.1]).

Theorem 1.3.2. *If $(A, \mathcal{D})$ is a standard digit system, then $\mathcal{F}(A, \mathcal{D})$ is a self-affine tile.*

The converse is not true, as shown in the following example. Let $A = 4$ and $\mathcal{D} = \{0, 1, 8, 9\}$, which is clearly not a residue set. Then we have

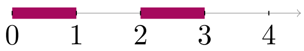
$$\mathcal{F} = \mathcal{F}(A, \mathcal{D}) = [0, 1] \cup [2, 3].$$

This can be easily seen by checking that $\mathcal{F}$ satisfies the set equation

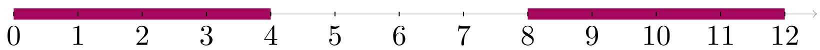
$$4\mathcal{F} = \mathcal{F} \cup (\mathcal{F}+1) \cup (\mathcal{F}+8) \cup (\mathcal{F}+9),$$

as is illustrated in the following pictures.

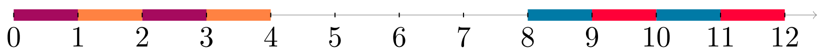
$\mathcal{F}$:



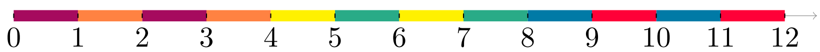
$4\mathcal{F}$:



$\mathcal{F} \cup (\mathcal{F}+1) \cup (\mathcal{F}+8) \cup (\mathcal{F}+9)$:



Therefore $\mathcal{F}$ is a self-affine set and it tiles the real line with translation set $\Gamma = 4\mathbb{Z} \cup (4\mathbb{Z}+1)$, as shown below.



Γ is exactly the set of integers that are expressed with the digits $\{0, 1, 8, 9\}$ in base 4, that is, Γ is the set of integer expansions. Although only two of the four residues modulo 4 appear in $\mathcal{D}$, it is still possible to expand all integers (and all real numbers). Note that Γ is not a lattice because it is not a group. What would happen if we wanted to translate $\mathcal{F}$ via a lattice? In this example, $\mathbb{Z}$ is the smallest lattice to contain Γ, and below we can see that each point (except for a measure zero set) belongs to exactly *two* different copies of $\mathcal{F}$: this is known as a multiple tiling of degree 2.

This type of example is known by Lagarias and Wang as a *product form digit set*. In this case, $\mathcal{D}$ can be decomposed as $\mathcal{D} = \{0, 1\} + 4\{0, 2\}$, where $\{0, 1\} + \{0, 2\} = \{0, 1, 2, 3\}$ is a residue set modulo 4. However, there are digit systems that are not of product form that also have the tiling property, like, for example, if we take base 4 and digits $\{0, 1, 8, 25\}$ (see [52]).

Lagarias and Wang [53] showed the existence of tilings and multiple tilings given by self-affine tiles associated with standard and non-standard digit systems for integer expanding matrices $A \in \mathbb{Z}^{n\times n}$. In Chapter 4, we generalize their results for the case of rational matrices.

1.4 Integer vs non-integer

Steiner and Thuswaldner [78] considered number systems whose base is an algebraic number, not necessarily an algebraic integer. They introduced the so-called *rational self-affine tiles*, defined in terms of these number systems, and proved the existence of lattice tilings arising from them. We will dive into the study of these objects.

We introduce an example that will appear frequently throughout the thesis. Consider $\alpha = \frac{-1+3i}{2}$, whose primitive minimal polynomial is $P_\alpha = 2X^2 + 2X + 5$, hence α is not an algebraic integer. The set $\mathcal{D} = \{0, 1, 2, 3, 4\}$ constitutes a residue set for $\mathbb{Z}[\alpha] \mod \alpha$. Consider the set of fractional parts

$$F := \Big\{ \sum_{j=1}^{\infty} \alpha^{-j} d_{-j} \,:\, d_{-j} \in \mathcal{D} \Big\} \subset \mathbb{C},$$

which satisfies

$$\alpha F = \bigcup_{d\in\mathcal{D}} (F + d).$$

This set is depicted on the left-hand side of Figure 1.5, and on the right-hand side we paint the points in different colors depending on the digit d_{-1}. The color choice used is red when $d_{-1} = 0$, green when $d_{-1} = 1$, yellow when $d_{-1} = 2$, purple when $d_{-1} = 3$ and blue when $d_{-1} = 4$. For each color we have a smaller copy of F, yet we can see that the different copies overlap. In other words, a complex number has multiple expansions in base α.

The explanation for the overlapping phenomenon is quite simple: the minimal polynomial of α over $\mathbb{Q}$ is $X^2+X+\frac{5}{2}$, and the independent coefficient of this polynomial equals the norm of α. So $N(\alpha) = \frac{5}{2}$, which is not an integer. Taking Lebesgue measure on the left side of the set equation we get $\lambda(\alpha F) = \frac{5}{2}\lambda(F)$, while on the right side, we have a union of 5 translations of F. If F has positive measure (which is the case) then the union cannot be essentially disjoint, so F is not self-affine. The goal is to explain how this can be "fixed", that is, how we can define a tiling. This will be explained in Chapters 2, 3, and 4 at increasing levels of generality. The "trick" is to think outside the box: instead of choosing the appropriate base to represent a given space, one can choose an appropriate space to represent the digit system. On the

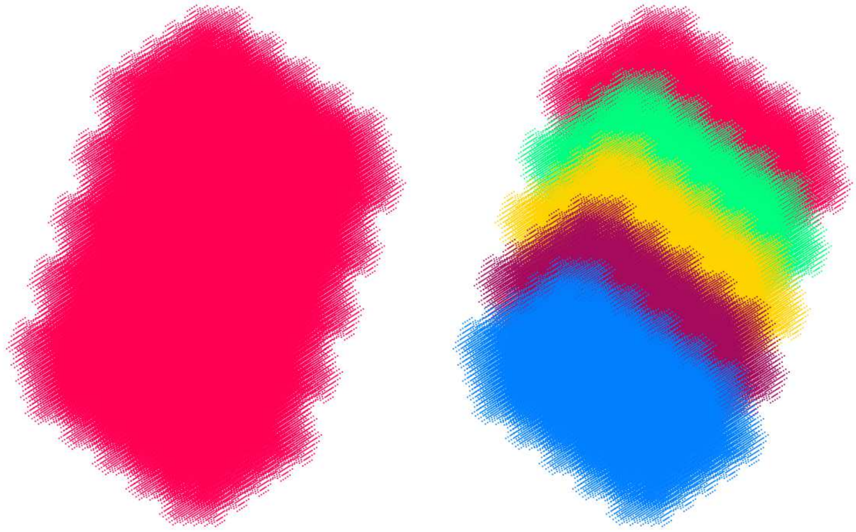


FIGURE 1.5: A set F related to the base $\frac{-1+3i}{2}$.

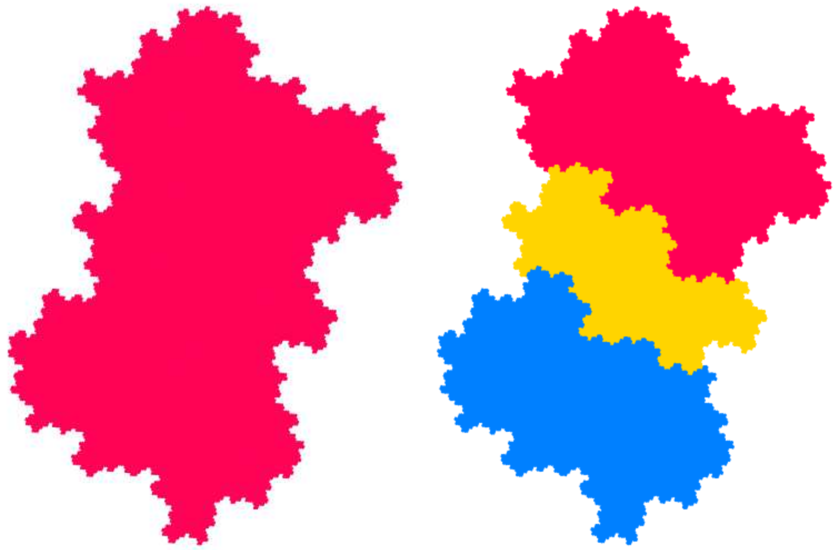


FIGURE 1.6: The zero slice $\mathcal{G}(0)$ subdivided into three subtiles.

left-hand side of Figure 1.6 we illustrate the *zero slice* $\mathcal{G}(0)$, which is obtained from F by selecting certain points. To give the reader an intuition of how this is done, on the right-hand side we subdivide $\mathcal{G}(0)$ into three subtiles that obey the same color coding as in Figure 1.5: the points of F that belong to $\mathcal{G}(0)$ satisfy $d_{-1} \in \{0, 2, 4\}$. In other words, a requirement for a series $x = \sum_{j=1}^{\infty} \alpha^{-j} d_{-j} \in F$ to be in $\mathcal{G}(0)$ is that d_{-1} is even. We will see in Chapter 3 that such points satisfy $|x|_{1+i} = 0$ where $|\cdot|_{1+i}$ is a $(1+i)$-adic absolute value defined in terms of the Gaussian prime $p = 1 + i$.

1.5 An art project

During the course of the Ph.D., we conducted a project at the intersection of mathematics and art that was mostly carried out at the University of Arkansas together with Edmund Harriss, who works both at the Mathematics department and the Arts department of this University. We present it now as a motivation, but we also understand it as an application of our research: on top of serving as an illustrative tool for mathematical objects and having an aesthetic aspect, this art piece is also interactive and contains a set of puzzles that can be entertaining and challenging for people regardless of their knowledge of the mathematics behind it. Moreover, having a physical object that one can play with enables the discovery of several patterns that are perhaps not visible in a computer illustration. Following the making of this art project, several questions arise that can be a starting point for future research.

Given the base $\alpha = \frac{-1+3i}{2}$ and the digits $\mathcal{D} = \{0, 1, 2, 3, 4\}$, we consider a self-affine tile $\mathcal{F}(\alpha, \mathcal{D})$ which is a subset of the space $\mathbb{K}_\alpha = \mathbb{C} \times K_{1+i}$, where K_{1+i} is a p-adic completion of the field $K = \mathbb{Q}(i)$ defined in terms of the Gaussian prime $p = 1 + i$. This example and the corresponding representation space are explained in full detail in Section 3.3. In [78, Figure 2] there is an illustration of a rational self-affine tile similar to our example, which was a starting point for the project, and it is essentially the only illustration of a three-dimensional rational self-affine tile that can be found in the literature. The lack of pictures in such a visual field of study was also a source of motivation to carry out the art project.

We proceed to give an overview of the process of creation and assembly of this piece, without going into the underlying mathematics, since that will be done in Chapter 3. Given an algebraic number system $(\alpha, \mathcal{D})$, we first come up with an algorithm to generate points belonging to the so-called *intersective tiles* associated with it, which are sets in $\mathbb{R}^n$ where n is the degree of the algebraic number α. This algorithm was written in SageMath, and it relies on writing intersective tiles as shift radix system tiles. Naturally, we mostly want to see examples when $n = 2$. With the goal in mind of having some of these sets laser cut, we looked for tiles that were topological disks and had an overall simple shape, even though the boundary has a fractal structure. This is how we arrived at the example $\alpha = \frac{-1+3i}{2}$ and computed several tiles (slices). Afterward, for each tile, we exported this list of points to the program Rhinoceros 3D, intended for 3D rendering. To approximate the boundary curve of the tiles, we used the notion of *concave hull*, which is a subset of the convex hull that is, in some sense, the smallest surface to contain all the points. This has no formal mathematical definition and is instead obtained through an algorithm given in terms of a parameter. We computed it using the visual programming language Grasshopper, which runs within the Rhinoceros 3D application. This allowed us to laser cut the shapes on 3mm wooden panels. They were subsequently hand painted with acrylic paint.

FIGURE 1.7: Photographs of wooden puzzles. Three-piece puzzle (left), nine-piece puzzle (middle) and 62-piece puzzle (right).

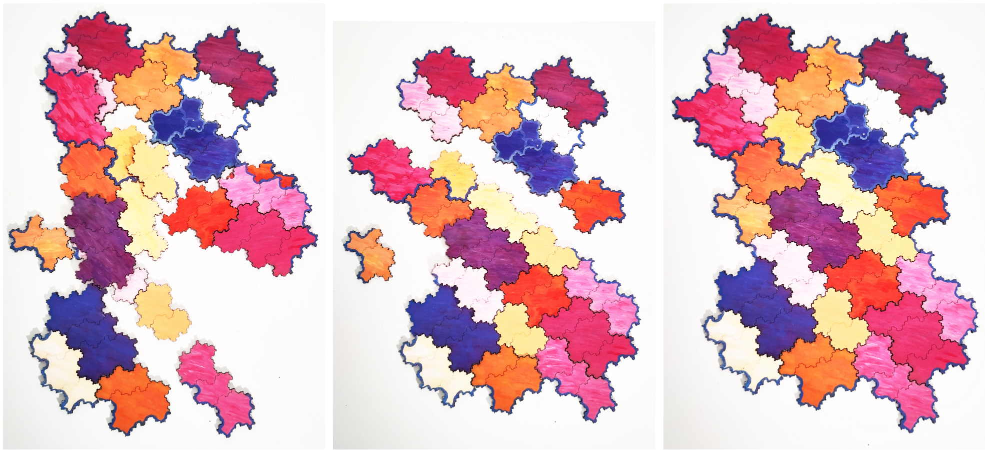


FIGURE 1.8: Photographs of 24-piece wooden puzzle. Different rearrangements.

All the puzzles that we show have the property of forming always the same shape. They are obtained by different stages of subdivisions of the *zero slice* from Figure 1.6. The first puzzle has three pieces, and is displayed on the left-hand side of Figure 1.7; this exhibits a property that we call *inter-affinity*: each piece can be subdivided as a union of other pieces. There is a nice application: one could hand the three-piece puzzle to someone, and tell them to put the pieces together to obtain the same shape as the red piece, which constitutes a different approach to solving puzzles. In the middle of Figure 1.7 is the nine-piece puzzle, and on the right-hand side is the 62-piece puzzle. All the puzzles have the same size and can be solved by stacking them up, beginning with the simplest one and using it as a hint to solve the next. Moreover, each puzzle contains a smaller copy of the puzzles with fewer pieces, which can be also used as a strategy for solving it. Figure 1.8 shows three different stages of assembly of the 24-piece puzzle.

The color choice is not random. One can see that tiles with similar colors are also similar in shape. Each tile, denoted $\mathcal{G}(z)$ is obtained by "slicing" the central tile $\mathcal{F}(\alpha, \mathcal{D})$ at some fixed height and is positioned on the plane on a point of a certain

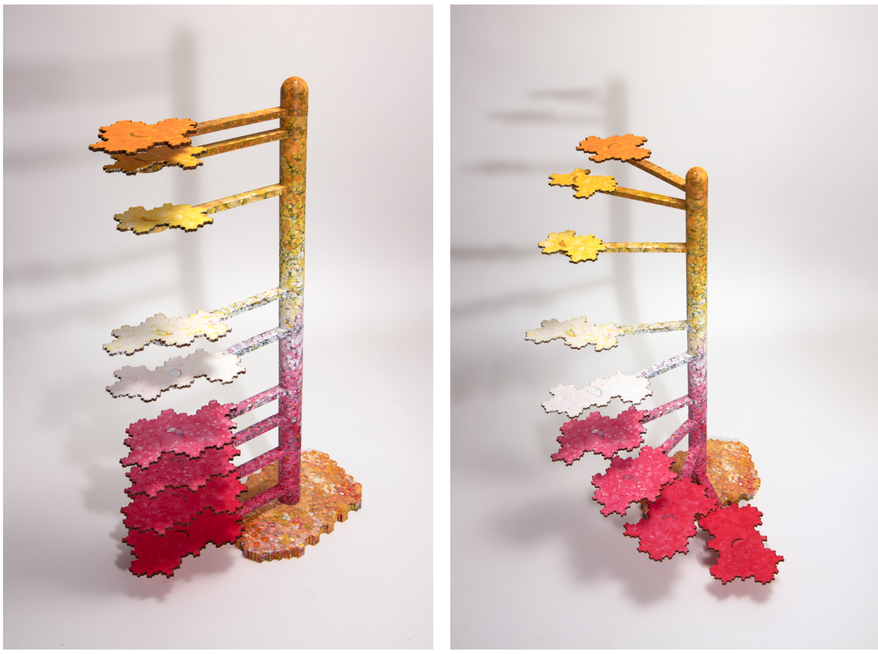


FIGURE 1.9: Photographs of sculpture with branches aligned and rotated.

lattice Λ. The height of the slice can be computed from the $(1+i)$-adic expansion of this point in Λ. We came up with an approximation of $\mathcal{F}(\alpha, \mathcal{D})$ by placing the tiles of the nine-piece puzzle in a vertical structure, and the vertical distances somewhat reflect the $(1+i)$-adic distance. This gave rise to the sculpture depicted in Figure 1.9. It consists of a central axe with nine arms, which can turn around. Each of the tiles has a hole with the shape of a phase of the moon, and each arm has a key that fits exactly in the hole, and there is a unique way to place the tiles in the structure (even if one flips or rotates them). The base of the sculpture is actually the set F from Figure 1.5. The design and construction of the sculpture were carried out together with Reilly-Dickens Hoffman from the Sculpture Department of the University of Arkansas. The central structure is 3D printed and the vertical axe contains a screw that fits into the base, which also has metal plates inside for stability.

1.6 Continued fractions and Sturmian sequences

Continued fractions provide a different way to represent real numbers as a string of digits (generally known as coefficients), which is in some sense more natural than number systems because it does not require the choice of a base; moreover, in general, continued fractions give better rational approximations of real numbers than digit expansions. However, the coefficients do not belong to a finite set of digits. The *classical continued fraction expansion* of a real number x is an expression of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

where a_0 is an integer and a_n is a positive integer for $n \geq 1$. Then x is the limit of the sequence of rational numbers obtained by truncating the fraction at the n-th coefficient a_n. It is usually denoted by $x = [a_0; a_1, a_2, \ldots]$, and $a_0, a_1, \ldots$ are called the *continued fraction coefficients* of x. The continued fraction expansion may be finite or infinite, and it is not hard to check that it is finite if and only if x is rational. For example, $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots]$. The expansion is unique for irrational numbers, whereas rational numbers have exactly two closely related different expansions. Some algebraic properties of real numbers can be visualized from the properties of the sequence of their continued fraction coefficients. For example, the continued fraction is eventually periodic precisely for the quadratic irrationals.

The continued fraction coefficients are obtained iteratively through the Gauss map $T : \mathbb{R} \to \mathbb{R}$, which is defined as $T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ for $x \neq 0$ and $T(0) = 0$, where $\lfloor \cdot \rfloor$ denotes the floor function. Then $a_n = \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor$ for $n \geq 0$ whenever $T^{n-1}(x) \neq 0$. If $T^{n-1}(x) = 0$ for some n, we obtain a finite continued fraction.

Sturmian sequences have been studied extensively (see, for instance, the survey [8]) and are closely linked to continued fractions. A sequence ω in $\mathcal{A}^{\mathbb{N}}$ (where $\mathcal{A}$ is a set known as an alphabet, usually taken to be finite) is known in combinatorics as an infinite word or a string. A factor of ω is a finite or infinite word that appears somewhere in ω. An infinite word ω is called Sturmian if, for any given $n \in \mathbb{N}$, there exist exactly $n+1$ pairwise different factors of ω of length n (the length of a word is how many letters it has). More formally, given a word ω, consider the *factor complexity function* $p_\omega : \mathbb{N} \to \mathbb{N}$ defined as $p_\omega(n) = \#\{u : u \text{ is a factor of } \omega \text{ of length } n\}$. Then ω is said to be a Sturmian sequence if $p_\omega(n) = n+1$ for all $n \in \mathbb{N}$. Sturmian sequences are the non-eventually periodic sequences with the smallest possible complexity (see *e.g.* [22, Corollary 4.3.2]). A Sturmian sequence ω has the property that, given any two factors u and v of ω of the same length and a letter $a \in \{0, 1\}$, the number of occurrences of a in u differs from the number of occurrences of a in v by at most one. This property is called 1-*balance* (or just *balance*) and characterizes Sturmian sequences. More precisely, the Sturmian sequences are exactly the non-eventually periodic balanced sequences over two letters (*cf. e.g.* [8, Theorem 6.1.8]). Moreover, Morse and Hedlund [62], as well as Coven and Hedlund [23], discovered a connection between Sturmian sequences and rotations by an irrational number x. Rauzy found a very elegant proof for this connection that relates Sturmian sequences to substitutions and the classical continued fraction algorithm (see [11] and Rauzy's earlier papers [69, 68]). Given an irrational number x, we can obtain a Sturmian word $\omega(x)$ which relates to the continued fraction coefficients of x, and one way to do this is using substitutions, a notion that we will introduce and study in Chapters 5 and 6.

A *generalized continued fraction* is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \ddots}},$$

where a_n and b_n can be any complex numbers. To motivate the use of generalized continued fractions, we can see that for some cases we obtain regular expansions for π, like, for example, when we let $b_n = (2n-1)^2$:

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \ddots}}}$$

We will study a type of generalized continued fractions called N-continued fractions (N is a fixed positive integer), obtained by letting $b_n = N$ for all $n \geq 1$, and in particular we will consider coefficients $a_n \geq N$. We will mention some properties of these expansions, but our study will focus on constructing infinite words that generalize Sturmian sequences for the case of N-continued fractions.

1.7 Structure of the thesis

The thesis can be seen to be divided into two parts. The first part is made up of Chapters 2, 3 and 4. It is devoted to the study of tilings related to number systems in the rational case. The setting of Chapter 2 is contained in the setting of Chapter 3, which is as well contained in the setting of Chapter 4. The second part of the thesis is made up of Chapters 5 and 6, and it is centered around N-continued fractions, the construction of generalizations of Sturmian sequences and Rauzy fractals for families of non-unimodular S-adic substitutions.

We begin our journey into the world of rational self-affine tiles in Chapter 2, which is meant as an introduction to the main notions of the theory. Most of it is devoted to a number system, which we call the *negasemiternary system* because it is done in base $-\frac{3}{2}$ with digits $\{0, 1, 2\}$. It is, in some sense, the simplest rational number system, and that is why we chose it as a starting point. We show how the 2-adic valuation is used to introduce a suitable representation space of the form $\mathbb{R} \times \mathbb{Q}_2$. We prove some theorems for this particular example, as it is useful to understand the main ideas before jumping to broader definitions. We introduce a set $\mathcal{F}$ that tiles $\mathbb{R} \times \mathbb{Q}_2$ and prove how this implies unique expansion almost everywhere. We link our work to shift radix systems, giving insight into the shape of $\mathcal{F}$. We also give some applications in two number theory problems: the Josephus problem and Mahler's problem. Afterward, we move to general rational bases $\frac{a}{b}$ and construct a space of the form $\mathbb{R} \times \mathbb{Q}_b$, where $\mathbb{Q}_b$ is a product of p-adic spaces that can alternatively be defined as a projective limit. We state some results in this new setting.

Chapter 3 is devoted to *algebraic number systems*, that is, number systems whose base is an algebraic number $\alpha \in \mathbb{C}$. Note that this contains Chapter 2 because rational numbers are algebraic numbers. We consider a digit set of the form $\{0, \dots, |a_0| - 1\}$, where a_0 is the independent coefficient of the minimal primitive polynomial of α in $\mathbb{Z}[X]$. Digit sets of this form were used by Steiner and Thuswaldner in [78], and we follow the terminology of that paper and state some of their results. This digit choice guarantees that the set of fractional parts of the number system $(\alpha, \mathcal{D})$ has positive measure, becoming a rational self-affine tile. We start by introducing several preliminary notions in algebraic number theory. In particular, we do a careful study of completions of number fields $K = \mathbb{Q}(\alpha)$ with respect to finite and infinite places $\mathfrak{p}$, denoted $K_{\mathfrak{p}}$, and relate $\mathfrak{p}$-adic completions to projective limits. We introduce Adèle rings because the representation spaces that we study are in fact subrings of Adèle rings. We introduce rational self-affine tiles and state some results for general algebraic number systems. Part of this chapter is centered around the example $\alpha = \frac{-1+3i}{2}$. We link our study to shift radix systems and show how this is useful in the computation of the tiles. Finally, we show an analogous way of expressing algebraic number systems in terms of matrices and vectors, motivating the next chapter.

Chapter 4 contains several original results and extends the existing theory of rational self-affine tiles in two directions: the *reducible case* and the *non-standard case*. For the reducible case we mean that we consider digit systems where the base is an expanding rational matrix A and we allow the characteristic polynomial to be reducible. We use results of linear algebra (in particular the Frobenius normal form of a matrix) to find the appropriate cardinality of digit sets. This is illustrated with an example. In the reducible case, we no longer have completions of algebraic number fields to introduce the representation space. Instead, we define the space in terms of a projective limit. In the non-standard case, we consider digits that are not necessarily a complete set of residues. We generalize the results obtained by Lagarias and Wang for the case of integer matrices: we show some topological properties of the central tile and prove a tiling theorem. Classical results of topology play a key role in the study of self-affine tiles, as well as measure-theoretic results involving Haar measures. The last part of Chapter 4 centers around the proof of the multiple tiling theorem, which is perhaps the strongest result presented in the thesis. The theory of locally compact abelian groups is needed to study the character group of the representation space and certain quotient groups. A preliminary example of the study of characters of p-adic spaces is done first. The key to the proof relies on a lemma that shows that the multiplication by A in a certain quotient group is ergodic.

In Chapter 5 we introduce N-continued fraction expansions of real numbers for $N \geq 1$, which corresponds to the classical continued fraction expansions when $N = 1$. We define two families of infinite words on a two-letter alphabet that generalize

Sturmian sequences, called *N-continued fraction sequences.* They are obtained as limit words of sequences of substitutions. We show that they are finitely balanced and compute their letter frequency. We use tools of combinatorics on words to find an explicit formula of their factor complexity function. Furthermore, we study the entropy and growth rate of N-continued fraction sequences. Ergodic theory plays a very important role in this section.

Finally, in Chapter 6 we consider Rauzy fractals in the setting of non-unimodular S-adic substitutions. We show some examples of the classical case and the S-adic case and proceed to show how self-affine Rauzy fractals for the non-unimodular case can be obtained using the same ideas we used for number systems, by introducing a representation space defined in terms of a projective limit. This is done as a form of closure to connect the two seemingly distinct topics of the thesis, one being self-affine sets related to number systems and the other being N-continued fraction sequences, which are in fact limit sequences for S-adic non-unimodular substitutions, so this theory allows us to define the corresponding Rauzy fractal. The theory of non-unimodular S-adic Rauzy fractals is not studied extensively, and even though we mention some results, most questions are still open and this will hopefully be a line for future research.

Chapter 2

Number systems with rational basis

In this first chapter, we tackle the issue of defining self-affine tiles related to digit expansions in rational bases. We show that a base $\frac{a}{b} \in \mathbb{Q}$ and the set of digits $\{0, 1, \ldots, |a|-1\}$ are suitable to represent points of a space of the form $\mathbb{R} \times \mathbb{Q}_b$, where $\mathbb{Q}_b$ is the ring of b-adic numbers (better known when b is a prime). We define a set $\mathcal{F}(\frac{a}{b}, \mathcal{D})$, called a rational self-affine tile, that induces a tiling of $\mathbb{R} \times \mathbb{Q}_b$ and shares properties with classical self-affine sets.

We will start our journey with the *negasemiternary system*, an example that was published in the expository article *A Number System With Base -3/2* (American Mathematical Monthly, 2022, [74]). It is presented now with further improvements. Number systems with rational bases similar to ours were introduced by Akiyama et al. [3] and generalizations of these have been studied in [19] and [75].

2.1 The negasemiternary system

The aim of this section is to define and explore a number system with base $-\frac{3}{2}$ and digits $\{0, 1, 2\}$ that we call the *negasemiternary number system*. This terminology, first used by Donald Knuth in [45], is due to the basis being negative (nega-), the denominator being 2 (-semi-) and the numerator being 3 (-ternary). We abbreviate it as NST. We chose to start in this manner because we believe that difficult concepts are more easily understood when a simple case is presented first, rather than directly jumping to abstract general definitions. This example has many beautiful properties that link different areas of mathematics and, nevertheless, can be studied in a way that is accessible to a broad readership. We mention that there are other ways to define a number system with base $-\frac{3}{2}$; see, for example, the one using the digit set $\{0, 1\}$ in [4].

We start our study of the NST number system by investigating expansions of real numbers and realize that the real line is somehow "too small". We introduce the space $\mathbb{K} = \mathbb{R} \times \mathbb{Q}_2$, where $\mathbb{Q}_2$ is the field of 2-adic numbers, and show how it naturally arises as a representation space. We relate to our number system a set $\mathcal{F}$, we study some of its topological properties and prove that it induces a tiling of the space $\mathbb{K}$ by translations. This tiling property is strongly related to the existence and uniqueness of expansions with respect to the base $-\frac{3}{2}$ in $\mathbb{K}$. We relate the slices of the tile $\mathcal{F}$ to shift radix systems. Afterward, we provide applications of the NST number system to Mahler's $\frac{3}{2}$-problem and the Josephus problem.

The minimal polynomial of $-\frac{3}{2}$ is $2X+3$, which is not monic; therefore, $-\frac{3}{2}$ is not an algebraic integer. This is a relevant feature because it means that, acting as the base of a number system, $-\frac{3}{2}$ is of a different nature than integers such as 10 and 2. When the base under consideration is not an algebraic integer, rational self-affine tiles live in subrings of certain adèle rings, which are important objects in algebraic number theory. In particular, the representation space $\mathbb{K}$ introduced here is an example of such a subring. The reason why we consider $\mathbb{K}$ instead of $\mathbb{R}$ is that $-\frac{3}{2}$ acts in some sense like an algebraic integer in $\mathbb{K}$ because its denominator is "multiplied away" in the 2-adic factor of $\mathbb{K}$. We will explain this in more detail later on.

Expansions in the real line.

In the decimal system, each integer can be expanded without using digits after the decimal point. In this section, we wish to define and explore such "integer expansions" in the number system with base $-\frac{3}{2}$ and digit set $\mathcal{D}=\{0,1,2\}$, which we denote as $(-\frac{3}{2},\mathcal{D})$.

We write

$$(d_k\dots d_0)_{-3/2} := \sum_{i=0}^{k} d_i\left(-\tfrac{3}{2}\right)^i \qquad (k\in\mathbb{N},\ d_i\in\mathcal{D};\ d_k\neq 0 \text{ whenever } k\geqslant 1)$$

and call $(d_k\dots d_0)_{-3/2}$ an *integer negasemiternary expansion.* For instance, $-6=(2110)_{-3/2}$ and $8=(21122)_{-3/2}$.

As a next step, we characterize the set

$$\mathcal{D}\left[-\tfrac{3}{2}\right]=\left\{(d_k\dots d_0)_{-3/2}\,:\,k\in\mathbb{N},\ d_i\in\mathcal{D};\ d_k\neq 0 \text{ whenever } k\geqslant 1\right\}$$

of all real numbers with an integer NST expansion (see also [75, Example 3.3]).

Theorem 2.1.1. *The set of all numbers having an integer NST expansion is* $\mathbb{Z}[\frac{1}{2}]$. *Here, as usual, we set* $\mathbb{Z}[\frac{1}{2}]=\{a2^{-\ell}\,:\,a\in\mathbb{Z},\,\ell\in\mathbb{N}\}$.

Proof. The inclusion $\mathcal{D}[-\frac{3}{2}]\subset\mathbb{Z}\left[\frac{1}{2}\right]$ is trivial. For the reverse inclusion, let $N_0\in\mathbb{Z}\left[\frac{1}{2}\right]$ be arbitrary. Then there exist $a_0\in\mathbb{Z}$ and $\ell\in\mathbb{N}$ such that $N_0=a_02^{-\ell}$. In order find an expansion of N_0 of the form

$$N_0=d_k\left(-\tfrac{3}{2}\right)^k+d_{k-1}\left(-\tfrac{3}{2}\right)^{k-1}+\cdots+d_1\left(-\tfrac{3}{2}\right)+d_0 \qquad (d_i\in\mathcal{D}) \tag{2.1}$$

for some $k \in \mathbb{N}$, we use the following algorithm. First, note that $2^\ell \mathcal{D} = \{0, 2^\ell, 2^{\ell+1}\}$ is a complete residue set modulo 3. Find the unique $d_0 \in \mathcal{D}$ so that $a_0 \equiv 2^\ell d_0 \pmod 3$. Write $N_0 = -\frac{3}{2}N_1 + d_0$. We have

$$N_1 = -\tfrac{2}{3}(N_0 - d_0) = -\tfrac{2}{3}(a_0 2^{-\ell} - d_0) = -\tfrac{2}{3}\frac{a_0 - 2^\ell d_0}{2^\ell}$$

and $a_0 - 2^\ell d_0$ is divisible by 3 because of the choice of d_0, therefore $N_1 \in \mathbb{Z}\left[\frac{1}{2}\right]$. Moreover, if we let $a_1 = \frac{-a_0 + 2^\ell d_0}{3} \in \mathbb{Z}$ then $N_1 = \frac{a_1}{2^{\ell-1}}$. We recursively define N_{i+1} for $i \geqslant 0$ by the equations

$$N_i = -\tfrac{3}{2}N_{i+1} + d_i \tag{2.2}$$

where $d_i \in \mathcal{D}$ is chosen so that $N_{i+1} \in \mathbb{Z}[\frac{1}{2}]$. Iterating (2.2) yields $N_i = a_i 2^{-\ell+i}$ for $a_i \in \mathbb{Z}$. When $i \in \{0, \ldots, \ell\}$, d_i is obtained by taking $a_i \equiv 2^{\ell-i} d_i \pmod 3$, and when $i > \ell$, d_i is obtained by taking $2^{i-\ell} a_i \equiv d_i \pmod 3$. One can easily see by induction on i that this yields

$$N_0 = \left(-\tfrac{3}{2}\right)^{i+1} N_{i+1} + d_i \left(-\tfrac{3}{2}\right)^i + \cdots + d_1 \left(-\tfrac{3}{2}\right) + d_0. \tag{2.3}$$

Note that $N_i = 0$ implies $N_{i+1} = 0$. If we can prove that $N_i = 0$ for i large enough, our algorithm gives the desired representation.

Since $N_i = a_i 2^{-\ell+i}$ with $a_i \in \mathbb{Z}$, we get $N_i \in 2\mathbb{Z}$ for $i > \ell$. Let $M_i = \frac{N_i}{2} \in \mathbb{Z}$ for $i > \ell$. We show that $M_i = 0$ for some i. We have $N_{i+1} = -\frac{2}{3}N_i + \frac{2}{3}d_i$, so dividing by 2 yields $M_{i+1} = -\frac{2}{3}M_i + \frac{1}{3}d_i$ with $d_i \in \mathcal{D}$; hence $|M_{i+1}| \leqslant \frac{2}{3}|M_i| + \frac{2}{3}$ and therefore $|M_{i+1}| < |M_i|$ holds for each $|M_i| \geqslant 3$. This implies that there is an $i \in \mathbb{N}$ with $|M_i| \leqslant 2$. Direct calculation shows that if $M_i = -2$, then $M_{i+1} = 2$, $M_{i+2} = -1$, $M_{i+3} = 1$, and $M_{i+4} = 0$. Thus there is $k_0 \in \mathbb{N}$ with $N_k = 0$ for all $k \geqslant k_0$, and this implies that $N_0 \in \mathcal{D}[-\frac{3}{2}]$.

□

By residue class considerations one can show that each $z \in \mathbb{Z}[\frac{1}{2}]$ has a unique integer NST expansion. For instance, $-\frac{3}{4} = (120)_{-3/2}$ and $\frac{7}{4} = (111)_{-3/2}$. The uniqueness holds because we are considering only integer NST expansions; otherwise, it is not true. We proceed to study NST expansions for arbitrary reals, allowing negative powers of the base. We consider the matter of uniqueness and motivate the subsequent construction of a tiling.

Let

$$(d_k \ldots d_0.d_{-1}d_{-2}\ldots)_{-3/2} := \sum_{i=-\infty}^{k} d_i \left(-\tfrac{3}{2}\right)^i \qquad (k \in \mathbb{N},\ d_i \in \mathcal{D}) \tag{2.4}$$

and assume that there are no padded zeros to the left of the "decimal point" (i.e., that $d_k \neq 0$ whenever $k \geq 1$). Consider

$$\Omega = \{(0.d_{-1}d_{-2}\ldots)_{-3/2} : d_{-i} \in \mathcal{D}\}, \tag{2.5}$$

the set of *fractional parts* of the NST number system in $\mathbb{R}$. One can prove that $\Omega = \left[-\frac{12}{5}, \frac{8}{5}\right]$. Clearly, Ω is bounded, and by a Cantor diagonal argument we can show that it is closed as well, so Ω is a compact set. We want to give an explicit characterization of Ω by establishing a *set equation*. If we multiply Ω by $-\frac{3}{2}$, this is tantamount to moving its "decimal point" one digit to the right. For an arbitrary element $\omega = (0.d_{-1}d_{-2}\ldots)_{-3/2} \in \Omega$ there are three possibilities: if $d_{-1} = 0$, then when multiplying ω by $-\frac{3}{2}$ it remains in Ω; if $d_{-1} = 1$, then $-\frac{3}{2}\omega \in \Omega + 1$, and if $d_{-1} = 2$, we have $-\frac{3}{2}\omega \in \Omega + 2$. This leads to the set equation

$$-\tfrac{3}{2}\Omega = \Omega \cup \big(\Omega + 1\big) \cup (\Omega + 2)$$

or equivalently,

$$\Omega = \left(-\tfrac{2}{3}\right)\Omega \cup \left(-\tfrac{2}{3}\right)\big(\Omega + 1\big) \cup \left(-\tfrac{2}{3}\right)(\Omega + 2), \tag{2.6}$$

which means that Ω can be expressed as a union of contracted copies of itself. By Hutchinson's Theorem [39], there exists a unique non-empty compact subset of $\mathbb{R}$ for which (2.6) holds. Since $\left[-\frac{12}{5}, \frac{8}{5}\right]$ is a solution of (2.6), the uniqueness of the solution yields $\Omega = \left[-\frac{12}{5}, \frac{8}{5}\right]$.

Each decomposition of a real number x as the sum of an element of $\mathbb{Z}\left[\frac{1}{2}\right]$ and an element of Ω leads to an expansion of x of the form (2.4), by Theorem 2.1.1 and the definition of Ω. The collection $\{\Omega + z : z \in \mathbb{Z}\left[\frac{1}{2}\right]\}$ covers the real line, so each real number can be written as such a sum, and hence it admits an expansion of the form (2.4). But since each $x \in \mathbb{R}$ is contained in multiple elements of the collection $\{\Omega + z : z \in \mathbb{Z}\left[\frac{1}{2}\right]\}$ (in fact, in infinitely many), it admits multiple expansions of the form (2.4). For example, $\frac{8}{5} = (0.020202\ldots)_{-3/2} = (2.11111\ldots)_{-3/2}$.

Different translations of Ω by elements of $\mathbb{Z}\left[\frac{1}{2}\right]$ overlap in sets of positive measure; in other words, we do not have the desired tiling property. This results in expansions that are not unique. In the subsequent sections, we will find a way to embed the collection $\{\Omega + z : z \in \mathbb{Z}\left[\frac{1}{2}\right]\}$ in a suitable space where it will give rise to a tiling.

2.2 Ambinumbers

The real line seems to be "too small" for the collection $\left\{\left[-\frac{12}{5}, \frac{8}{5}\right] + z : z \in \mathbb{Z}\left[\frac{1}{2}\right]\right\}$, so we wish to enlarge the space $\mathbb{R}$ to resolve the issue of overlaps. Another way, which we do not pursue here, would be to restrict the "admissible" digit strings; see [3]. Indeed, our next goal is to define a new space, called $\mathbb{K}$, in which the NST number system induces a tiling in a natural way. We call its elements *ambinumbers*, a term also coined by Donald Knuth in [45]. The idea behind this is as follows: the overlaps occur because the three digits $\{0, 1, 2\}$ are "too many" for a base whose modulus is $\frac{3}{2}$. Such a base would need one and a half digits, which is, of course, not doable. What causes all the problems is the denominator 2. Roughly speaking, this denominator accumulates powers of 2 that are responsible for the overlaps. It turns out that these overlaps can be "unfolded" by adding a 2*-adic factor* to our representation space. The strategy of enlarging the representation space by p-adic factors that we are about to present was used in the setting of substitution dynamical systems, e.g., by Siegel [77] and in a much more general framework than ours in [78].

We begin by introducing the 2-adic numbers. Consider a non-zero rational number y and write $y = 2^\ell \frac{p}{q}$ where $\ell \in \mathbb{Z}$ and both p and q, are odd. The 2-adic absolute value in $\mathbb{Q}$ is defined by

$$|y|_2 = \begin{cases} 2^{-\ell}, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0, \end{cases}$$

and the 2-adic distance between two rationals x and y is given by $|x-y|_2$. Two points are close under this metric if their difference is divisible by a large positive power of 2.

We define $\mathbb{Q}_2$ as the completion of $\mathbb{Q}$ with respect to $|\cdot|_2$. The space $\mathbb{Q}_2$ is a field called the *field of* 2*-adic numbers*. Every non-zero $y \in \mathbb{Q}_2$ can be written uniquely as a series

$$y = \sum_{i=\ell}^{\infty} c_i 2^i \qquad (\ell \in \mathbb{Z},\ c_i \in \{0, 1\},\ c_\ell \neq 0).$$

This series converges in $\mathbb{Q}_2$ because large powers of two have a small 2-adic absolute value. Indeed, we have $|y|_2 = 2^{-\ell}$.

We define our *representation space* as $\mathbb{K} = \mathbb{R} \times \mathbb{Q}_2$, with the additive group structure given by component-wise addition. Moreover, $\mathbb{Z}\left[\frac{1}{2}\right]$ acts on $\mathbb{K}$ by multiplication; more precisely, if $\alpha \in \mathbb{Z}\left[\frac{1}{2}\right]$ and $(x_1, x_2) \in \mathbb{K}$, then $\alpha \cdot (x_1, x_2) = (\alpha x_1, \alpha x_2)$.

For every $(x_1, x_2), (y_1, y_2) \in \mathbb{K}$ define

$$\mathbf{d}((x_1, x_2), (y_1, y_2)) := \max\{|x_1 - y_1|, |x_2 - y_2|_2\}.$$

Then $\mathbf{d}$ is a metric on $\mathbb{K}$. Intuitively, two points in $\mathbb{K}$ are far apart if either their real components are far apart or their 2-adic components are far apart.

We define the embedding

$$\varphi : \mathbb{Q} \to \mathbb{K}, \qquad z \mapsto (z, z).$$

Consider the image of $\mathbb{Z}\left[\frac{1}{2}\right]$ under φ. Despite both coordinates of $\varphi(z)$ being the same, the set $\varphi(\mathbb{Z}\left[\frac{1}{2}\right])$ is not on a "diagonal" as it would be if it were embedded in $\mathbb{R}^2$. Indeed, the points of $\mathbb{Z}\left[\frac{1}{2}\right]$ that are close in $\mathbb{R}$ are far apart in the 2-adic distance. In particular, we will show that the points of $\varphi(\mathbb{Z}\left[\frac{1}{2}\right])$ form a *lattice*.

A subset Λ of $\mathbb{K}$ is a *lattice* if it satisfies the three following conditions.

1. Λ is a group.
2. Λ is uniformly discrete, which means that there exists $r > 0$ such that every open ball of radius r in $\mathbb{K}$ contains at most one point of Λ.
3. Λ is relatively dense, which means there exists $R > 0$ such that every closed ball of radius R in $\mathbb{K}$ contains at least one point of Λ.

Lemma 2.2.1. *$\varphi(\mathbb{Z}[\frac{1}{2}])$ is a lattice in $\mathbb{K}$.*

Proof. 1. The fact that $\varphi(\mathbb{Z}[\frac{1}{2}])$ is a group follows from the additive group structure of $\mathbb{Z}[\frac{1}{2}]$ because φ is a group homomorphism.

2. To get uniform discreteness of $\varphi(\mathbb{Z}[\frac{1}{2}])$, we first show that $\mathbf{d}(\varphi(z), \varphi(0)) \geqslant 1$ holds for every non-zero $z \in \mathbb{Z}[\frac{1}{2}]$. Recall that $\mathbf{d}(\varphi(z), \varphi(0)) = \max\{|z|, |z|_2\}$. If $|z| < 1$, there exist $a, \ell \in \mathbb{Z}$ with a odd and $\ell \geq 1$ such that $z = a2^{-\ell}$, so $|z|_2 = 2^\ell > 1$. Due to the group structure, this implies that the distance between any two elements of $\varphi(\mathbb{Z}[\frac{1}{2}])$ is at least one; hence $\varphi(\mathbb{Z}[\frac{1}{2}])$ is uniformly discrete.

3. For the relative denseness, consider an arbitrary element $(x_1, x_2) \in \mathbb{K}$. We claim that there exists $z \in \mathbb{Z}[\frac{1}{2}]$ such that $\mathbf{d}(\varphi(z), (x_1, x_2)) \leqslant 2$. Let $z_1 \in \mathbb{Z}$ be one of the integers closest to x_1. If $x_2 = \sum_{i=\ell}^{\infty} c_i 2^i$ with $\ell \in \mathbb{Z}$ and $c_i \in \{0, 1\}$, then set $z_2 = \sum_{i=\ell}^{-1} c_i 2^i \in \mathbb{Z}[\frac{1}{2}]$ (note that $z_2 = 0$ if $\ell \geqslant 0$). Therefore,

$$\mathbf{d}((x_1, x_2), (z_1, z_2)) = \max\{|x_1 - z_1|, |x_2 - z_2|_2\} \leqslant 1.$$

Now we set $z = z_1 + z_2 \in \mathbb{Z}[\frac{1}{2}]$. Because z_1 is an integer, $|z_1|_2 \leqslant 1$, and since $z_2 \in [0, 1]$, we have $|z_2| \leqslant 1$. Thus $\mathbf{d}(\varphi(z), (z_1, z_2)) = \max\{|z_2|, |z_1|_2\} \leqslant 1$, and so $\mathbf{d}(\varphi(z), (x_1, x_2)) \leqslant 2$ by the triangle inequality; hence $\varphi(\mathbb{Z}[\frac{1}{2}])$ is relatively dense. □

Figure 2.1 illustrates some points of $\varphi(\mathbb{Z}[\frac{1}{2}])$. Drawing pictures in this setting is not straightforward: the space $\mathbb{K}$ is non-Euclidean, so we need to represent it in $\mathbb{R}^2$ while somehow maintaining the 2-adic nature of the second component. We do this in the following way: any point $y \in \mathbb{Q}_2$ has a 2-adic expansion $y = \sum_{i=\ell}^{\infty} c_i 2^i$ with $\ell \in \mathbb{Z}$, $c_i \in \{0, 1\}$. We consider the mapping

$$\gamma : \mathbb{Q}_2 \to \mathbb{R}, \qquad \sum_{i=\ell}^{\infty} c_i 2^i \mapsto \sum_{i=\ell}^{\infty} c_i 2^{-i}, \tag{2.7}$$

which is well-defined since the sum on the right-hand side converges in $\mathbb{R}$. An ambinumber $(x_1, x_2) \in \mathbb{K}$ is now represented as $(x_1, \gamma(x_2)) \in \mathbb{R}^2$. The reason for the choice of this embedding is that, in $\mathbb{Q}_2$, multiplying by 2 has in some sense the same effect as dividing by 2 in $\mathbb{R}$. We note also that in [78] a very similar embedding was used to illustrate rational self-affine tiles.

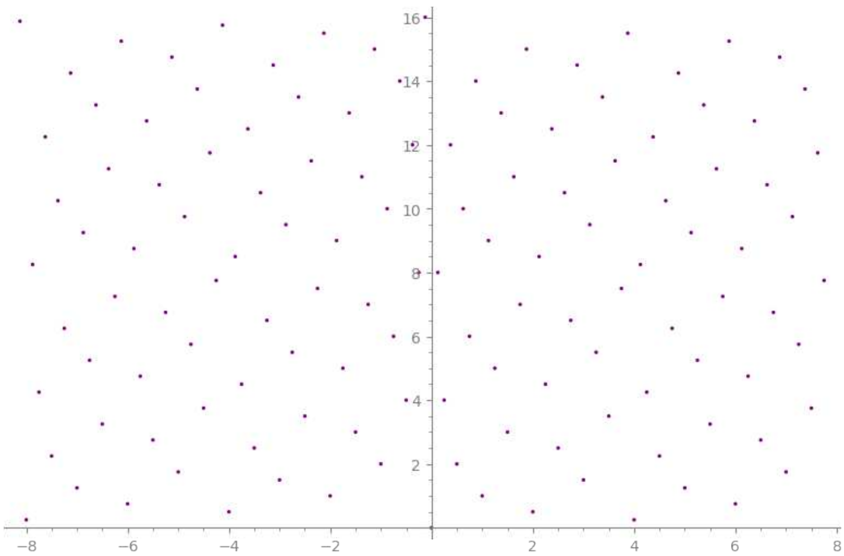


FIGURE 2.1: Representation of the lattice points $\varphi(\mathbb{Z}[\frac{1}{2}]) \in \mathbb{K}$ using the embedding $\gamma : \mathbb{Q}_2 \to \mathbb{R}$.

The lattice $\varphi(\mathbb{Z}[\frac{1}{2}])$, which will play the role of the "integers" in $\mathbb{K}$, turns out to be a proper translation set for a tiling of $\mathbb{K}$ related to the number system $(-\frac{3}{2}, \mathcal{D})$.

In the real case, when defining a tiling we allowed overlaps as long as it was on a set of Lebesgue measure zero. To generalize this, we need to define a natural measure on $\mathbb{K}$. Such a *Haar measure* is a translation-invariant Borel measure that is finite for compact sets. It can be defined in spaces with a sufficiently "nice" structure. (More specifically, it is defined on locally compact topological groups.) A Haar measure is unique up to a scaling factor. The Lebesgue measure λ on $\mathbb{R}$ is a Haar measure.

Let $\mathbb{Z}_2 = \left\{ \sum_{i=0}^{\infty} c_i 2^i \,:\, c_i \in \{0,1\} \right\} \subset \mathbb{Q}_2$ be the ring of 2-adic integers, and let μ_2 be the Haar measure on $\mathbb{Q}_2$ that satisfies $\mu_2(2^\ell \mathbb{Z}_2) = 2^{-\ell}$. Note that multiplying a set by large powers of 2 makes its measure μ_2 small.

Let $\mu = \lambda \times \mu_2$ be the product measure of λ and μ_2 on $\mathbb{K} = \mathbb{R} \times \mathbb{Q}_2$; that is, if $M_1 \subset \mathbb{R}$ and $M_2 \subset \mathbb{Q}_2$ are respectively measurable, then the sets of the form $M = M_1 \times M_2$ generate the σ-algebra of μ, and $\mu(M) := \lambda(M_1)\mu_2(M_2)$. One can show that μ is a Haar measure on $\mathbb{K}$. For any measurable set $M = M_1 \times M_2 \subset \mathbb{K}$, we have $\lambda(-\frac{3}{2}M_1) = \frac{3}{2}\lambda(M_1)$ and $\mu_2(-\frac{3}{2}M_2) = 2\mu_2(M_2)$,

which yields

$$\mu\left(-\tfrac{3}{2}M\right) = \lambda\left(-\tfrac{3}{2}M_1\right)\mu_2\left(-\tfrac{3}{2}M_2\right) = 3\mu(M). \tag{2.8}$$

Thus multiplying any measurable set $M \subset \mathbb{K}$ by the base $-\frac{3}{2}$ enlarges the measure by the factor 3, which can be interpreted as having "enough space" for three digits.

2.3 A tiling of a non-euclidean space

Recall that in (2.5) we defined the set Ω of real numbers of the form $(0.d_{-1}d_{-2}\ldots)_{-3/2}$. We now embed the digits in $\mathbb{K}$, obtaining the set

$$\mathcal{F} := \left\{ \sum_{i=1}^{\infty} \left(-\tfrac{3}{2}\right)^{-i} \varphi(d_{-i}) \;:\; d_{-i} \in \mathcal{D} \right\}. \tag{2.9}$$

The set $\mathcal{F}$ is a compact subset of $\mathbb{K}$. Indeed, given any sequence in $\mathcal{F}$, we can use a Cantor diagonal argument to find a convergent subsequence. Let $x \in \mathcal{F}$. If we multiply x by the base $-\frac{3}{2}$, we obtain $-\frac{3}{2}x \in \mathcal{F} + \varphi(d_{-1})$ with $d_{-1} \in \{0, 1, 2\}$. Thus $\mathcal{F}$ satisfies the *set equation*

$$-\tfrac{3}{2}\mathcal{F} = \mathcal{F} \cup (\mathcal{F} + \varphi(1)) \cup \big(\mathcal{F} + \varphi(2)\big) \tag{2.10}$$

in $\mathbb{K}$, which can be written more compactly as $-\frac{3}{2}\mathcal{F} = \mathcal{F} + \varphi(\mathcal{D})$. It turns out that this set equation completely characterizes $\mathcal{F}$. Note that (2.10) is equivalent to

$$\mathcal{F} = \left(-\tfrac{2}{3}\right)\mathcal{F} \cup \left(-\tfrac{2}{3}\right)(\mathcal{F} + \varphi(1)) \cup \left(-\tfrac{2}{3}\right)\big(\mathcal{F} + \varphi(2)\big), \tag{2.11}$$

and multiplying by $-\frac{2}{3}$ is a uniform contraction in $\mathbb{K}$: it is a contraction in $\mathbb{R}$ because $|-\frac{2}{3}| < 1$ and also in $\mathbb{Q}_2$ because $|-\frac{2}{3}|_2 = \frac{1}{2} < 1$. Thus (2.11) states that $\mathcal{F}$ is equal to the union of three contracted copies of itself. Due to this contraction property, we may apply Hutchinson's theorem (see [39]) which says that there exists a unique non-empty compact subset of $\mathbb{K}$ that satisfies the set equation (2.11). The set $\mathcal{F}$ is an example of a *rational self-affine tile* in the sense of [78].

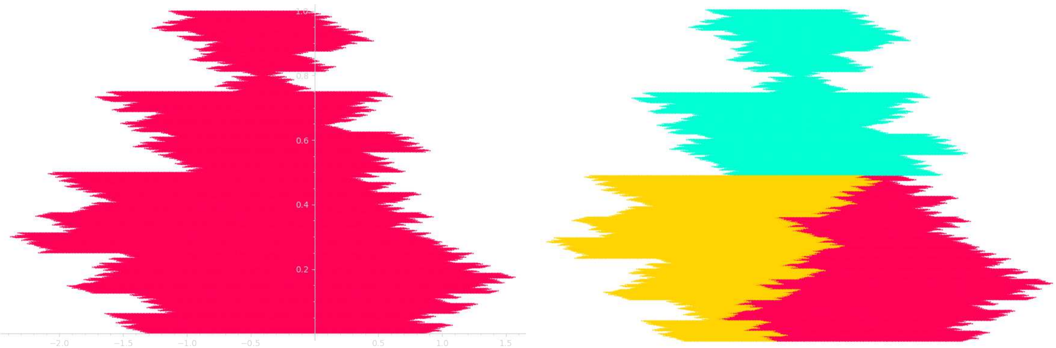


FIGURE 2.2: The tile $\mathcal{F}$ related to the number system with base $-\frac{3}{2}$.

Figure 2.2 (left) shows a representation of $\mathcal{F}$ in $\mathbb{R}^2$, again using the function γ from (2.7) to map $\mathbb{Q}_2$ to $\mathbb{R}$. The right-hand side illustrates the self-affinity of $\mathcal{F}$. The copies of $\mathcal{F}$ appear to have different shapes; this is due to the embedding of $\mathbb{K}$ in $\mathbb{R}^2$, and there is essentially no way to avoid this. When depicting p-adic spaces or ultrametric spaces, there is always something "lost in translation". We color red the points such that $d_{-1} = 0$, green when $d_{-1} = 1$, and yellow when $d_{-1} = 2$. Note how the green tile is on top because these points have residue one modulo two, and hence are further away from zero in the 2-adic distance.

Since according to Theorem 2.1.1 the set $\mathbb{Z}\left[\frac{1}{2}\right]$ is the analog of $\mathbb{Z}$ in the NST number system, we define the analog of the collection in (1.1) by setting

$$\mathcal{C} = \{\mathcal{F} + \varphi(z) \,:\, z \in \mathbb{Z}\left[\tfrac{1}{2}\right]\}.$$

Then $\mathcal{C}$ is a collection of copies of $\mathcal{F}$ translated by elements of the lattice $\varphi(\mathbb{Z}\left[\frac{1}{2}\right])$. We will show in Theorem 2.3.3 that $\mathcal{C}$ is a *tiling* of $\mathbb{K}$, meaning that:

1. $\mathcal{C}$ is a *covering* of $\mathbb{K}$, i.e., $\langle \mathcal{C} \rangle = \mathbb{K}$, where $\langle \mathcal{C} \rangle = \mathcal{F} + \varphi(\mathbb{Z}\left[\frac{1}{2}\right])$ is the union of the elements of $\mathcal{C}$.
2. Almost every point in $\mathbb{K}$ (with respect to the measure μ) is contained in exactly one element of $\mathcal{C}$. In particular, the elements of $\mathcal{C}$ have disjoint interiors.

In Figure 2.3, we have a subdivision of $\mathcal{F}$ into nine smaller copies of itself, which again appear to be different because of the way the embedding is defined. Or, equivalently, we can say that we have placed nine copies of $\mathcal{F}$ to form an enlarged version of $\mathcal{F}$. This is an illustration of the idea behind the proof of the existence of a tiling because the union of these nine copies can be seen as a patch of a covering of the whole space $\mathbb{K}$. The color choice in this example is not related to the digits.

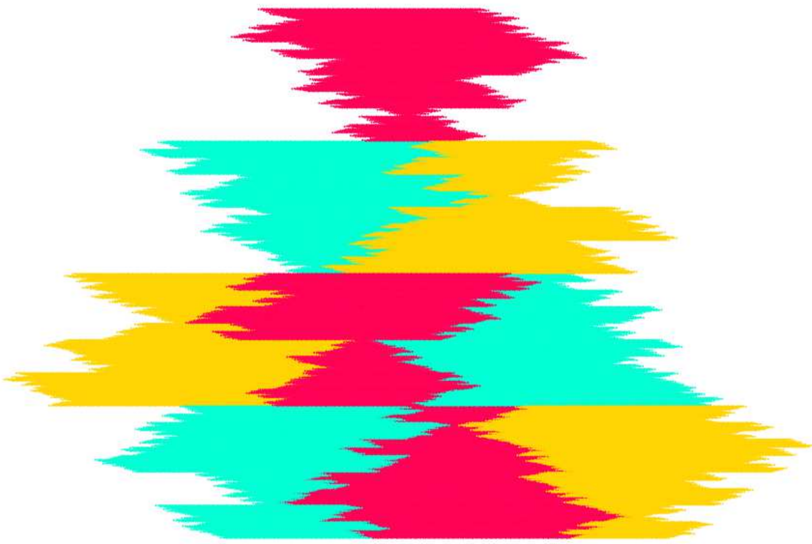


FIGURE 2.3: A patch of the tiling $\mathcal{C}$ of $\mathbb{K}$ by translates of $\mathcal{F}$.

To define a tiling, it is necessary that our central tile $\mathcal{F}$ has reasonably nice topological properties. In particular, we prove that $\mathcal{F}$ is the closure of its interior and that its boundary $\partial\mathcal{F}$ has measure zero. In a general setting, this result is contained in [78, Theorem 1].

Theorem 2.3.1. *$\mathcal{F}$ is the closure of its interior.*

Proof. We first prove that $\mathcal{C}$ is a covering of $\mathbb{K}$, i.e., $\langle\mathcal{C}\rangle = \mathbb{K}$, where $\langle\mathcal{C}\rangle$ is the union of the elements of $\mathcal{C}$. Applying the set equation (2.10), we obtain the following.

$$-\tfrac{3}{2}\langle\mathcal{C}\rangle = -\tfrac{3}{2}\mathcal{F} - \tfrac{3}{2}\varphi\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) = \mathcal{F} + \varphi(\mathcal{D}) - \tfrac{3}{2}\varphi\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big).$$

Note that $\mathcal{D}$ is a complete set of representatives of residue classes of $\mathbb{Z}\big[\frac{1}{2}\big]/(-\frac{3}{2})\mathbb{Z}\big[\frac{1}{2}\big]$, so $\varphi(\mathcal{D}) - \frac{3}{2}\varphi(\mathbb{Z}\big[\frac{1}{2}\big]) = \varphi(\mathbb{Z}\big[\frac{1}{2}\big])$. Thus $-\frac{3}{2}\langle\mathcal{C}\rangle = \langle\mathcal{C}\rangle$ and, a fortiori, for any $k \in \mathbb{N}$ we have $(-\frac{2}{3})^k\langle\mathcal{C}\rangle = \langle\mathcal{C}\rangle$. Recall that multiplying by $-\frac{2}{3}$ is a contraction in $\mathbb{K}$. We have shown in Lemma 2.2.1 that $\varphi(\mathbb{Z}\big[\frac{1}{2}\big])$ is a relatively dense set in $\mathbb{K}$, and therefore so is $\langle\mathcal{C}\rangle$, meaning there is some $R > 0$ for which every closed ball of radius R intersects $\langle\mathcal{C}\rangle$. But since $\langle\mathcal{C}\rangle$ is invariant under contractions by $(-\frac{2}{3})^k$, this implies that any ball of radius $(\frac{2}{3})^k R$ with $k \in \mathbb{N}$ intersects $\langle\mathcal{C}\rangle$; hence, it is dense in $\mathbb{K}$.

Consider now an arbitrary ambinumber $x \in \mathbb{K}$ and a bounded neighborhood V of x. Since $\varphi(\mathbb{Z}\big[\frac{1}{2}\big])$ is uniformly discrete and V is bounded, V intersects only a finite number of translates of $\mathcal{F}$, each of which is compact. Since $\langle\mathcal{C}\rangle$ is dense in $\mathbb{K}$, x cannot be at positive distance from all these translates of $\mathcal{F}$. Thus x is contained in some translate of $\mathcal{F}$ and, hence, $x \in \langle\mathcal{C}\rangle$. Since x was arbitrary, this implies that $\langle\mathcal{C}\rangle = \mathbb{K}$.

Next, we show that $\operatorname{int}\mathcal{F} \neq \varnothing$. Assume on the contrary that $\operatorname{int}\mathcal{F} = \varnothing$. Consider the sets $U_z := \mathbb{K} \setminus (\mathcal{F} + \varphi(z))$ with $z \in \mathbb{Z}\big[\frac{1}{2}\big]$. By assumption, U_z is dense in $\mathbb{K}$ for each $z \in \mathbb{Z}\big[\frac{1}{2}\big]$, and $\{U_z \mid z \in \mathbb{Z}\big[\frac{1}{2}\big]\}$ is a countable collection. Baire's theorem asserts that a countable intersection of dense sets is dense. But

$$\bigcap_{z\in\mathbb{Z}[\frac{1}{2}]} U_z = \mathbb{K} \setminus \bigcup_{z\in\mathbb{Z}[\frac{1}{2}]} \mathcal{F} + \varphi(z) = \mathbb{K} \setminus \langle\mathcal{C}\rangle = \varnothing,$$

which is clearly not dense. This contradiction yields $\operatorname{int}\mathcal{F} \neq \varnothing$.

We now prove the result. Iterating the set equation (2.10) for $k \in \mathbb{N}$ times yields

$$\mathcal{F} = \big(-\tfrac{2}{3}\big)^k \mathcal{F} + \big(-\tfrac{2}{3}\big)^k \varphi(\mathcal{D}) + \big(-\tfrac{2}{3}\big)^{k-1} \varphi(\mathcal{D}) + \cdots + \big(-\tfrac{2}{3}\big)\varphi(\mathcal{D}).$$

Setting

$$\mathcal{D}_k := \mathcal{D} + \big(-\tfrac{3}{2}\big)\mathcal{D} + \cdots + \big(-\tfrac{3}{2}\big)^{k-1}\mathcal{D}, \tag{2.12}$$

this becomes

$$\mathcal{F} = (-\tfrac{2}{3})^k\big(\mathcal{F} + \varphi(\mathcal{D}_k)\big) \qquad (k \in \mathbb{N}), \tag{2.13}$$

which means we can write $\mathcal{F}$ as a finite union of arbitrarily small shrunken and translated copies of itself. We know that $\mathcal{F}$ has an inner point x, and therefore each copy of the form $(-\frac{2}{3})^k(\mathcal{F} + \varphi(d))$, $d \in \mathcal{D}_k$, has an inner point. Thus for any $y \in \mathcal{F}$ and any $\varepsilon > 0$ we can choose $k \in \mathbb{N}$ and $d \in \mathcal{D}_k$ so that $\operatorname{diam}((-\frac{2}{3})^k(\mathcal{F} + \varphi(d))) < \varepsilon$ and $y \in (-\frac{2}{3})^k(\mathcal{F} + \varphi(d))$. Thus there is an inner point at distance less than ε from y. Since $y \in \mathcal{F}$ and $\varepsilon > 0$ were arbitrary, this proves the result. $\square$

Theorem 2.3.2. *The boundary of $\mathcal{F}$ has zero measure.*

Proof. Let x be an inner point of $\mathcal{F}$ and $B_\varepsilon(x) \subset \mathcal{F}$ an open ball of radius $\varepsilon > 0$ centered at x. Because multiplication by $-\frac{2}{3}$ is a uniform contraction in $\mathbb{K}$, there is $k \in \mathbb{N}$ such that $\operatorname{diam}(-\frac{2}{3})^k\mathcal{F} < \varepsilon$. Thus by (2.13) there is $d_0 \in \mathcal{D}_k$ such that

$$\left(-\tfrac{2}{3}\right)^k (\mathcal{F} + \varphi(d_0)) \subset B_\varepsilon(x) \subset \operatorname{int} \mathcal{F}.$$

Let $y \in \partial\big(\left(-\frac{2}{3}\right)^k (\mathcal{F}+\varphi(d_0))\big) \subset \operatorname{int}\mathcal{F}$. Since y is also an inner point of $\mathcal{F}$, and (2.13) exhibits $\mathcal{F}$ as a finite union of compact sets, y must lie in $(-\frac{2}{3})^k(\mathcal{F} + \varphi(d))$ for some $d \in \mathcal{D}_k \setminus \{d_0\}$. Thus the boundary $\partial((-\frac{2}{3})^k(\mathcal{F} + \varphi(d_0)))$ is covered at least twice by the collection $\{(-\frac{2}{3})^k(\mathcal{F} + \varphi(d)) \,:\, d \in \mathcal{D}_k\}$. This entails that

$$\begin{aligned}\mu(\mathcal{F}) &= \mu\Big(\bigcup_{d\in\mathcal{D}_k} \left(-\tfrac{2}{3}\right)^k (\mathcal{F} + \varphi(d))\Big)\\ &\leqslant \sum_{d\in\mathcal{D}_k} \mu\big(\left(-\tfrac{2}{3}\right)^k (\mathcal{F} + \varphi(d))\big) - \mu\big(\partial\big(\left(-\tfrac{2}{3}\right)^k (\mathcal{F} + \varphi(d_0))\big)\big).\end{aligned}$$

Note that (as a Haar measure) μ is translation invariant, the cardinality of $\mathcal{D}_k$ is 3^k, and from (2.8) it follows that $\mu((-\frac{2}{3})^k\mathcal{F}) = 3^{-k}\mu(\mathcal{F})$. All this combined yields

$$\mu(\mathcal{F}) \leqslant \sum_{d\in\mathcal{D}_k} \mu\big(\left(-\tfrac{2}{3}\right)^k \mathcal{F}\big) - \mu\big(\partial\big(\left(-\tfrac{2}{3}\right)^k \mathcal{F}\big)\big) \leqslant \mu(\mathcal{F}) - \mu\big(\partial\big(\left(-\tfrac{2}{3}\right)^k \mathcal{F}\big)\big)$$

and therefore $\mu(\partial((-\frac{2}{3})^k\mathcal{F})) = 0$. This implies that $\mu(\partial\mathcal{F}) = 0$. $\square$

This section contains a tiling theorem for the NST number system. This result is contained in [78, Theorem 2] in a more general setting. For our special case, the proof is much simpler. We will also show that the tiling property relates to the uniqueness (almost everywhere) of NST expansions in the $(-\frac{3}{2})$-number system embedded in $\mathbb{K}$.

Theorem 2.3.3. *The collection* $\mathcal{C} = \{\mathcal{F} + \varphi(z) \,:\, z \in \mathbb{Z}[\frac{1}{2}]\}$ *forms a tiling of* $\mathbb{K}$.

Proof. We have shown in the proof of Theorem 2.3.1 that $\mathcal{C}$ is a covering of $\mathbb{K}$. It remains to show that almost every point of $\mathbb{K}$ is covered by exactly one element of the collection $\mathcal{C}$. Recall that for each $k \geqslant 1$, the sets $\mathcal{D}_k$ (see (2.12)) consist of all the integer NST expansions with at most k digits. According to Theorem 2.1.1, the set $\mathbb{Z}[\frac{1}{2}]$ is the set of all integer NST expansions. This implies that $\mathbb{Z}[\frac{1}{2}] = \bigcup_{k\geqslant 1}\mathcal{D}_k$ and, hence, $\mathbb{K} = \mathcal{F} + \varphi(\mathbb{Z}[\frac{1}{2}]) = \bigcup_{k\geqslant 1}\mathcal{F} + \varphi(\mathcal{D}_k)$. Therefore, it suffices to prove that the collection $\{\mathcal{F} + \varphi(d) \,:\, d \in \mathcal{D}_k\}$ has *essentially disjoint* elements for each $k \geqslant 1$, that is, if $d, d' \in \mathcal{D}_k$ are distinct then $\mu((\mathcal{F} + \varphi(d)) \cap (\mathcal{F} + \varphi(d'))) = 0$. Applying (2.13) we obtain

$$\begin{aligned}3^k\mu(\mathcal{F}) = \mu\left(\left(-\tfrac{3}{2}\right)^k \mathcal{F}\right) = \mu\Big(\bigcup_{d\in\mathcal{D}_k} \mathcal{F} + \varphi(d)\Big) &\leqslant \sum_{d\in\mathcal{D}_k} \mu\left(\mathcal{F} + \varphi(d)\right)\\ &= 3^k\mu(\mathcal{F}).\end{aligned}$$

This implies equality everywhere and, hence, different $\varphi(\mathcal{D}_k)$-translates of $\mathcal{F}$ only overlap in sets of measure zero. Thus the same is true for different $\mathbb{Z}\left[\frac{1}{2}\right]$-translates of $\mathcal{F}$. So the tiles in $\mathcal{C}$ are essentially disjoint, and $\mathcal{C}$ is a tiling. $\square$

Corollary 2.3.4. *Almost every point $x \in \mathbb{K}$ has a unique expansion of the form*

$$x = \sum_{i=-\infty}^{k} \left(-\tfrac{3}{2}\right)^i \varphi(d_i) \qquad (k \in \mathbb{N},\ d_i \in \mathcal{D};\ d_k \neq 0 \text{ whenever } k \geqslant 1). \tag{2.14}$$

Proof. Let $x \in \mathbb{K}$ and suppose it has two different expansions

$$x = \sum_{i=-\infty}^{k} \left(-\tfrac{3}{2}\right)^i \varphi(d_i) = \sum_{i=-\infty}^{k} \left(-\tfrac{3}{2}\right)^i \varphi(d_i'),$$

where $d_k \neq 0$ for $k \geq 1$ and where we pad the second expansion with zeros if necessary. Let $m \leqslant k$ be the largest integer such that $d_m \neq d_m'$, and consider the point $(-\frac{3}{2})^{-m}x$. Recall that multiplying x by $(-\frac{3}{2})^{-m}$ is the analog of moving the decimal point m places to the left if m is positive and to the right if it is negative. Let $\omega := (d_k \dots d_{m+1} d_m)_{-3/2}$ and $\omega' := (d_k' \dots d_{m+1}' d_m')_{-3/2}$. Then $\omega, \omega' \in \mathbb{Z}\left[\frac{1}{2}\right]$ are distinct, and it follows from our assumption and the definition of the tile $\mathcal{F}$ that $(-\frac{3}{2})^{-m}x - \varphi(\omega), (-\frac{3}{2})^{-m}x - \varphi(\omega') \in \mathcal{F}$. Hence, we obtain

$$\left(-\tfrac{3}{2}\right)^{-m} x \in (\mathcal{F} + \varphi(\omega)) \cap (\mathcal{F} + \varphi(\omega')).$$

As tiles only overlap on their boundaries, this implies that $x \in (-\frac{3}{2})^m \partial(\mathcal{F} + \varphi(\omega))$. Therefore, a point $x \in \mathbb{K}$ has two different expansions if and only if $x \in \mathit{\Gamma}$, where $\mathit{\Gamma} := \bigcup_{m\in\mathbb{Z}} (-\frac{3}{2})^m \partial(\mathcal{F} + \varphi(\mathbb{Z}\left[\frac{1}{2}\right]))$. Since $\mathbb{Z}\left[\frac{1}{2}\right]$ is countable, $\mathit{\Gamma}$ is a countable union of the sets $(-\frac{3}{2})^m \partial(\mathcal{F} + \varphi(z))$, $m \in \mathbb{Z}$, $z \in \mathbb{Z}\left[\frac{1}{2}\right]$, each of which has measure 0. Thus $\mu(\mathit{\Gamma}) = 0$, which gives the result. $\square$

2.4 Slices of the central tile

A natural question that can arise at this point is: what can we say about the shape of $\mathcal{F}$? We can partially answer that by looking at its *slices*. By a slice, we mean the following: consider the space $\mathbb{R} \times \{0\} \subset \mathbb{K}$, which is clearly isomorphic to $\mathbb{R}$. Given $z \in \mathbb{Z}\left[\frac{1}{2}\right]$, we can consider the intersection of $\mathcal{F} + \varphi(z)$ and $\mathbb{R} \times \{0\}$, and naturally embed it in $\mathbb{R}$. We denote this set by $\mathcal{G}(z)$, and call the the *slice at height* z. Namely,

$$\mathcal{G}(z) := \{x \in \mathbb{R} : (x, 0) \in \mathcal{F} + \varphi(z)\}. \tag{2.15}$$

It is a consequence of [78, Proposition 6.1] that $\mathcal{F}$ can be seen as a union of such slices, each of them placed in the corresponding 2-adic height, so

$$\mathcal{F} = \bigcup_{z\in\mathbb{Z}\left[\frac{1}{2}\right]} (\mathcal{G}(z)\times\{0\} - \varphi(z)).$$

In particular, it follows from [78, Lemma 6.2] that $\mathcal{G}(z) = \varnothing$ for all $z \in \mathbb{Z}\left[\frac{1}{2}\right]\setminus 2\mathbb{Z}$. A good deal of information can be obtained by carefully examining the congruence properties of fractional expansions. We follow some ideas from Donald Knuth's preprint “Ambidextrous Numbers” [45], where several properties of ambinumbers are displayed. In particular, it is shown in [45, Theorem 10] that the slices of $\mathcal{F}$ are closed real intervals, and algorithms are given to compute their lengths. Moreover, the NST expansion of points of each slice $\mathcal{G}(z)$ is characterized in terms of a directed graph.

Let us start with the slice at height zero. A point $x \in \mathbb{R}$ belongs to $\mathcal{G}(0)$ if and only if $(x, 0) \in \mathcal{F}$. From Corollary 2.3.4, and using that $\mathcal{F}$ is the set of fractional parts, we know that there exists an expansion (that could start with padded zeros) of the form

$$(x,0) = \sum_{i=1}^{\infty} (-\tfrac{3}{2})^{-i}\varphi(d_{-i}), \qquad d_{-i}\in\mathcal{D}.$$

Therefore, x is the limit (in $\mathbb{R}$) of the series

$$\sum_{i=1}^{\infty} \left(-\tfrac{3}{2}\right)^{-i} d_{-i}. \tag{2.16}$$

Moreover, the series (2.16) converges to 0 in the 2-adic metric, due to the definition of $\mathcal{G}(0)$. We claim that this holds if and only if, for every $n \geq 1$,

$$\Big|\sum_{i=1}^{n} \left(-\tfrac{3}{2}\right)^{-i} d_{-i}\Big|_2 \leq \frac{1}{2^{n+1}}. \tag{2.17}$$

Suppose this was not true, then we could find some n so that

$$\Big|\sum_{i=1}^{n} \left(-\tfrac{3}{2}\right)^{-i} d_{-i}\Big|_2 \geq \frac{1}{2^{n}}.$$

Note that $d_{-n-1}(-\frac{3}{2})^{-n-1} + d_{-n-2}(-\frac{3}{2})^{-n-2} + \ldots$ is a (rational) multiple of 2^{n+1}, hence this last tail of the expansion will not affect the value of the 2-adic absolute value, yielding $|\sum_{i=1}^{\infty}\left(-\frac{3}{2}\right)^{-i} d_{-i}|_2 \geq \frac{1}{2^n}$, a contradiction.

In other words, we can establish conditions on the digits to ensure that their 2-adic absolute value is small enough, and we use this idea to express $\mathcal{G}(0)$. Equation (2.17) can be rephrased as 2^{n+1} divides $\sum_{i=1}^{n}(-\frac{3}{2})^{-i}d_{-i}$, which holds if and only if 2 divides $\sum_{i=1}^{n}(-\frac{3}{2})^{n-i}d_{-i}$ (here, the notion of division is in a rational sense). This yields

$$\mathcal{G}(0) = \Big\{ \sum_{i=1}^{\infty} \left(-\tfrac{3}{2}\right)^{-i} d_{-i} \in \mathbb{R} \,:\, d_{-i} \in \mathcal{D} \text{ and } \sum_{i=1}^{n} (-\tfrac{3}{2})^{n-i} d_{-i} \in 2\mathbb{Z} \text{ for all } n \geq 1 \Big\}.$$

Consider the map

$$T_{-3/2} : \mathbb{Z}\left[\tfrac{1}{2}\right] \to \mathbb{Z}\left[\tfrac{1}{2}\right], \qquad x \mapsto -\tfrac{2}{3}(x-d),$$

where d is the unique element of $\mathcal{D}$ such that $T_{-3/2}(x) \in \mathbb{Z}[\frac{1}{2}]$. It is known as the *backward division mapping*. Note that $T_{-3/2}(x) = y$ if $x = -\frac{3}{2}y + d$. Moreover, this map is related to the digits of the integer NST expansion of dyadic rationals that we used in Theorem 2.1.1.

It follows from [78, Lemma 6.11] that $\mathcal{G}(0)$ is the Hausdorff limit

$$\mathcal{G}(0) = \operatorname*{Lim}_{k\to\infty} (-\tfrac{3}{2})^{-k} (T_{-3/2}^{-k}(0) \cap 2\mathbb{Z}).$$

In general, for $y \in 2\mathbb{Z}$ we get

$$\mathcal{G}(y) = \operatorname*{Lim}_{k\to\infty} (-\tfrac{3}{2})^{-k} (T_{-3/2}^{-k}(y) \cap 2\mathbb{Z}).$$

As a direct consequence, we have for every $z \in 2\mathbb{Z}$ that

$$\mathcal{G}(z) = \bigcup_{y \in T_{-3/2}^{-1}(z)} -\tfrac{2}{3}\mathcal{G}(y).$$

Moreover, the collection $\{\mathcal{G}(y) \,:\, y \in 2\mathbb{Z}\}$ forms a tiling of $\mathbb{R}$ (see [78, Theorem 3]).

The study of the slices of $\mathcal{F}$ can be made more precise with the help of *shift radix systems*, or SRS for short. These are simple dynamical systems that provide a unified notion for several types of number systems and admit an interesting tiling theory (see [44] for a complete survey). In particular, Knuth's Twin Dragon from Figure 1.1 is an example of a *shift radix system tile*. For a given parameter $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$, a shift radix system (abbreviated SRS) is a map

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \qquad (x_1, \ldots, x_d) \mapsto (x_2, \ldots, x_d, -\lfloor r_1 x_1 + \cdots + r_d x_d \rfloor),$$

where $\lfloor y \rfloor = \max\{n \in \mathbb{Z} \,:\, n \leq y\}$ is the *floor function*. The NST number system is related to the shift radix system $\tau_{2/3} : \mathbb{Z} \to \mathbb{Z}$, which is given by

$$\tau_{2/3}(N) = -\left\lfloor \tfrac{2}{3} N \right\rfloor.$$

It can be seen by some simple computations that

$$T_{-3/2}^{-1}(y) \cap 2\mathbb{Z} = \tau_{2/3}^{-1}(y).$$

We define the set

$$\mathcal{T}_{2/3}(y) := \operatorname*{Lim}_{k\to\infty} (-\tfrac{3}{2})^{-k} (\tau_{2/3}^{-k}(y)).$$

This is an example of an SRS tile. Then

$$\mathcal{G}(2y) = 2\mathcal{T}_{2/3}(y).$$

This implies that we can approximate $\mathcal{G}(y)$ by taking the preimages of a map. For a large k, it is much faster to compute $\tau_{2/3}^{-k}(y)$ than to compute all points of $T_{-3/2}^{-k}(y)$ and then select the ones that belong to $2\mathbb{Z}$. Later on, when we deal with examples in more dimensions and compute pictures, this becomes a huge advantage because it saves hours of compiling time and allows for much more accurate pictures of these sets. We will see in what follows that very interesting shapes arise in greater dimensions.

2.5 Applications in number theory

We relate our object of study to some problems in number theory.

Mahler's Problem.

Rational base number systems can be used to shed some light on *Mahler's $\frac{3}{2}$-problem* (see [59]), a well-known question in number theory concerning the distribution of powers of rationals modulo 1. It is formulated as follows: We say that $z \in \mathbb{R}$ is a *Z-number* if the sequence of fractional parts $\{z(\frac{3}{2})^n\}$, $n \geq 1$, is contained in the interval $[0, \frac{1}{2}]$. The (still open) question stated by Mahler is: Do Z-numbers exist? In [59], Mahler proves that the set of Z-numbers is at most countable.

More generally, given a rational number $\frac{p}{q}$ with $|\frac{p}{q}| > 1$ and a set $I \subsetneq [0,1]$, one can ask for which $z \in \mathbb{R}$ the sequence $\{z(\frac{p}{q})^n\}$, $n \geq 1$, is *eventually contained* in I, meaning $\{z(\frac{p}{q})^n\} \in I$ for all sufficiently large n. This can be answered for particular sets I using rational base number systems (see [3]). Indeed, consider the number system $(\frac{3}{2}, \mathcal{D})$ with $\mathcal{D} = \{0,1,2\}$; it is defined analogously as $(-\frac{3}{2}, \mathcal{D})$. Associated with this number system is the tile $\mathcal{F}(\frac{3}{2}, \mathcal{D})$, defined by replacing $-\frac{3}{2}$ by $\frac{3}{2}$ in the set equation (2.9). A *k-subtile* is an element of the k-th iteration of the set equation of $\mathcal{F}(\frac{3}{2}, \mathcal{D})$. Let

$$Z_{-3/2}(I) = \left\{ z \in \mathbb{R} \;\middle|\; \left\{ z \left(-\tfrac{3}{2}\right)^n \right\} \text{ is eventually contained in } I \right\}.$$

By [3, Theorem 49] the set $Z_{-3/2}([0,\frac{1}{3}]\cup[\frac{2}{3},1])$ is countably infinite and equals the set of all $z\in\mathbb{R}$ such that $(z,0)\in\mathbb{K}$ has multiple $\frac{3}{2}$-expansions. (The "$-$" sign makes no difference because $[0,\frac{1}{3}]\cup[\frac{2}{3},1]\pmod 1$ is symmetric around 0.) With our theory we can give a geometric interpretation of this result! Indeed, analogously to Corollary 2.3.4, one can show that points with multiple $\frac{3}{2}$-expansions correspond to points on the real line that are located in more than one *k-subtile* of $\mathcal{F}(\frac{3}{2},\mathcal{D})$ for some $k\in\mathbb{N}$. Thus $Z_{-3/2}([0,\frac{1}{3}]\cup[\frac{2}{3},1])$ consists of all points in $\mathbb{R}$ in which two k-subtiles of $\mathcal{F}(\frac{3}{2},\mathcal{D})$ meet, and we arrive at the following result.

Corollary 2.5.1. *The sequence* $(\{z(-\frac{3}{2})^n\})_{n\geq 1}$ *stays eventually in* $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$ *if and only if* $(z,0)\in\mathbb{K}$ *is contained in more than one k-subtile of* $\mathcal{F}(\frac{3}{2},\mathcal{D})$ *for some* $k\in\mathbb{N}$.

Josephus Problem.

This famous riddle goes back to the Jewish historian Flavius Josephus (see [30, Book 3, Chapter 8, Part 7]), and has a simple formulation: given two positive integers m and p, consider a group of m people standing in a circle, numbered clockwise from 1 to m. Starting at position 1, the first $p-1$ people are skipped, and the pth person is executed. The procedure is repeated with the remaining people, starting with the next person, going clockwise, and skipping $p-1$ people, until only one person remains, who is freed. Where should a person be positioned in the initial circle in order to avoid execution? We denote the answer to this question by $J_p(m)$.

The solution is not hard for $p=2$, that is when every other person gets executed, and it is given explicitly by $J_2(m)=1+2(m-2^{\lfloor\log_2(m)\rfloor})$. In the case $p=3$, according to [64], the solution is given by

$$J_3(m)=3m+1-\left\lfloor K(3)\left(\frac{3}{2}\right)^{\left\lceil \log_{3/2}\frac{2m+1}{K(3)}\right\rceil}\right\rfloor,$$

where $\lceil y\rceil=\min\{n\in\mathbb{Z}: n\geq y\}$ is the *ceiling function*, and $K(3)=1.62227\ldots$ is a constant. Interestingly, the value of $K(3)$ is related to SRS tiles: consider the map $\tau_{-2/3}(N)=-\lfloor -\frac{2}{3}N\rfloor$ and the SRS tile $\mathcal{T}_{-2/3}(0):=\mathrm{Lim}_{k\to\infty}(-\frac{3}{2})^{-k}(\tau_{-2/3}^{-k}(0))$. (Just as in the NST case, the SRS tile $\mathcal{T}_{-2/3}(0)$ is half the length of the intersective tile $\mathcal{G}(0)$ of $\mathcal{F}(\frac{3}{2},\mathcal{D})$). Then $K(3)=\lambda(\mathcal{T}_{-2/3})$ (see [78, Section 2] and [3, Section 4.4 and Theorem 2]).

2.6 Rational base number systems

In the last section of this chapter, we consider number systems with any rational base. We generalize the theory that we developed for the NST number theory and state some results.

Let $\frac{a}{b}\in\mathbb{Q}$ with a and b coprime integers, $|a|\geq b\geq 2$. A complete residue set of $\mathbb{Z}[\frac{a}{b}]\mod\frac{a}{b}$ is given by $\mathcal{D}=\{0,\ldots,|a|-1\}$. By taking the base $\frac{a}{b}$ and the digit set $\mathcal{D}$, can we define a number system that has the tiling property? Can we define a suitable representation space as in the $-\frac{3}{2}$ case? The answer is yes, however, it is perhaps not obvious at first.

We recall some notions on p-adic numbers first; for more on this topic, we refer the reader to [38]. Let p be a (positive) prime number. The p-adic valuation is defined on $\mathbb{Q}$ as follows: given a non-zero rational number x, we write it uniquely as $y = p^\ell \frac{s}{t}$ where $s, t \in \mathbb{Z}$ are coprime with p, and $\ell \in \mathbb{Z}$. We define on $\mathbb{Q}$ the p-adic valuation $\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ as

$$\nu_p(y) := \begin{cases} \ell, & y \neq 0, \\ \infty, & y = 0. \end{cases} \tag{2.18}$$

On $\mathbb{Q}$ the *p-adic absolute value* is defined by

$$|y|_p := p^{-\nu_p(y)}, \tag{2.19}$$

with the convention that $p^{-\infty} = 0$. We mention that the p-adic absolute value is not a norm because it does not satisfy the property of homogeneity. We define the *field of p-adic numbers* $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. Every non-zero $y \in \mathbb{Q}_p$ can be written uniquely as a series

$$y = \sum_{i=\nu_p(y)}^{\infty} c_i p^i \qquad (c_i \in \{0, \dots, p-1\},\ c_{\nu_p(y)} \neq 0). \tag{2.20}$$

This series converges in $\mathbb{Q}_p$. Here $\nu_p(y)$ extends the p-adic valuation to all of $\mathbb{Q}_p$, and naturally (2.19) holds in $\mathbb{Q}_p$. The *p-adic metric* is defined on $\mathbb{Q}_p$ by

$$\mathbf{d}_p(y, y') := |y - y'|_p = p^{-\nu_p(y-y')}. \tag{2.21}$$

The p-adic metric satisfies the *ultrametric inequality* (also known as the strong triangle inequality), namely

$$\mathbf{d}_p(y, y') \leqslant \max\{\mathbf{d}_p(y, y''), \mathbf{d}_p(y'', y')\}$$

for every $y, y', y'' \in \mathbb{Q}_p$. This ultrametric property will be extremely important in what follows, in particular when we come up with ways to visualize rational self-affine tiles. The reason is that points in ultrametric spaces are not to be thought of as lying in a continuous, Euclidean-like space; instead, they can be seen as having different orders of magnitude: the metric is defined in terms of a valuation, and the valuation takes discrete values.

The field of p-adic numbers is a complete (ultra) metric space, which is also a locally compact abelian group. Moreover, it is totally disconnected. The ring of integers of $\mathbb{Q}_p$ is denoted by $\mathbb{Z}_p$ and called the ring of p-adic integers, which are exactly the points $y \in \mathbb{Q}_p$ such that $\nu_p(y) \geq 0$. There is a Haar measure μ_p on $\mathbb{Q}_p$ which is normalized in a way that $\mu_p(\mathbb{Z}_p) = 1$, and we call it the *p-adic measure*: it is translation invariant, finite on compact sets, and positive on open sets. If $M \subset \mathbb{Q}_p$ is a measurable set, then, for every $k \in \mathbb{Z}$,

$$\mu_p(p^k M) = p^{-k}\mu_p(M). \tag{2.22}$$

Back to the number system in base $\frac{a}{b}$ and digits $\mathcal{D} = \{0, \dots, |a| - 1\}$, suppose that $b = p$ is a prime number. Then we can consider the representation space $\mathbb{R} \times \mathbb{Q}_p$, and everything works the same as in our example from the previous section, with the only difference that we may need the minus sign at the beginning of negative numbers' expansions whenever a is positive. The tiling property and unique expansion almost everywhere hold.

If b is not a prime number, it is perhaps not so obvious how to proceed. Imagine that we have $b = 6$: what would be the 6-adic absolute value of, say, $x = \frac{2}{3}$? For non-prime b, there are two different yet equivalent ways to proceed in the definition of the representation space. One is by looking at the product of all $\mathbb{Q}_p$ for p dividing b. This was done in [78] in a more general context, and we begin by explaining it in this form. The second, equivalent way, is to look at what we will call b-adic numbers. We can define a space $\mathbb{Q}_b$ also for non-prime $b \geq 2$, however, this space is no longer a field. To show the equivalence between the two spaces, we introduce the notion of projective or inverse limits, which will be crucial in what follows. We present both ways to explain this in order to establish a bridge between the setting that has already been defined and studied and the new setting that we have worked out.

The b-adic numbers.

Let $b \in \mathbb{Z}$, $b \geq 2$ and suppose

$$b = p_1^{r_1} \cdots p_k^{r_k}$$

for distinct positive primes $p_1, \dots, p_k$ and $r_1, \dots, r_k \geq 1$. Consider the fields $\mathbb{Q}_{p_1}, \dots, \mathbb{Q}_{p_k}$ endowed with their respective metrics and measures. Then the product $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_k}$ inherits the product topology and product measure. The ring $\mathbb{Z}[\frac{a}{b}]$ can be diagonally embedded via

$$\varphi : \mathbb{Z}[\tfrac{a}{b}] \to \mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_k}, \qquad z \mapsto (z, \dots, z).$$

Note how the exponents r_j do not affect the definition of the space, but only the prime factors do. It holds in fact that $\varphi(\mathbb{Z}[\frac{a}{b}])$ is a lattice in the product topology, yet we will not prove it now, because we will show the results after we have introduced the space $\mathbb{Q}_b$. However, it is quite intuitive after all we have learned about $-\frac{3}{2}$: points of $\mathbb{Z}[\frac{a}{b}]$ that are close to 0 in the Euclidean metric need to have a power of at least one prime p_j in the denominator, making the corresponding p_j-adic absolute value large for this number.

Definition 2.6.1 (b-adic valuation and absolute value)**.** *Define the b-adic valuation*

$$\nu_b : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}, \qquad \nu_b(y) := \min_{p|b}\{\nu_p(y)\}.$$

This induces the b-adic absolute value $|y|_b = b^{-\nu_b(y)}$. We define the space of b-adic numbers $\mathbb{Q}_b$ as the completion of $\mathbb{Q}$ with respect to $|\cdot|_b$.

For example, $\nu_6(\frac{2}{3}) = \min\{\nu_2(\frac{2}{3}), \nu_3(\frac{2}{3})\} = \min\{1, -1\} = -1$. The ring $\mathbb{Q}_b$ is not necessarily a field when b is not prime, and that is why these spaces are not so well known; for our purposes, the field property is irrelevant. The valuation ν_b is extended to all of $\mathbb{Q}_b$, and the ring of integers is denoted $\mathbb{Z}_b$. It follows that $\mathbb{Q}_{p^r} \simeq \mathbb{Q}_p$ and $\mathbb{Z}_{p^r} \simeq \mathbb{Z}_p$ for any prime p and any $r \geq 1$.

For each $n \in \mathbb{N}$, consider the quotients $\mathbb{Z}/b^n\mathbb{Z}$. We can define the canonical projections

$$\pi_n : \mathbb{Z}/b^{n+1}\mathbb{Z} \to \mathbb{Z}/b^n\mathbb{Z}, \qquad x \mapsto x \mod b^n.$$

Therefore, we have the following sequence which is known as a *projective system*:

$$\ldots \longrightarrow \mathbb{Z}/b^{n+1}\mathbb{Z} \xrightarrow{\pi_n} \mathbb{Z}/b^n\mathbb{Z} \xrightarrow{\pi_{n-1}} \mathbb{Z}/b^{n-1}\mathbb{Z} \longrightarrow \ldots \xrightarrow{\pi_1} \mathbb{Z}/b\mathbb{Z}.$$

This entitles the definition of the *projective limit* (or *inverse limit*), given by

$$\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/b^n\mathbb{Z} = \{(y_n)_{n\in\mathbb{N}} \,:\, y_n \in \mathbb{Z}/b^n\mathbb{Z} \text{ and } \pi_n(y_{n+1}) = y_n \text{ for every } n\}.$$

Take $y = (y_n)_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/b^n\mathbb{Z}$. Consider the set of residues $\{0, \ldots, b-1\}$. There are unique $c_0, c_1, \ldots \in \{0, \ldots, b-1\}$ such that

$$\begin{aligned} y_0 &= c_0 \mod b \\ y_1 &= c_0 + bc_1 \mod b^2 \\ &\vdots \\ y_n &= c_0 + \cdots + b^n c_n \mod b^{n+1} \\ &\vdots \end{aligned} \tag{2.23}$$

and thus for every $n \in \mathbb{N}$ it holds $\pi_n(y_{n+1}) = y_n$. Therefore, we can express y as a formal series

$$y = \sum_{i=0}^{\infty} c_i b^i \qquad (c_i \in \{0, \ldots, b-1\}). \tag{2.24}$$

This series converges under the b-adic metric. We get the isomorphism $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/b^n\mathbb{Z} \simeq \mathbb{Z}_b$ (see [63, Proposition 4.5]). The whole space of b-adic numbers can be reached by "shifting" the starting point: it holds that

$$\mathbb{Q}_b = \bigcup_{\ell \geq 1} b^{-\ell}\mathbb{Z}_b \simeq \bigcup_{\ell \geq 1} \varprojlim_{n \in \mathbb{N}} b^{-\ell}(\mathbb{Z}/b^n\mathbb{Z}).$$

We can also regard the elements of $\mathbb{Q}_b$ as Laurent series: each non-zero b-adic number can be expressed uniquely as

$$y = \sum_{i=\nu_b(y)}^{\infty} c_i b^i \qquad (c_i \in \{0, \dots, b-1\},\ c_{\nu_b(y)} \neq 0). \tag{2.25}$$

For example, $\frac{2}{3} = 4 \cdot 6^{-1} + 0 + 0 \cdot 6 + 0 \cdot 6^2 + \cdots$, which shows that $\nu_6(\frac{2}{3}) = -1$ since the 6-adic expansion is unique.

Let $b = p_1^{r_1} \cdots p_k^{r_k}$. We now show an isomorphism $\mathbb{Q}_b \simeq \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_k}$. From the Chinese Remainder Theorem, it holds, for every $n \in \mathbb{N}$, that

$$\mathbb{Z}/b^n\mathbb{Z} = \mathbb{Z}/p_1^{nr_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{nr_k}$$

and it follows using $\mathbb{Q}_{p^r} \simeq \mathbb{Q}_p$ and $\mathbb{Z}_{p^r} \simeq \mathbb{Z}_p$ (the intermediate steps are left for the reader) that

$$\varprojlim_{n \in \mathbb{N}} \mathbb{Z}/b^n\mathbb{Z} \simeq \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p_1^n\mathbb{Z} \times \cdots \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p_k^n\mathbb{Z},$$

yielding $\mathbb{Q}_b \simeq \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_k}$ and $\mathbb{Z}_b \simeq \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$.

For each basis $\frac{a}{b}$, consider the representation space $\mathbb{K}_{\frac{a}{b}} := \mathbb{R} \times \mathbb{Q}_b$. (We use a subindex to distinguish different spaces, so the space $\mathbb{K}$ from before is, in fact, $\mathbb{K}_{-\frac{3}{2}}$). This space is endowed with a metric

$$\mathbf{d}((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|_b\}$$

Let $\mu = \lambda \times \mu_b$ be the product measure on $\mathbb{K}_{\frac{a}{b}}$, where λ is the Lebesgue measure on $\mathbb{R}$ and μ_b is the Haar measure on $\mathbb{Q}_b$ satisfying $\mu_b(\mathbb{Z}_b) = 1$.

The embedding φ can be now written as

$$\varphi : \mathbb{Q} \to \mathbb{K}_{\frac{a}{b}}, \qquad x \mapsto (x, x)$$

Proposition 2.6.2. *The set $\varphi(\mathbb{Z}[\frac{a}{b}])$ is a lattice in $\mathbb{K}_{\frac{a}{b}}$.*

Proof. 1. The fact that $\varphi(\mathbb{Z}[\frac{a}{b}])$ is a group follows from the additive group structure of $\mathbb{Z}[\frac{a}{b}]$ because φ is a group homomorphism.

2. To obtain the uniform discreteness of $\varphi(\mathbb{Z}[\frac{a}{b}])$, we first show that $\mathbf{d}(\varphi(z), \varphi(0)) \geqslant 1$ holds for every non-zero $z \in \mathbb{Z}[\frac{a}{b}]$. Recall that $\mathbf{d}(\varphi(z), \varphi(0)) = \max\{|z|, |z|_b\}$. If $|z| < 1$, there exists a prime p that divides b that appears in the denominator of z, so $\nu_p(z) \leq -1$. By the definition of ν_b, we have $\nu_b(z) \leq -1$, and hence

$|z|_b = b^{-\nu_b(z)} > 1$. Due to the group structure, this implies that the distance between any two elements of $\varphi(\mathbb{Z}[\frac{a}{b}])$ is at least one; therefore $\varphi(\mathbb{Z}[\frac{a}{b}])$ is uniformly discrete.

3. For relative density, consider an arbitrary element $(x_1, x_2) \in \mathbb{K}_{\frac{a}{b}}$. We claim that there exists $z \in \mathbb{Z}[\frac{a}{b}]$ such that $\mathbf{d}(\varphi(z), (x_1, x_2)) \leqslant 2$. Let $z_1 \in \mathbb{Z}$ be one of the integers closest to x_1. If $x_2 = \sum_{i=l}^{\infty} c_i b^i$ with $l \in \mathbb{Z}$ and $c_i \in \{0, \dots, b-1\}$, then set $z_2 = \sum_{i=l}^{-1} c_i b^i \in \mathbb{Z}[\frac{a}{b}]$ (note that $z_2 = 0$ if $l \geqslant 0$). Therefore,

$$\mathbf{d}((x_1, x_2), (z_1, z_2)) = \max\{|x_1 - z_1|, |x_2 - z_2|_b\} \leqslant 1.$$

Now we set $z = z_1 + z_2 \in \mathbb{Z}[\frac{a}{b}]$. Because z_1 is an integer, $|z_1|_b \leqslant 1$, and since $z_2 \in [0,1]$, we have $|z_2| \leqslant 1$. Thus $\mathbf{d}(\varphi(z), (z_1, z_2)) = \max\{|z_2|, |z_1|_b\} \leqslant 1$, and so $\mathbf{d}(\varphi(z), (x_1, x_2)) \leqslant 2$ by the triangle inequality; hence $\varphi(\mathbb{Z}[\frac{a}{b}])$ is relatively dense.

□

Let $\mathcal{H}(K)$ be the family of non-empty compact subsets of $\mathbb{K}_{\frac{a}{b}}$. Consider the map Ψ defined on $\mathcal{H}(K)$ by

$$\Psi : \mathcal{H}(K) \to \mathcal{H}(K), \qquad X \mapsto \bigcup_{d \in \mathcal{D}} \tfrac{b}{a}(X + \varphi(d)). \tag{2.26}$$

Since multiplication by $\frac{b}{a}$ is a contraction in $\mathbb{K}_{\frac{a}{b}}$, the mapping Ψ is a contraction in $\mathcal{H}(\mathbb{K}_{\frac{a}{b}})$ with respect to the induced Hausdorff metric, and the attractor of an iterated function system is the Hausdorff limit of the sequence of compact sets $\Psi^k(X)$, for any compact set X. This implies that the following definition makes sense.

Definition 2.6.3 (Set of fractional parts)**.** *Let $\mathcal{D} = \{0, \dots, |a|-1\}$ be a digit set for the base $\frac{a}{b}$. The set equation on $\mathbb{K}_{\frac{a}{b}}$ given by*

$$\tfrac{a}{b}\mathcal{F} = \bigcup_{d \in \mathcal{D}} (\mathcal{F} + \varphi(d)) \tag{2.27}$$

has a unique non-empty compact solution, which is given explicitly by

$$\mathcal{F} = \mathcal{F}(\tfrac{a}{b}, \mathcal{D}) = \Big\{ \sum_{j=1}^{\infty} (\tfrac{a}{b})^{-j} \varphi(d_j) \,:\, d_j \in \mathcal{D} \Big\}.$$

We say that $\mathcal{F}$ is the set of fractional parts of this number system.

We state some results analogous to the ones in the previous section for the case $\frac{a}{b} = -\frac{3}{2}$. Proofs can be found in [78, Theorems 1 and 2] in a more general case. Moreover, they are direct consequences of the results we state in Chapter 4 in even more generality.

Theorem 2.6.4. *Let $\mathcal{D} = \{0, \ldots, |a| - 1\}$ be a digit set for the base $\frac{a}{b}$ and let $\mathcal{F} = \mathcal{F}(\frac{a}{b}, \mathcal{D})$ be the set of fractional parts. Then $\mathcal{F}$ has positive measure and its boundary $\partial\mathcal{F}$ has measure zero.*

Theorem 2.6.5. *The collection $\mathcal{C} = \{\mathcal{F} + \varphi(z) \,:\, z \in \mathbb{Z}[\frac{a}{b}]\}$ forms a tiling of $\mathbb{K}_{\frac{a}{b}}$.*

Rational number systems can be associated with degree one polynomials $aX - b$ where the zero has modulus greater than 1. In the next chapter, we will consider integer polynomials of arbitrary degree, and define number systems with more dimensions.

Chapter 3

Number systems with algebraic basis

This chapter is devoted to the study of number systems where the base is an expanding algebraic number α, and we call this an *algebraic number system*. We consider digit sets of the form $\mathcal{D} = \{0, 1, \ldots, |a_0| - 1\}$, where a_0 is the independent coefficient of the minimal primitive polynomial of α. We are going to use a lot of machinery from algebraic number theory, and notions regarding the factorization of ideals will play a key role; these topics will be introduced in Section 3.1. We will introduce certain completions of the field $\mathbb{Q}(\alpha)$ that are analogous to p-adic completions of $\mathbb{Q}$. In section 3.2, we will define rational self-affine tiles for algebraic bases as subsets of a representation space $\mathbb{K}_\alpha$. *Rational self-affine tiles* are introduced in [78], and the results from this chapter, as well as the terminology, derive from that paper. However, the theory here is presented with a thorough study of the underlying mathematical notions, numerous pictures, and explicit calculations.

Section 3.3 centers around the example $\alpha = \frac{-1+3i}{2}$ and $\mathcal{D} = \{0, 1, 2, 3, 4\}$. That section is self-contained and the definitions can be read without having to learn all the preliminaries in algebraic number theory because the computation of the representation space will be fairly easier than in the general case. Working with $\mathbb{C}$ instead of $\mathbb{R}$ will result in more interesting pictures. The central tile $\mathcal{F}(\alpha, \mathcal{D})$ for this example can be depicted in three dimensions, and we visualize it as a pile of slices. We will carefully study how these slices distribute to form a tiling of $\mathbb{C}$ and how each can be expressed as a union of contracted copies of other slices, a notion that we call *inter-affinity*.

Finally, in Section 3.4, we relate intersective tiles to shift radix systems, which have huge computational advantages. Moreover, we show how algebraic number systems can be defined in terms of rational matrices and vectors; this motivates Chapter 4, where we will consider arbitrary expanding rational matrices. This shows that the setting of this chapter is contained in the setting of the next one.

3.1 Preliminaries in algebraic number theory

We follow [63] and [83].

Algebraic number fields.

Consider an algebraic number field, that is, a field that is a finite extension of $\mathbb{Q}$. Then it is always of the form $\mathbb{Q}(\alpha)$ for some algebraic number α, which means the smallest field to contain both $\mathbb{Q}$ and α. The degree n of the primitive minimal polynomial P_α is exactly the degree of the extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, which means that $\mathbb{Q}(\alpha)$ is an n-dimensional $\mathbb{Q}$-vector space with basis given by $\{1, \alpha, \ldots, \alpha^{n-1}\}$. The ring of integers of $\mathbb{Q}(\alpha)$ is given by the elements that are algebraic integers. Moreover, $\mathbb{Q}(\alpha) \simeq \mathbb{Q}[X]/(P_\alpha)$.

The field $\mathbb{Q}$ has the property of unique factorization, which means that every element can be uniquely expressed as a product of irreducible elements, which are the prime numbers (sometimes called rational primes to distinguish them from a more general notion of primes) and a unit, which can be ± 1. However, an arbitrary algebraic number field $\mathbb{Q}(\alpha)$ is usually not a unique factorization domain. For that reason, the theory of ideals allows one to define a generalization of the notion of prime factorization by showing that an algebraic number field is a Dedekind domain, and hence every ideal has a unique factorization into prime ideals.

By Ostrowski's theorem [65], every absolute value in $\mathbb{Q}$ is either equivalent to the usual absolute value, denoted by $|\cdot|_\infty$, or it is equivalent to a p-adic absolute value $|\cdot|_p$ for some prime $p \in \mathbb{Z}$ (recall that two absolute values $|\cdot|$ and $|\cdot|'$ are equivalent if $|\cdot|' = |\cdot|^c$ for some $c > 0$). Each prime number p induces a maximal ideal in $\mathbb{Q}$ denoted by (p). We could consider "∞" to be another prime. In other words, we define the set $\mathfrak{M} = \{\infty, 2, 3, \ldots, p, \ldots\}$ as the set of *places* (or *primes*) of $\mathbb{Q}$, each of which corresponds to an equivalent class of absolute values. Each place defines a completion of $\mathbb{Q}$ with respect to the corresponding absolute value, which is given by the field of p-adic numbers $\mathbb{Q}_p$ for each rational prime p, and to $\mathbb{R}$ for the infinite prime ∞.

Places.

Consider an algebraic number $\alpha \in \mathbb{C}$ of degree n, let $K = \mathbb{Q}(\alpha)$ with ring of integers $\mathcal{O}_K$. A *place* or *prime* $\mathfrak{p}$ of K is an equivalent class of absolute values of K. We denote by $\mathfrak{M}$ the set of places of K. An absolute value $|\cdot|$ is said to be archimedean if $\{|n| : n \in \mathbb{N}\}$ is unbounded, and non-archimedean otherwise. A place $\mathfrak{p}$ is called infinite, denoted by $\mathfrak{p} \mid \infty$, if it is a class of archimedean absolute values, and it is called finite, denoted by $\mathfrak{p} \nmid \infty$, if it is a class of non-archimedean absolute values. We can identify $\mathfrak{p}$ with a prime ideal of $\mathcal{O}_K$ noting that there is a prime ideal given by $\{x \in K : |x|_{\mathfrak{p}} < 1\}$ that is independent of the representative $|\cdot|_{\mathfrak{p}}$ and, therefore, we identify this prime ideal with the equivalence class and, with abuse of notation, also denote it by $\mathfrak{p}$. Let $K_\mathfrak{p}$ denote the completion of K with respect to $|\cdot|_\mathfrak{p}$. Its ring of integers is given by $\mathcal{O}_\mathfrak{p} = \{x \in K_\mathfrak{p} : |x|_\mathfrak{p} \leq 1\}$. $K_\mathfrak{p}$ is also a (locally compact) ring.

Let us first study the infinite places of K. Since K is a field extension of $\mathbb{Q}$, and the only infinite prime of $\mathbb{Q}$ is the usual absolute value, each infinite place of K must equal the usual absolute value when restricted to $\mathbb{Q}$. The *Galois conjugates* of α are the n roots of the minimal polynomial of α, which are given by $\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \overline{\alpha_{r+1}}, \ldots, \alpha_{r+s}, \overline{\alpha_{r+s}}$, where $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ are the real conjugates, $\alpha_{r+1}, \overline{\alpha_{r+1}}, \ldots, \alpha_{r+s}, \overline{\alpha_{r+s}} \in \mathbb{C} \setminus \mathbb{R}$ are the imaginary conjugates. Then $r + 2s = n$. An embedding from K to $\mathbb{R}$ or $\mathbb{C}$ (that is, an injective field homomorphism) leaves $\mathbb{Q}$ unchanged and is uniquely determined by the image of α. Consider the *Galois embeddings*

$$\begin{aligned} \tau_j &: K \to \mathbb{R}, \qquad \alpha \mapsto \alpha_j, \qquad 1 \leq j \leq r \\ \tau_j &: K \to \mathbb{C}, \qquad \alpha \mapsto \alpha_j, \qquad r+1 \leq j \leq r+s. \end{aligned} \tag{3.1}$$

When $r + 1 \leq j \leq r + s$ (that is, when the Galois conjugate is imaginary), we can consider also the conjugate maps $\overline{\tau_j}$ such that $\overline{\tau_j}(\alpha) = \overline{\alpha_j}$. Given an embedding $\tau_j : K \to \mathbb{R}$ with $1 \leq j \leq r$, there is an infinite prime $\mathfrak{p}$ of K given by the representative $|x|_\mathfrak{p} := |\tau_j(x)|$, and the completion of K with respect to this absolute value is isomorphic to $\mathbb{R}$. Given an embedding $\tau_j : K \to \mathbb{C}$ for $r+1 \leq j \leq r+s$, there is an infinite prime $\mathfrak{p}$ of K given by the representative $|x|_\mathfrak{p} := \tau_j(x)\overline{\tau_j}(x)$ and the completion of K with respect to this absolute value is $\mathbb{C}$. It turns out that every real embedding $\tau : K \to \mathbb{R}$ is equivalent to τ_j for $1 \leq j \leq r$, and every complex embedding $\tau : K \to \mathbb{C}$ is equivalent to τ_j or $\overline{\tau_j}$ for $r+1 \leq j \leq r+s$. Therefore, we have exactly $r+s$ infinite primes in K. The Galois embeddings form a group called the *Galois group*, which we denote as $\mathrm{Gal}(K/\mathbb{Q})$.

When the primitive polynomial P_α has degree 2, the Galois conjugates of α are α and $\bar{\alpha}$, and hence the Galois embeddings are the identity and the conjugate map. This yields only one archimedean absolute value, given by $|x| = x \cdot \bar{x}$, namely the usual complex norm. The completion of $\mathbb{Q}(\alpha)$ with respect to this absolute value is of course $\mathbb{C}$.

Prime factorization and norms.

We now proceed to study the finite primes of K. For each prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, the quotient $\mathcal{O}_K/\mathfrak{p}$ is a field whose characteristic is a rational prime number p. It holds that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$, so $\mathfrak{p}$ is a divisor of the ideal $p\mathcal{O}_K$; we say in this case that $\mathfrak{p}$ *lies over* $p\mathbb{Z}$. Then $\mathcal{O}_K/\mathfrak{p}$ is a finite extension of $\mathbb{Z}/p\mathbb{Z}$, and we denote by $f_\mathfrak{p}$ the degree of this extension and call it the *degree of inertia* of $\mathfrak{p}$.

Definition 3.1.1 (Norm of a prime ideal). *We define the norm of the prime ideal* $\mathfrak{p}$ *of* $\mathcal{O}_K$ *as* $\mathfrak{N}(\mathfrak{p}) := p^{f_{\mathfrak{p}}}$*, where* p *is the rational prime such that* $\mathfrak{p}$ *lies over* $p\mathbb{Z}$ *and* $f_{\mathfrak{p}}$ *is the degree of inertia.*

If $\mathfrak{q}$ is an ideal of $\mathcal{O}_K$, then there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ and positive integers $e_1, \dots, e_r$ such that

$$\mathfrak{q} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}.$$

The factorization is unique (up to rearrangement). Then the norm of the ideal $\mathfrak{q}$ is given by $\mathfrak{N}(\mathfrak{q}) = \mathfrak{N}(\mathfrak{p}_1)^{e_1} \cdots \mathfrak{N}(\mathfrak{p}_r)^{e_r}$. It follows form the definition of $\mathfrak{N}(\mathfrak{p})$ and the Chinese Remainder Theorem that $\mathfrak{N}(\mathfrak{q}) = |\mathcal{O}_K/\mathfrak{q}|$ (this could alternatively be given as a definition).

The notion of unique decomposition can be also applied to the so-called fractional ideals of $\mathcal{O}_K$: this generalizes the fact that rational numbers can also be expressed uniquely as a product of rational primes if one allows negative exponents. A *fractional ideal* of $\mathcal{O}_K$ is an ideal $\mathfrak{q}$ of K such that there exists $a \in K$ satisfying $a\mathfrak{q} \subset \mathcal{O}_K$. Then there exist two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathcal{O}_K$ such that $\mathfrak{q} = \frac{\mathfrak{a}}{\mathfrak{b}}$. This quotient is unique if we require $(\mathfrak{a}, \mathfrak{b}) = \mathcal{O}_K$, that is if there is no finite prime $\mathfrak{p}$ that is a factor of both $\mathfrak{a}$ and $\mathfrak{b}$. We say that $\mathfrak{a}$ and $\mathfrak{b}$ are coprime, and they satisfy $\mathfrak{a} + \mathfrak{b} = \mathcal{O}_K$. Then we can define the norm of a fractional ideal by setting $\mathfrak{N}(\mathfrak{q}) = \frac{\mathfrak{N}(\mathfrak{a})}{\mathfrak{N}(\mathfrak{b})}$.

Given an invertible element $x \in K^\times$, let $(x) := x\mathcal{O}_K$. Then (x) is a fractional ideal of $\mathcal{O}_K$, and we can express it as a quotient of coprime integer ideals

$$(x) = \frac{\mathfrak{a}}{\mathfrak{b}}.$$

Both $\mathfrak{a}$ and $\mathfrak{b}$ factor in a unique way as a finite product of prime ideals, which yields

$$(x) = \frac{\prod_{\mathfrak{p}|\mathfrak{a}} \mathfrak{p}^{e_{\mathfrak{p}}}}{\prod_{\mathfrak{p}|\mathfrak{b}} \mathfrak{p}^{e_{\mathfrak{p}}}}.$$

If we rewrite this factorization by setting a negative exponent for $\mathfrak{p}$ whenever $\mathfrak{p} \mid \mathfrak{b}$, and by setting the exponent of $\mathfrak{p}$ to be zero whenever $\mathfrak{p}$ does not divide $\mathfrak{a}$ nor $\mathfrak{b}$ (which occurs for almost all primes), then we can find a unique factorization for the ideal (x) as

$$(x) = \prod_{\mathfrak{p} \nmid \infty} \mathfrak{p}^{\nu_{\mathfrak{p}}(x)}. \tag{3.2}$$

This entitles the following definition:

Definition 3.1.2 ($\mathfrak{p}$-adic valuation and absolute value). *Let* $\mathfrak{p}$ *be a prime ideal of* $\mathcal{O}_K$. *We define the* $\mathfrak{p}$*-adic valuation as*

$$\nu_{\mathfrak{p}} : K \to \mathbb{Z}, \qquad x \mapsto \nu_{\mathfrak{p}}(x),$$

where $\nu_{\mathfrak{p}}(x)$ *is given by the exponent of* $\mathfrak{p}$ *in the decomposition of the ideal* (x) *given in* (3.2) *whenever* $x \in K^\times$*, and* $\nu_{\mathfrak{p}}(0) = \infty$*. We define the* $\mathfrak{p}$*-adic absolute value*

$$|\cdot|_{\mathfrak{p}} : K \to \mathbb{R}, \qquad x \mapsto \mathfrak{N}(\mathfrak{p})^{-\nu_{\mathfrak{p}}(x)},$$

with the convention that $|0|_{\mathfrak{p}} = \mathfrak{N}(\mathfrak{p})^{-\infty} = 0$.

In this way, we see how the prime ideal $\mathfrak{p}$ induces the corresponding absolute value (place). Then

$$\mathfrak{N}((x)) = \prod_{\mathfrak{p} \nmid \infty} \mathfrak{N}(\mathfrak{p})^{\nu_{\mathfrak{p}}(x)}.$$

This yields the following important formula: if $\mathfrak{M}$ is the set of all (finite and infinite) places of K, then for every $x \in K$ it holds that

$$\prod_{\mathfrak{p} \in \mathfrak{M}} |x|_{\mathfrak{p}} = 1.$$

Given a prime $\mathfrak{p}$, the corresponding Haar measure $\mu_{\mathfrak{p}}$ in $K_{\mathfrak{p}}$ is defined as follows: When $\mathfrak{p} \mid \infty$, the completion $K_{\mathfrak{p}}$ is either $\mathbb{R}$ or $\mathbb{C}$, which is endowed with the real or complex Lebesgue measure, respectively. When $\mathfrak{p} \nmid \infty$, we let $\mu_{\mathfrak{p}}$ be the Haar measure satisfying $\mu_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}) = 1$ (recall that the Haar measure is unique up to a scaling factor). For each prime $\mathfrak{p}$ of K, for every $x \in K_{\mathfrak{p}}$ and every measurable set $M \subset K_{\mathfrak{p}}$, we have

$$\mu_{\mathfrak{p}}(x \cdot M) = |x|_{\mathfrak{p}} \mu_{\mathfrak{p}}(M).$$

The norm of $x \in K$, seeing K as an extension of $\mathbb{Q}$, is defined as

$$N_{K/\mathbb{Q}}(x) := \prod_{\tau \in \mathrm{Gal}(K/\mathbb{Q})} \tau(x).$$

The norm of a principal ideal of K coincides with the norm of the generator, that is, $\mathfrak{N}((x)) = N_{K/\mathbb{Q}}(x)$.

Moreover, one can also define an *ideal norm* on fractional ideals of $\mathcal{O}_K$ as follows: given a ring R, we denote by $\mathcal{I}_R$ the group of non-zero fractional ideals of R. Each $\mathfrak{q} \in \mathcal{I}_R$ can be written uniquely as $\mathfrak{q} = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$ where $e_{\mathfrak{p}} \in \mathbb{Z}$ and $\mathfrak{p}$ ranges over all prime ideals of R. Then we have the ideal norm

$$N_{K/\mathbb{Q}} : \mathcal{I}_{\mathcal{O}_K} \to \mathcal{I}_{\mathbb{Z}}, \qquad N_{K/\mathbb{Q}}\Big(\prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}\Big) := \prod_{p} \prod_{\mathfrak{p} \mid (p)} p^{f_{\mathfrak{p}} e_{\mathfrak{p}}} \mathbb{Z},$$

where p ranges over all rational primes, $\mathfrak{p} \mid (p)$ means that $\mathfrak{p}$ lies over $p\mathbb{Z}$ and $f_{\mathfrak{p}}$ is the inertia degree. For principal ideals, the ideal norm is compatible with the field norm of a generator, hence

$$N_{K/\mathbb{Q}}((x)) = N_{K/\mathbb{Q}}(x)\mathbb{Z} = \mathfrak{N}((x))\mathbb{Z}.$$

Each embedding τ defined in $\mathbb{Q}(\alpha)$ can naturally be extended to ideals of $\mathbb{Q}(\alpha)$. From [63, Proposition 1.6 (iv)], the ideal norm satisfies

$$N_{K/\mathbb{Q}}(\mathfrak{p})\mathcal{O}_K = \prod_{\tau \in \mathrm{Gal}(K/\mathbb{Q})} \tau(\mathfrak{p}). \tag{3.3}$$

If $P_\alpha = a_n X^n + \ldots + a_1 X + a_0 \in \mathbb{Z}[X]$ is the primitive minimal polynomial of α, then

$$\mathfrak{N}((\alpha)) = N_{K/\mathbb{Q}}(\alpha) = \frac{|a_0|}{a_n} \tag{3.4}$$

(recall that $a_n > 0$ by assumption). The next formula regarding the norm of $\mathfrak{a}$ and $\mathfrak{b}$ is proven following [78, Lemma 3.3].

Lemma 3.1.3. *If* $(\alpha) = \frac{\mathfrak{a}}{\mathfrak{b}}$ *then*

$$\mathfrak{N}(\mathfrak{a}) = |a_0| \text{ and } \mathfrak{N}(\mathfrak{b}) = a_n.$$

Proof. Given a polynomial $f \in K[X]$, we define the content of f, denoted by $c(f)$, as the (fractional) ideal of $\mathcal{O}_K$ generated by the coefficients of f. We have

$$\frac{1}{a_n}\mathcal{O}_K = c\left(\frac{P_\alpha(X)}{a_n}\right)$$

because the coefficients of P_α are coprime by definition. The monic polynomial $\frac{P_\alpha(X)}{a_n} \in \mathbb{Q}[X]$ has the Galois conjugates of α as roots, so we get

$$\frac{P_\alpha(X)}{a_n} = \prod_{\tau \in \mathrm{Gal}(K/\mathbb{Q})} (X - \tau(\alpha)).$$

In our setting, non-zero ideals of $\mathcal{O}_K$ are always invertible, so it follows from the Dedekind-Mertens Lemma (see [6, Theorem 8.1]) that

$$c(fg) = c(f)c(g)$$

for all $f, g \in K[X]$. This yields

$$c\left(\frac{P_\alpha(X)}{a_n}\right) = c\left(\prod_{\tau \in \mathrm{Gal}(K/\mathbb{Q})} (X - \tau(\alpha))\right) = \prod_{\tau \in \mathrm{Gal}(K/\mathbb{Q})} c(X - \tau(\alpha)).$$

Note that

$$c(X - \tau(\alpha)) = \tau(c(X - \alpha)).$$

From (3.3) follows that

$$\prod_{\tau \in \mathrm{Gal}(K/\mathbb{Q})} \tau(c(X - \alpha)) = N_{K/\mathbb{Q}}(c(X - \alpha))\mathcal{O}_K.$$

The content of the polynomial $X - \alpha$ is the ideal $(\mathcal{O}_K, \alpha\mathcal{O}_K)$, and using that $\mathfrak{a}$ and $\mathfrak{b}$ are coprime we get

$$(\mathcal{O}_K, \alpha\mathcal{O}_K) = \left(\mathcal{O}_K, \frac{\mathfrak{a}}{\mathfrak{b}}\right) = \frac{1}{\mathfrak{b}}.$$

Combining everything, we obtain

$$\frac{1}{a_n}\mathcal{O}_K = \prod_{\tau \in \mathrm{Gal}(K/\mathbb{Q})} c(X - \tau(\alpha)) = N_{K/\mathbb{Q}}(c(X-\alpha))\mathcal{O}_K = \mathfrak{N}\left(\frac{1}{\mathfrak{b}}\right)\mathcal{O}_K = \frac{1}{\mathfrak{N}(\mathfrak{b})}\mathcal{O}_K.$$

Hence $\mathfrak{N}(\mathfrak{b}) = a_n$, and from (3.4) follows $\mathfrak{N}(\mathfrak{a}) = |a_0|$. □

As a consequence, we get $|\mathcal{O}_K/\mathfrak{b}| = a_n$.

Adèle rings.

Let K be an algebraic number field and $\mathcal{O}_K$ its ring of integers. Consider the set $\mathfrak{M}$ of all finite and infinite places of K. We will introduce the Adèle ring $\mathbb{A}_K$ of K, which will be used later for defining rational self-affine tiles related to number systems. The space $\mathbb{A}_K$ will have a particular metric so that K will be embedded in it as a Delone set (i.e., a uniformly discrete and relatively dense set, similar to a lattice). The existence of this lattice will allow us to define lattice tilings in certain subsets of Adèle rings. Both the topological and the algebraic properties are of interest.

We would like to embed K into a product of its completions $K_\mathfrak{p}$, in a way that every $x \in K$ is a $\mathfrak{p}$-adic integer for all but finitely many $\mathfrak{p} \in \mathfrak{M}$. And that is exactly how we define $\mathbb{A}_K$. Let

$$\mathbb{A}_K := \left\{(a_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{M}} \in \prod_{\mathfrak{p}\in\mathfrak{M}} K_\mathfrak{p} \,:\, a_\mathfrak{p} \in \mathcal{O}_\mathfrak{p} \text{ for all but finitely many } \mathfrak{p} \in \mathfrak{M}\right\}.$$

This is also known as a restricted product. Note that the restriction of being in the ring of integers for almost all places is only relevant for finite places, since there is a finite number of infinite places.

What is most relevant is the topology assigned to $\mathbb{A}_K$, which is not the subspace topology of the product topology. Instead, we take as a basis of the topology the set

$$\mathcal{B} := \left\{\prod_{\mathfrak{p}\in\mathfrak{M}} U_\mathfrak{p}\right\}$$

where $U_\mathfrak{p} \subset K_\mathfrak{p}$ is open and $U_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ for all but finitely many $\mathfrak{p} \in \mathfrak{M}$. Alternatively, we can define

$$\mathcal{B} = \left\{\prod_{\mathfrak{p}\in S} U_\mathfrak{p} \times \prod_{\mathfrak{p}\notin S} \mathcal{O}_\mathfrak{p} \,:\, S \subset \mathfrak{M} \text{ is finite}\right\}.$$

The Adèle ring $\mathbb{A}_K$ is also a measure space endowed with the product measure of the Haar measures $\mu_\mathfrak{p}$ of each factor. Since K is canonically embedded in each completion $K_\mathfrak{p}$, it can be embedded into $\mathbb{A}_K$ by

$$K \hookrightarrow \mathbb{A}_K, \qquad x \hookrightarrow (x)_{\mathfrak{p}\in\mathfrak{M}}.$$

Then K is embedded into $\mathbb{A}_K$ as a discrete, co-compact set. Moreover, K acts multiplicatively on $\mathbb{A}_K$ by $\xi \cdot (x_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{M}} = (\xi \cdot x_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{M}}$.

Projective limits.

Let $K = \mathbb{Q}(\alpha)$ and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$. Consider the quotients $\mathcal{O}_K/\mathfrak{p}^k$ for $k \in \mathbb{N}$ and the canonical projections

$$\pi_k : \mathcal{O}_K/\mathfrak{p}^{k+1} \to \mathcal{O}_K/\mathfrak{p}^k.$$

Therefore, we have the projective system

$$\dots \longrightarrow \mathcal{O}_K/\mathfrak{p}^{k+1} \xrightarrow{\pi_k} \mathcal{O}_K/\mathfrak{p}^k \xrightarrow{\pi_{k-1}} \dots \xrightarrow{\pi_1} \mathcal{O}_K/\mathfrak{p}.$$

This entitles the definition of the projective limit (or inverse limit), given by

$$\varprojlim_{k\in\mathbb{N}} \mathcal{O}_K/\mathfrak{p}^k = \{y = (y_k)_{k\in\mathbb{N}} \,:\, y_k \in \mathcal{O}_K/\mathfrak{p}^k \text{ and } \pi_k(y_{k+1}) = y_k \text{ for every } k \in \mathbb{N}\}.$$

This definition can be stated in terms of any ideal of $\mathcal{O}_K$.

Lemma 3.1.4. *Let $\mathcal{O}_\mathfrak{p}$ be the ring of integers of $K_\mathfrak{p}$. Then*

$$\mathcal{O}_\mathfrak{p} \simeq \varprojlim_{k\in\mathbb{N}} \mathcal{O}_K/\mathfrak{p}^k$$

Proof. See [63, Proposition 4.5]. □

As a consequence,

$$K_\mathfrak{p} = \bigcup_{\ell\geq 1} \mathfrak{p}^{-\ell}\mathcal{O}_\mathfrak{p} \simeq \bigcup_{\ell\geq 1} \varprojlim_{k\in\mathbb{N}} \mathfrak{p}^{-\ell}(\mathcal{O}_K/\mathfrak{p}^k).$$

Next, we choose a *uniformizer* β for the ideal $\mathfrak{p}$, that is, an element of $\mathbb{Q}(\alpha)$ satisfying $\nu_\mathfrak{p}(\beta) = 1$. Let $\mathcal{E}$ be a complete set of residues of $\mathcal{O}_K \mod \mathfrak{p}$. We can regard the elements of $K_\mathfrak{p}$ as Laurent series: each non-zero $x \in K_\mathfrak{p}$ can be expressed uniquely as

$$y = \sum_{i=\nu_\mathfrak{p}(y)}^{\infty} c_i\beta^i \qquad (c_i \in \mathcal{E},\ c_{\nu_b(y)} \neq 0). \tag{3.5}$$

In the setting where $K = \mathbb{Q}$ and $\mathfrak{p} = p\mathbb{Z}$ for a rational prime p, the projective limit obtained is the field of p-adic numbers. The uniformizer is canonically taken to be p and the residue set is $\mathcal{E} = \{0, 1, \ldots, p-1\}$. For that reason, we will refer to the elements of $K_\mathfrak{p}$ as $\mathfrak{p}$-adic numbers and to the elements of $\mathcal{O}_\mathfrak{p}$ as $\mathfrak{p}$-adic integers.

3.2 Rational self-affine tiles

We have established the necessary preliminaries to arrive at the main definitions of this chapter, namely of representation space and of rational self-affine tiles for the setting of algebraic number systems. We will assume that α is an expanding algebraic number as defined next. We follow the notations of Section 3.1.

Definition 3.2.1 (Expanding algebraic number)**.** *We say that $\alpha \in \mathbb{C}$ is an expanding algebraic number of degree n if α is a root of an irreducible polynomial $P_\alpha(X) = a_n X^n + \ldots + a_1 X + a_0 \in \mathbb{Z}[X]$ and all roots of P_α have modulus greater than* 1.

Assume, as before, that the coefficients of P_α are coprime and the leading coefficient a_n is positive. Consider the number field $K = \mathbb{Q}(\alpha)$ and its ring of integers $\mathcal{O}_K$. Let (α) be the fractional ideal of $\mathcal{O}_K$ generated by α that factorizes as $(\alpha) = \frac{\mathfrak{a}}{\mathfrak{b}}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are coprime ideals of $\mathcal{O}_K$ (that is, $\mathfrak{a} + \mathfrak{b} = \mathcal{O}_K$). Let $\mathfrak{M}$ be the collection of all places of K. Recall there are finitely many infinite primes in $\mathfrak{M}$, given by the r real Galois conjugates and the s pairs of imaginary Galois conjugates of α. Recall there are finitely many finite primes dividing $\mathfrak{b}$.

Definition 3.2.2 (Representation space and embedding)**.** *Let*

$$S_\alpha := \{\mathfrak{p} \in \mathfrak{M} \,:\, \mathfrak{p} \mid \infty \text{ or } \mathfrak{p} \mid \mathfrak{b}\}.$$

Define the representation space

$$\mathbb{K}_\alpha := \prod_{\mathfrak{p} \in S_\alpha} K_\mathfrak{p}.$$

$\mathbb{K}_\alpha$ is endowed with the product topology and the product Haar measure, denoted μ_α. Define the diagonal embedding

$$\varphi_\alpha : K \to \mathbb{K}_\alpha, \qquad x \mapsto (x)_{\mathfrak{p} \in S_\alpha}. \tag{3.6}$$

Definition 3.2.3 (Euclidean and $\mathfrak{b}$-adic components)**.** *Let*

$$\mathbb{K}_\infty := \prod_{\mathfrak{p} \mid \infty} K_\mathfrak{p}, \qquad K_\mathfrak{b} := \prod_{\mathfrak{p} \mid \mathfrak{b}} K_\mathfrak{p}.$$

Then

$$\mathbb{K}_\alpha = \mathbb{K}_\infty \times K_\mathfrak{b}.$$

We consider the corresponding embeddings

$$\varphi_\infty : K \to K_\infty, \qquad x \mapsto (x)_{\mathfrak{p}|\infty} \tag{3.7}$$

and

$$\varphi_{\mathfrak{b}} : K \to K_{\mathfrak{b}}, \qquad x \mapsto (x)_{\mathfrak{p}|\mathfrak{b}}. \tag{3.8}$$

We say that $\mathbb{K}_\alpha$ has an archimedean component (or Euclidean component) given by $\mathbb{K}_\infty$, and a non-archimedean component (or a $\mathfrak{b}$-adic component) given by $K_{\mathfrak{b}}$.

The field $\mathbb{Q}(\alpha)$ acts multiplicatively in the space $\mathbb{K}_\alpha$: if $\xi \in \mathbb{Q}(\alpha)$ then

$$\xi \cdot (x_{\mathfrak{p}})_{\mathfrak{p}\in S_\alpha} = (\xi \cdot x_{\mathfrak{p}})_{\mathfrak{p}\in S_\alpha}.$$

Our goal is to introduce a number system in base α that is embedded in $\mathbb{K}_\alpha$. For this matter, we look at series of powers of α where the powers go from $-\infty$. Since $(\alpha^{-1}) = \frac{\mathfrak{b}}{\mathfrak{a}}$, it turns out that α^{-1} generates the ideal $\mathfrak{p}^e$ whenever $\mathfrak{p} \mid \mathfrak{b}$ for the exponent e of $\mathfrak{p}$ in the prime factorization of $\mathfrak{b}$. Therefore, power series of α^{-1} are convergent in $\mathbb{K}_{\mathfrak{p}}$.

Proposition 3.2.4. *We have the isomorphisms*

$$\mathbb{K}_\infty \simeq \mathbb{R}^r \times \mathbb{C}^s \simeq \mathbb{R}^n$$

and

$$K_{\mathfrak{b}} \simeq \bigcup_{\ell \geq 1} \varprojlim_{k\in\mathbb{N}} \mathfrak{b}^{-\ell}(\mathcal{O}_K/\mathfrak{b}^k).$$

Proof. For the first isomorphism, recall that there are r real Galois embeddings and s pairs of imaginary Galois embeddings and that $r + 2s = n$.

For the second, let $\mathfrak{b} = \prod_{\mathfrak{p}|\mathfrak{b}} \mathfrak{p}^{e_{\mathfrak{p}}}$ with $e_{\mathfrak{p}} \geq 1$. It follows from the Chinese Remainder Theorem that, for every k,

$$\mathcal{O}_K/\mathfrak{b}^k = \prod_{\mathfrak{p}|\mathfrak{b}} \mathcal{O}_K/\mathfrak{p}^{e_{\mathfrak{p}}k}.$$

From the definition of projective limit, it follows that

$$\varprojlim_{k\in\mathbb{N}} \mathcal{O}_K/\mathfrak{p}^{e_{\mathfrak{p}}k} \simeq \varprojlim_{k\in\mathbb{N}} \mathcal{O}_K/\mathfrak{p}^k.$$

Therefore

$$\varprojlim_{k\in\mathbb{N}} \mathcal{O}_K/\mathfrak{b}^k \simeq \prod_{\mathfrak{p}|\mathfrak{b}} \varprojlim_{k\in\mathbb{N}} \mathcal{O}_K/\mathfrak{p}^k.$$

One can check that $\frac{1}{\mathfrak{b}}\mathcal{O}_{\mathfrak{p}} = \frac{1}{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$, which implies

$$\bigcup_{\ell\geq 1}\varprojlim_{k\in\mathbb{N}} \mathfrak{b}^{-\ell}(\mathcal{O}_K/\mathfrak{b}^k) = \prod_{\mathfrak{p}|\mathfrak{b}}\bigcup_{\ell\geq 1}\varprojlim_{k\in\mathbb{N}} \mathfrak{p}^{-\ell}\mathcal{O}_K/\mathfrak{p}^k.$$

Hence, Lemma 3.1.4 implies

$$\bigcup_{\ell\geq 1}\varprojlim_{k\in\mathbb{N}} \mathfrak{b}^{-\ell}(\mathcal{O}_K/\mathfrak{b}^k) \simeq \prod_{\mathfrak{p}|\mathfrak{b}} K_{\mathfrak{p}} = K_{\mathfrak{b}}.$$

□

Note that we can embed $\mathbb{Q}(\alpha)$ in $\mathbb{R}^r\times\mathbb{C}^s$ using the Galois embeddings τ_j for $1\leq j\leq r+s$ defined in (3.1), as

$$\varphi_\infty : K \to \mathbb{R}^r\times\mathbb{C}^s, \qquad x\mapsto (\tau_1(x),\dots,\tau_{r+s}(x)). \tag{3.9}$$

The action of $\mathbb{Q}(\alpha)$ on $\mathbb{R}^r\times\mathbb{C}^s$ would be seen as follows: given $t\in\mathbb{Q}(\alpha)$ and $(x_1,\dots,x_{r+s})\in\mathbb{R}^r\times\mathbb{C}^s$, set

$$t\cdot(x_1,\dots,x_{r+s}) := (\tau_1(t)\,x_1,\dots,\tau_{r+s}(t)\,x_{r+s}).$$

In particular, $\alpha\cdot(x_1,\dots,x_{r+s}) = (\alpha_1\,x_1,\dots,\alpha_{r+s}\,x_{r+s})$.

Definition 3.2.5 ($\mathfrak{b}$-adic valuation and absolute value)**.** *Define the $\mathfrak{b}$-adic valuation as*

$$\nu_{\mathfrak{b}} : K_{\mathfrak{b}} \to \mathbb{Z}\cup\{\infty\}, \qquad \nu_{\mathfrak{b}}(y) := \min_{\mathfrak{p}|\mathfrak{b}}\{\nu_{\mathfrak{p}}(y)\}.$$

This induces the $\mathfrak{b}$-adic absolute value

$$|y|_{\mathfrak{b}} = \mathfrak{N}(\mathfrak{b})^{-\nu_{\mathfrak{b}}(y)},$$

which is the same as the one from Definition 3.2.3.

Recall that $\mathfrak{N}(\mathfrak{b}) = |\mathcal{O}_K/\mathfrak{b}| = a_n$ follows from Lemma 3.1.3.

We equip $\mathbb{K}_\alpha$ with the following metric: let $(x,y),(x',y')\in\mathbb{K}_\infty\times K_{\mathfrak{b}} = \mathbb{K}_\alpha$. Then

$$d((x,y),(x',y')) = \max\{|x-y|_\infty, |x'-y'|_{\mathfrak{b}}\}.$$

We can regard $\mathbb{K}_\alpha$ as an open subring of $\mathbb{A}_K$ via the embedding

$$\mathbb{K}_\alpha \hookrightarrow \mathbb{A}_K, \qquad (x_{\mathfrak{p}})_{\mathfrak{p}\in S_\alpha} \hookrightarrow (x_{\mathfrak{p}})_{\mathfrak{p}\in S_\alpha}\times(0)_{\mathfrak{p}\in\mathfrak{M}\setminus S_\alpha}.$$

The space $\mathbb{K}_\alpha$ inherits the subspace topology from $\mathbb{A}_K$, but because $\mathbb{K}_\alpha$ is a finite product, it coincides with the product topology of its factors. This is exactly the same topology as the one induced by the metric we just defined. This metric is chosen so that $\varphi_\alpha(\mathcal{O}_K)$ is discrete.

Definition 3.2.6 (Rational self-affine tiles)**.** *Let α be an expanding algebraic number. Let $\mathcal{D} \subset \mathbb{Z}[\alpha]$. The non-empty compact set $\mathcal{F} = \mathcal{F}(\alpha, \mathcal{D}) \subset \mathbb{K}_\alpha$ defined by the set equation*

$$\alpha\mathcal{F} = \bigcup_{d \in \mathcal{D}} (\mathcal{F} + \varphi_\alpha(d))$$

is called a rational self affine tile if $\mu_\alpha(\mathcal{F}) > 0$.

Definition 3.2.7 (Standard and primitive digit sets)**.** *Given an expanding algebraic number α and a digit set $\mathcal{D} \subset \mathbb{Z}[\alpha]$, we say that $\mathcal{D}$ is a standard digit set if it is a complete set of representatives of $\mathbb{Z}[\alpha]/\alpha\mathbb{Z}[\alpha]$. We say that $\mathcal{D}$ is a primitive digit set if $\mathbb{Z}[\alpha]$ is the smallest α-invariant $\mathbb{Z}$-submodule of $\mathbb{Z}[\alpha]$ containing the difference set $\mathcal{D} - \mathcal{D}$.*

For now, we are going to consider standard digit sets of a particular form.

Lemma 3.2.8. *Let α be a root of the primitive minimal polynomial $P_\alpha(X) = a_n X^n + \ldots + a_1 X + a_0 \in \mathbb{Z}[X]$. Then the set $\mathcal{D} = \{0, 1, \ldots, |a_0| - 1\}$ is a complete set of residues of $\mathbb{Z}[\alpha] \mod \alpha$, that is, $(\alpha, \mathcal{D})$ is a standard digit system.*

Proof. Let $x \in \mathbb{Z}[\alpha]$. Then we can write

$$x = x_k \alpha^k + \cdots + x_1 \alpha + x_0$$

with $x_k, \ldots, x_1, x_0 \in \mathbb{Z}$ and $k \in \mathbb{N}$. Note that $a_0 \neq 0$, otherwise P_α would be reducible. Then there exists a unique $d \in \mathcal{D}$ such that $x_0 = a_0 \widetilde{x}_0 + d$ for $\widetilde{x}_0 \in \mathbb{Z}$. This yields

$$\begin{aligned} x &= x_k \alpha^k + \cdots + x_1 \alpha + a_0 \widetilde{x}_0 + d \\ &= x_k \alpha^k + \cdots + x_1 \alpha - (a_n \alpha^n + \cdots + a_1 \alpha) \widetilde{x}_0 + d \end{aligned}$$

so $x \in \alpha\mathbb{Z}[\alpha] + d$ and this choice is unique. □

Recall that, given an expanding matrix with integer determinant in the classical setting, we could always obtain a self-affine tile by considering a residue set for the digits (this was stated in Theorem 1.3.2). We now state some theorems that can be found in [78, Theorem 1 and 2].

Theorem 3.2.9. *Given an expanding algebraic number α and a standard digit set $\mathcal{D} \subset \mathbb{Z}[\alpha]$, the rational self-affine tile $\mathcal{F}(\alpha, \mathcal{D})$ is the closure of its interior and its boundary has measure zero.*

Theorem 3.2.10. *Consider an expanding algebraic number α and a standard and primitive digit set $\mathcal{D} \subset \mathbb{Z}[\alpha]$. The collection*

$$\{\mathcal{F}(\alpha, \mathcal{D}) + \varphi_\alpha(z) \, : \, z \in \mathbb{Z}[\alpha]\}$$

forms a tiling of $\mathbb{K}_\alpha$.

3.2.1 Intersective tiles

After all the considerations of the previous sections, we have a more clear understanding of how to represent $\mathcal{F}(\alpha, \mathcal{D})$ by introducing a space where it becomes self-affine. Nevertheless, this does not solve the problem of how to depict it. The factor $K_{\mathfrak{b}}$ of $\mathbb{K}_\alpha$ is endowed with an ultrametric, so there is essentially no completely faithful way to portray subsets of $\mathbb{K}_\alpha$.

Recall from Definition 3.2.2 that the space $\mathbb{K}_\alpha$ is of the form $\mathbb{K}_\alpha = \prod_{S_\alpha} K_{\mathfrak{p}}$, where S_α are the places of K that divide either ∞ or $\mathfrak{b}$. Moreover, we can write $\mathbb{K}_\alpha = \mathbb{K}_\infty \times K_{\mathfrak{b}}$ and from Proposition 3.2.4 we have $\mathbb{K}_\infty \simeq \mathbb{R}^n$. From now on, we identify $\mathbb{K}_\infty$ with $\mathbb{R}^n$. We write $(x, y) \in \mathbb{K}_\alpha$ where $x \in \mathbb{R}^n$, $y \in K_{\mathfrak{b}}$. If we think of $K_{\mathfrak{b}}$ as a one-dimensional space, we can picture $\mathbb{K}_\alpha$ as an $n+1$ dimensional space, where copies of $\mathbb{R}^n$ are indexed by points of $K_{\mathfrak{b}}$.

Given a rational self-affine tile $\mathcal{F}(\alpha, \mathcal{D})$, Steiner and Thuswaldner [78] introduced intersective tiles as follows.

Definition 3.2.11 (Intersective tile)**.** *Given $z \in \mathbb{Z}[\alpha]$, we define the intersective tile of $\mathcal{F}(\alpha, \mathcal{D})$ at height z as*

$$\mathcal{G}(z) := \{x \in \mathbb{R}^n \, : \, (x, 0) \in \mathcal{F}(\alpha, \mathcal{D}) + \varphi(z)\}. \tag{3.10}$$

If we identify $\mathbb{K}_\infty \times \{0\}$ with $\mathbb{R}^n$, the set $\mathcal{G}(z)$ is the intersection of $\mathcal{F}(\alpha, \mathcal{D}) + \varphi(z)$ with $\mathbb{K}_\infty \times \{0\}$. For this reason, $\mathcal{G}(z)$ is called an intersective tile, or a slice of $\mathcal{F}(\alpha, \mathcal{D})$. In particular, the zero slice $\mathcal{G}(0)$ is made up of the points of $\mathcal{F}(\alpha, \mathcal{D})$ whose $\mathfrak{b}$-adic part equals zero, and we identify them with points in $\mathbb{R}^n$.

One important remark is that $\mathcal{F}(\alpha, \mathcal{D})$ is bounded, hence $(\mathcal{F} + \varphi(z)) \cap (\mathbb{K}_\infty \times \{0\})$ is empty whenever z is "too far away" from 0 in the distance induced by $|\cdot|_{\mathfrak{b}}$. In fact, we can exactly characterize for which heights the intersective tile is empty. Consider the set

$$\Lambda := \mathbb{Z}[\alpha] \cap \alpha^{-1}\mathbb{Z}[\alpha^{-1}].$$

It is endowed with the structure of a $\mathbb{Z}$-module. This set is very important because we will use it to decide which points belong to each slice of $\mathcal{F}(\alpha, \mathcal{D})$. Moreover, as we will see later, it is related to a tiling of $\mathbb{R}^n$ given by the intersective tiles of $\mathcal{F}(\alpha, \mathcal{D})$. In the negasemiternary number system, this module corresponds to $2\mathbb{Z}$. It is shown in [78, Lemma 6.2] that $\mathcal{G}(z) = \varnothing$ for all $z \in \mathbb{Z}[\alpha] \setminus \Lambda$. This implies the following proposition.

Proposition 3.2.12. *We can write $\mathcal{F}(\alpha, \mathcal{D})$ as a union of intersective tiles in the form*

$$\mathcal{F}(\alpha, \mathcal{D}) = \bigcup_{z \in \Lambda} (\mathcal{G}(z) \times \{0\} - \varphi(z)).$$

Proof. See [78, Proposition 6.1] and [78, Lemma 6.2]. $\square$

Each intersective tile can be obtained as a *Hausdorff limit* of a sequence of sets, that is, a limit taken with respect to the Hausdorff distance. This sequence of sets will be defined in terms of Λ and of the map

$$T_\alpha : \mathbb{Z}[\alpha] \to \mathbb{Z}[\alpha], \qquad x \mapsto \frac{x-d}{\alpha}, \tag{3.11}$$

where d is the unique element of $\mathcal{D}$ such that $T_\alpha(x) \in \mathbb{Z}[\alpha]$ (this holds because $\mathcal{D}$ is a complete set of residues modulo α). Then T_α is known as the *backward division mapping*, or as a *shift* map.

Proposition 3.2.13. *Let $z \in \Lambda$. Then*

$$\mathcal{G}(z) = \underset{k\to\infty}{Lim}\, \varphi_\infty(\alpha^{-k}(T_\alpha^{-k}(z) \cap \Lambda)), \tag{3.12}$$

where the limit is a Hausdorff limit.

Proof. See [78, Lemma 6.11]. $\square$

It follows for every $k \geq 1$ that

$$\mathcal{G}(z) = \bigcup_{y \in T_\alpha^{-k}(z)\cap\Lambda} \alpha^{-k}\mathcal{G}(y), \tag{3.13}$$

and this union is disjoint.

An illustration is shown in Figure 3.1 for $\alpha = \frac{-5-\sqrt{15}\,i}{4}$, which is a root of $2x^2+5x+5$, and digits $\{0,1,2,3,4\}$. It corresponds to the previous equation when $z=0$, and $k=1$, and shows the union of the slices $\mathcal{G}(0)$, $\mathcal{G}(2)$, and $\mathcal{G}(4)$ forming an enlarged copy of $\mathcal{G}(0)$.

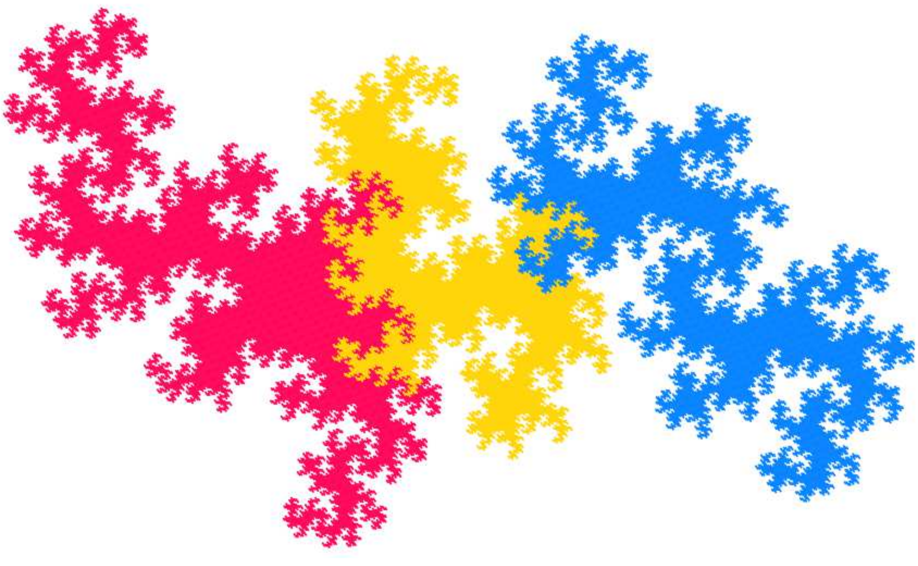


FIGURE 3.1: The union of the slices $\mathcal{G}(0)$ (left), $\mathcal{G}(2)$ (middle) and $\mathcal{G}(4)$ (right) for $\alpha = \frac{-5-\sqrt{15}\,i}{4}$

Theorem 3.2.14. *The collection $\{\mathcal{G}(z) : z \in \Lambda\}$ forms a tiling of $\mathbb{R}^n$.*

Proof. See [78, Theorem 4]. □

We can summarize the ideas regarding intersective tiles as follows: $\mathcal{F}(\alpha, \mathcal{D})$ gives a tiling of $\mathbb{K}_\alpha$, and we can think of $\mathbb{K}_\alpha$ as copies of $\mathbb{R}^n$ indexed by $K_{\mathfrak{b}}$. Suppose that we consider the tiling given by $\mathcal{F}(\alpha, \mathcal{D})$ and we intersect it with one of the copies of $\mathbb{R}^n$ that make up $\mathbb{K}_\alpha$. This is like slicing the tiling at some fixed height. Then different translations of $\mathcal{F}(\alpha, \mathcal{D})$ get intersected at different heights, and what we obtain is a tiling of $\mathbb{R}^n$ given by the different slices of $\mathcal{F}(\alpha, \mathcal{D})$. Moreover, because $\mathcal{F}(\alpha, \mathcal{D})$ is self-affine, it can be subdivided into smaller copies of itself, and when taking a slice of the space each of the slices of $\mathcal{F}(\alpha, \mathcal{D})$ becomes a union of other slices. Because the smaller copies of $\mathcal{F}(\alpha, \mathcal{D})$ were essentially disjoint, so are the slices.

3.3 An example with base (-1+3i)/2

Let $\alpha = \frac{-1+3i}{2}$, root of $P_\alpha = 2X^2 + 2X + 5$, and $\mathcal{D} = \{0, 1, 2, 3, 4\}$. The study of this particular example can be made simpler than the general case, so this section can be read independently, as it does not require the definitions or results introduced as preliminaries in Section 3.1.

For $\alpha = \frac{-1+3i}{2}$ we have the underlying field

$$\mathbb{Q}(\alpha) = \mathbb{Q}(i) = \{a + bi \in \mathbb{C} : a, b \in \mathbb{Q}\}.$$

The ring of integers of $\mathbb{Q}(i)$ is the ring of Gaussian integers

$$\mathbb{Z}[i] = \{a + bi \in \mathbb{C} : a, b \in \mathbb{Z}\}.$$

There are exactly four units of this ring, namely $\{\pm 1, \pm i\}$. For a general commutative ring, a prime p is a non-zero element that is not a unit such that whenever p divides a product ab, then p divides a or p divides b. The primes of $\mathbb{Z}[i]$ are called *Gaussian primes*. In particular, Gaussian integers whose norm is a prime number are Gaussian primes. Given a Gaussian prime p and a unit u, the number up is a Gaussian prime *associated* to u.

We can express α as a quotient of Gaussian integers as

$$\alpha = \frac{-2 + i}{1 + i}.$$

Note that $N(-2 + i) = 5$ and $N(1 + i) = 2$, where N is the usual complex norm defined as $N(a + bi) = a^2 + b^2$. This shows that they are both Gaussian primes.

Let $K = \mathbb{Q}(i)$. We will define a completion of K in terms of $p = 1 + i$ denoted K_{1+i}. The current example is simpler than the general case of completions of algebraic number fields, because $\mathbb{Z}[i]$ is a principal ideal domain, so the notion of p-adic in this space can be defined without using the notion of ideal.

Given a non-zero $x \in \mathbb{Q}(i)$, it can be expressed as a product

$$x = u\,(1+i)^{e_{1+i}}\, p_1^{e_1} \cdots p_k^{e_k} \tag{3.14}$$

where $u \in \{\pm 1, \pm i\}$, $e_{1+i}, e_1, \ldots \in \mathbb{Z}$, and $p_1, \ldots, p_k$ are Gaussian primes different from $1+i$, and by different we mean that none of them is an associated prime of $1+i$. Note that the exponents e_j might be negative because $\mathbb{Q}(i)$ is the field of fractions of $\mathbb{Z}[i]$. In fact, this factorization is unique up to multiplication by units if we take all the Gaussian primes to be different. For example, note that 2 is not prime in $\mathbb{Z}[i]$, and it factors as $2 = -i(1+i)^2$.

Definition 3.3.1 ($(1+i)$-adic valuation and absolute value)**.** *Consider the Gaussian prime $p = 1+i$. We define the $(1+i)$-adic valuation*

$$\nu_{1+i} : \mathbb{Q}(i) \to \mathbb{Z} \cup \{\infty\}, \qquad x \mapsto e_{1+i}$$

for x as in (3.14), and $\nu_{1+i}(0) = \infty$. This induces the $(1+i)$-adic absolute value

$$|x|_{1+i} := 2^{-\nu_{1+i}(x)}.$$

For example, we get $\nu_{1+i}(2) = 2$ and hence $|2|_{1+i} = \frac{1}{4}$. We define K_{1+i} as the completion of K with respect to this absolute value. The residue field $\mathbb{Z}[i]/(1+i)$ is isomorphic to the finite field $\mathbb{F}_2$ of characteristic 2, so we can choose $\{0,1\}$ as a set of residues. We can regard the elements $x \in K_{1+i}$ as infinite series

$$x = \sum_{j=\nu_{1+i}(x)}^{\infty} (1+i)^j c_j, \qquad c_j \in \{0,1\}, \tag{3.15}$$

where ν_{1+i} extends the valuation to all of K_{1+i} and satisfies $c_{\nu_{1+i}(x)} \neq 0$. The space K_{1+i} is endowed with a Haar measure, normalized so that its ring of integers (that is, the set of points x so that $\nu_{1+i}(x) \geq 1$) has measure 1.

Suppose now that we take $x \in \mathbb{Q}(i)$ and consider a series expansion in base $\alpha = \frac{-1+3i}{2}$ of the form

$$\sum_{j=k}^{\infty} \alpha^{-j} d_{-j} \tag{3.16}$$

where $d_{-j} \in \mathcal{D} = \{0,1,2,3,4\}$. Note that $\alpha^{-1} = \frac{1+i}{-2+i}$, and so $|\alpha^{-1}|_{1+i} = \frac{1}{2}$, hence the series (3.16) is convergent in $|\cdot|_{1+i}$, namely it converges to an element of K_{1+i}, but it also converges in $\mathbb{C}$, since $|\alpha^{-1}| = \frac{2}{5}$. These two limits are different, so the idea is that we can regard (3.16) as a pair, where the first coordinate is the limit in $\mathbb{C}$ and

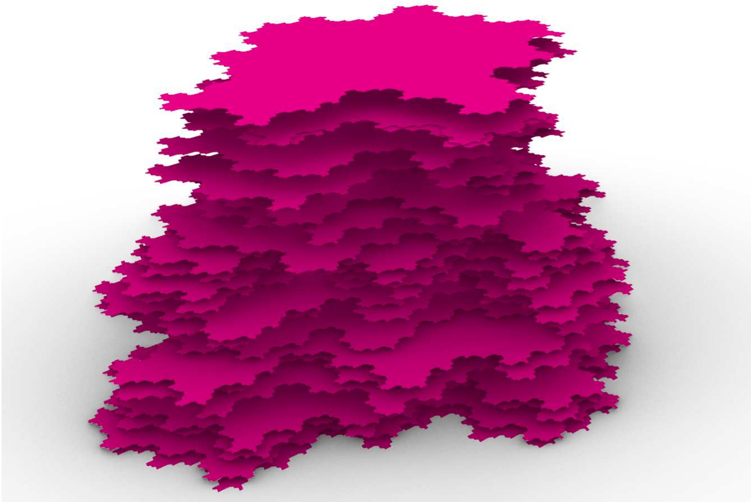


FIGURE 3.2: Front view of $\mathcal{F}(\alpha, \mathcal{D})$ for $\alpha = \frac{-1+3i}{2}$.

the second one is the limit in K_{1+i}. More formally, let φ_α be the embedding

$$\varphi_\alpha : \mathbb{Q}(i) \to \mathbb{C} \times K_{1+i}, \qquad z \mapsto (z, z).$$

Then, instead of looking at a series (3.16) in $\mathbb{C}$ we look at the series

$$(x_1, x_2) = \sum_{j=k}^{\infty} \alpha^{-j} \varphi_\alpha(d_{-j}), \tag{3.17}$$

where $d_{-j} \in \mathcal{D}, x_1 \in \mathbb{C}$ and $x_2 \in K_{1+i}$. Let

$$\mathcal{F}(\alpha, \mathcal{D}) := \left\{ \sum_{j=1}^{\infty} \alpha^{-j} \varphi_\alpha(d_{-j}) \;:\; d_{-j} \in \mathcal{D}. \right\} \tag{3.18}$$

Then $\mathcal{F}(\alpha, \mathcal{D})$ is the attractor of the set equation

$$\alpha \mathcal{F}(\alpha, \mathcal{D}) = \bigcup_{d \in \mathcal{D}} (\mathcal{F}(\alpha, \mathcal{D}) + \varphi_\alpha(d)). \tag{3.19}$$

It is illustrated from different angles in Figures 3.2 and 3.3. The plane of the floor corresponds to $\mathbb{C}$, and K_{1+i} is illustrated in the vertical direction. $\mathcal{F}(\alpha, \mathcal{D})$ is depicted as a union of slices that pile up. The embedding Ψ used to depict points of K_{1+i} in a vertical line is obtained from the $(1+i)$-adic expansion defined in (3.15) as follows:

$$\Psi : K_{1+i} \to \mathbb{R}, \qquad \sum_{j=\nu_{1+i}(x)}^{\infty} (1+i)^j c_j \mapsto \sum_{j=\nu_{1+i}(x)}^{\infty} 2^{-j} c_j$$

(recall that $c_j \in \{0, 1\}$).

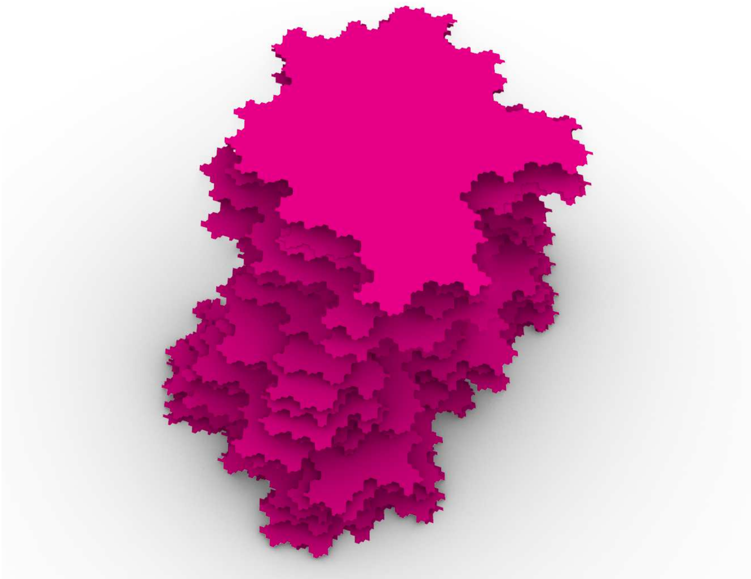


FIGURE 3.3: Top view of $\mathcal{F}(\alpha, \mathcal{D})$ for $\alpha = \frac{-1+3i}{2}$.

Let μ_α be the product Haar measure on $\mathbb{C} \times K_{1+i}$. It follows from [78, Lemma 3.3] that

$$\mu_\alpha(\alpha \cdot M) = |a_0|\, \mu_\alpha(M)$$

for every measurable set $M \subset \mathbb{C} \times K_{1+i}$, where a_0 is the independent coefficient of P_α. Moreover, by Lemma 3.2.8, we have $|\mathcal{D}| = |a_0|$. For this example, $a_0 = 5$. This implies that the union on the right of the set equation is essentially disjoint. Moreover, $\mathcal{F}(\alpha, \mathcal{D})$ is a compact set that is the closure of its interior and its boundary has measure zero. Therefore, $\mathcal{F}(\alpha, \mathcal{D})$ has positive measure, and it is in fact a self affine set. Consider the set of integer expansions

$$\varphi_\alpha(\mathcal{D})[\alpha] := \Big\{ \sum_{j=1}^{k} \alpha^j \varphi_\alpha(d_j) \, : \, k \in \mathbb{Z},\, d_j \in \mathcal{D} \Big\}.$$

For this particular choice of $\mathcal{D} = \{0, 1, 2, 3, 4\}$ we have $\varphi_\alpha(\mathcal{D})[\alpha] = \varphi_\alpha(\mathbb{Z}[\alpha])$. Then the collection $\mathcal{F}(\alpha, \mathcal{D}) + \varphi_\alpha(\mathcal{D})[\alpha]$ is a tiling of $\mathbb{C} \times K_{1+i}$ (see [78, Theorem 2]).

3.3.1 Inter-affine tiles

We continue with the example $\alpha = \frac{-1+3i}{2}$ and $\mathcal{D} = \{0, 1, 2, 3, 4\}$. We are going to study the intersective tiles $\mathcal{G}(z)$ of $\mathcal{F}(\alpha, \mathcal{D})$ as in Definition 3.2.11. A useful feature of this example is that there is only one Archimedean completion of $\mathbb{Q}(\alpha)$, which implies that the map φ_∞ is the canonical embedding of $\mathbb{Q}(\alpha)$ in $\mathbb{C}$.

Consider the space $\mathbb{C} \times \{0\} \subset \mathbb{C} \times K_{1+i}$. Clearly, $\mathbb{C} \times \{0\} \simeq \mathbb{C}$. We take intersections of the form $(\mathcal{F}(\alpha, \mathcal{D}) + \varphi(z)) \cap (\mathbb{C} \times \{0\})$. Hence, we can naturally illustrate the slices of $\mathcal{F}(\alpha, \mathcal{D})$ in the plane. We show how this set is subdivided by means of equation (3.13). The $\mathbb{Z}$-module $\Lambda = \mathbb{Z}[\alpha] \cap \alpha^{-1}\mathbb{Z}[\alpha^{-1}]$ can be computed in terms of a special $\mathbb{Z}$-basis called the Brunotte basis (we will study it in more detail in the next section). The Brunotte basis for this example is given by $w_0 = 2$ and $w_1 = 3i + 1$, so Λ is given by

$$\Lambda = \mathrm{span}_{\mathbb{Z}}\{2, 3i + 1\} \subset \mathbb{C}$$

and is depicted in Figure 3.4.

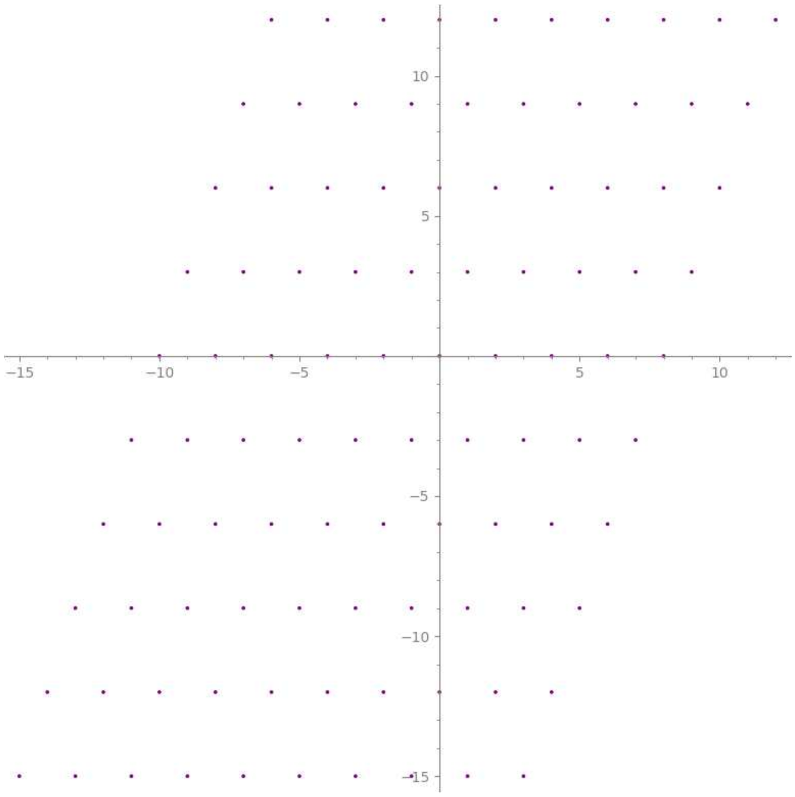


FIGURE 3.4: The lattice Λ in $\mathbb{C}$ for $\alpha = \frac{-1+3i}{2}$.

Let $z \in \Lambda$ and consider the backward division map T_α defined in (3.11). Then

$$\mathcal{G}(z) = \operatorname*{Lim}_{k\to\infty} \alpha^{-k}(T_\alpha^{-k}(z) \cap \Lambda). \tag{3.20}$$

In particular, this implies that the zero slice $\mathcal{G}(0)$ is the Hausdorff limit

$$\mathcal{G}(0) = \operatorname*{Lim}_{k\to\infty} \alpha^{-k}(T_\alpha^{-k}(0) \cap \Lambda). \tag{3.21}$$

This gives a way of approximating $\mathcal{G}(0)$: given $k \geq 1$, we consider points with integer expansions in base α of at most k digits, select which ones belong to Λ and renormalize by multiplying by α^{-k}.

From equation (3.13), we obtain that the first subdivision of the zero tile is given by

$$\mathcal{G}(0) = \bigcup_{y \in T_\alpha^{-1}(0) \cap \Lambda} \alpha^{-1}\mathcal{G}(y).$$

Note that $T_\alpha^{-1}(0) = \mathcal{D}$, so we have 5 candidates and we need to find which ones are in Λ. We see that $\Lambda \cap \mathcal{D} = \{0, 2, 4\}$. It follows

$$\mathcal{G}(0) = \alpha^{-1}(\mathcal{G}(0) \cup \mathcal{G}(2) \cup \mathcal{G}(4)).$$

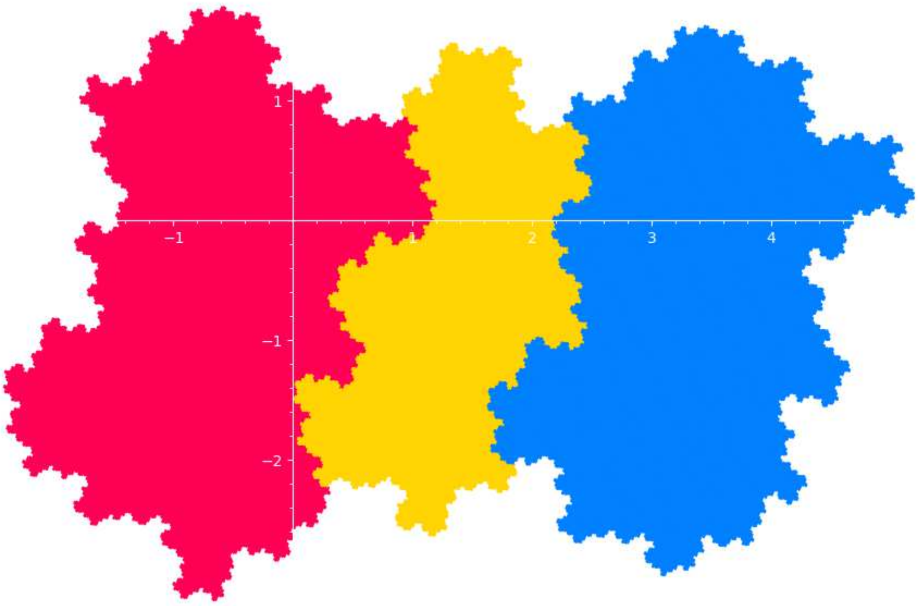


FIGURE 3.5: The union of the slices $\mathcal{G}(0)$ (left), $\mathcal{G}(2)$ (middle) and $\mathcal{G}(4)$ (right) for $\alpha = \frac{-1+3i}{2}$.

The union $\mathcal{G}(0) \cup \mathcal{G}(2) \cup \mathcal{G}(4)$ is shown in Figure 3.5. If we multiply this union by $\alpha^{-1} = \frac{-1-3i}{5}$, we obtain $\mathcal{G}(0)$.

A relevant feature of the zero slice is that

$$\{0\} \subset \cdots \subset \alpha^{-1}\mathcal{G}(0) \subset \mathcal{G}(0) \subset \alpha\mathcal{G}(0) \subset \alpha^2\mathcal{G}(0) \subset \cdots$$

This is not hard to check. A point x is in $\mathcal{G}(0)$ if and only if $(x, 0)$ can be expanded as $(x, 0) = \varphi(d_{-1})\alpha^{-1} + \varphi(d_{-2})\alpha^{-2} + \ldots$ We get that $\alpha^{-1}(x, 0) = (\alpha^{-1}x, 0) = \varphi(d_{-1})\alpha^{-2} + \varphi(d_{-2})\alpha^{-3} + \ldots$, therefore $\alpha^{-1}x \in \mathcal{G}(0)$. Hence, $\alpha^{-1}\mathcal{G}(0) \subset \mathcal{G}(0)$ and the rest of the inclusions follow. This is illustrated in Figure 3.6.

We can continue iterating the process of subdividing the zero slice into a union of other slices. We already had the first iteration for $k = 1$, and for the next step, we would have to consider $T_\alpha^{-2}(0)$ and find which of those points belong to Λ. That gives us

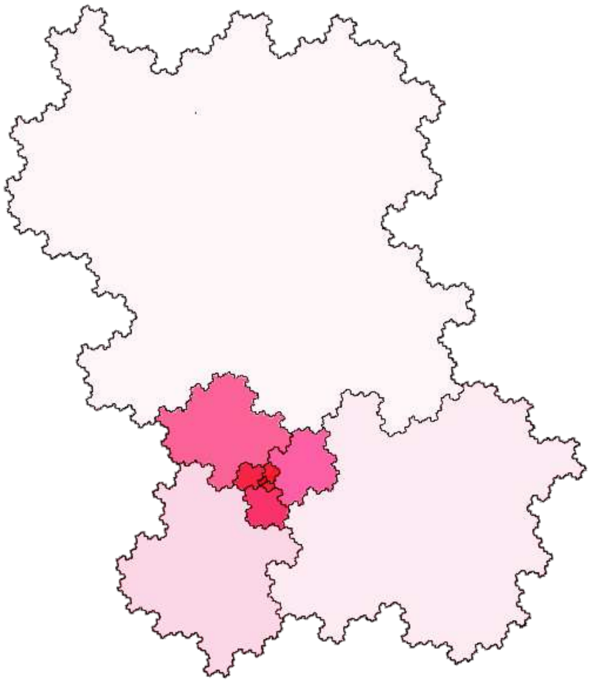


FIGURE 3.6: Inclusion of affine copies of $\mathcal{G}(0)$.

$$T_\alpha^{-2}(0) \cap \Lambda = \{0, 2, 4, -1+3i, 1+3i, 3+3i, -2+6i, 6i, 2+6i\},$$

so there are 9 tiles in the second iteration. Each of these complex numbers has a $(1+i)$-adic expansion and this expansion tells us the height where we can find the corresponding slice in $\mathcal{F}(\alpha, \mathcal{D})$.

We introduce a notation for the $(1+i)$-adic expansion of a $(1+i)$-adic integer as

$$\sum_{j=0}^{\infty}(1+i)^j c_j =: [c_0, c_1, c_2, \ldots]_{1+i}$$

with $c_j \in \{0, 1\}$. Then

$$\begin{aligned} 0 &= [0,0,0,0,0,0,0,0,0,0,\ldots]_{1+i} \\ 2 &= [0,0,1,1,0,1,1,1,1,1,\ldots]_{1+i} \\ 4 &= [0,0,0,0,1,0,1,1,1,1,\ldots]_{1+i} \\ -1+3i &= [0,1,0,1,0,0,0,0,0,0,\ldots]_{1+i} \\ 1+3i &= [0,1,1,0,0,0,0,0,0,0,\ldots]_{1+i} \\ 3+3i &= [0,1,0,1,1,0,1,1,1,1,\ldots]_{1+i} \\ -2+6i &= [0,0,0,1,1,1,0,1,1,1,\ldots]_{1+i} \\ 6i &= [0,0,1,0,1,1,0,1,1,1,\ldots]_{1+i} \\ 2+6i &= [0,0,0,1,0,1,0,1,1,1,\ldots]_{1+i}. \end{aligned} \tag{3.22}$$

All of the previous expansions are eventually periodic, so the three dots indicate that the rest of the coefficients are all 0 or all 1, corresponding to the last written digit. They can be computed in the same way as any p-adic expansion is computed in $\mathbb{Z}$. Note that writing it in this manner gives us a natural order for points in Λ with respect to their $(1+i)$-adic expansion, because we can consider the lexicographic order of $\{0,1\}^{\mathbb{N}}$. This yields

$$0 \prec 4 \prec 6i+2 \prec 6i-2 \prec 6i \prec 2 \prec 3i-1 \prec 3i+3 \prec 3i+1.$$

We color the zero slice with red and move towards the blue in the color spectrum as we get away from zero. As is reasonable, tiles that are closer are more similar in shape.

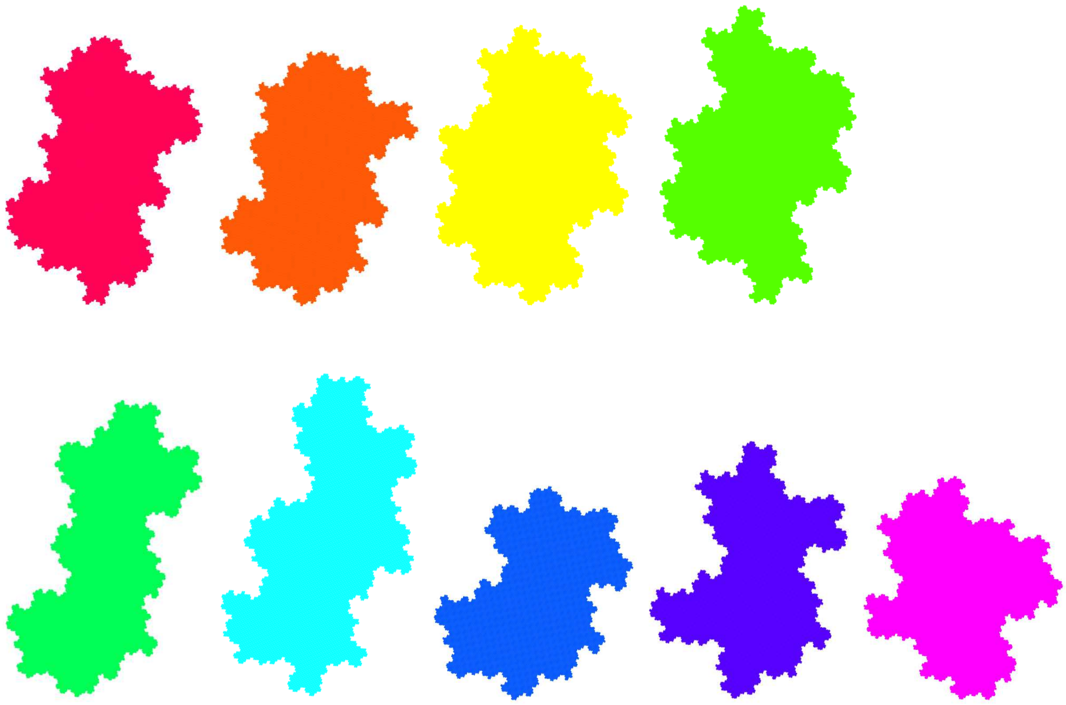


FIGURE 3.7: Ordered slices for $0 \prec 4 \prec 6i+2 \prec 6i-2 \prec 6i \prec 2 \prec 3i-1 \prec 3i+3 \prec 3i+1$.

These complex numbers all belong to the lattice Λ. If we place each tile at the corresponding position, we get a patch of a tiling of $\mathbb{C}$ illustrated in Figure 3.8.

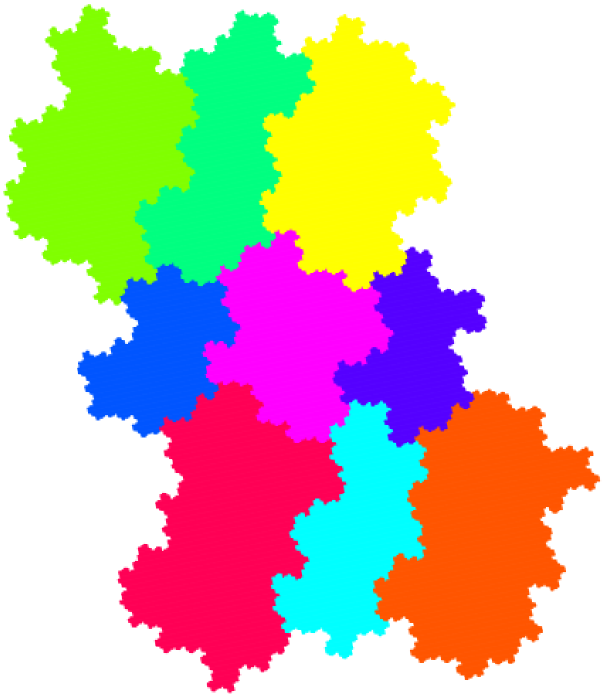


FIGURE 3.8: Patch of tiling with 9 tiles.

Each of these 9 tiles can be subdivided as well, and in fact we get

$$\begin{aligned}
\mathcal{G}(0) &= \alpha^{-1}(\mathcal{G}(0) \cup \mathcal{G}(2) \cup \mathcal{G}(4)) \\
\mathcal{G}(4) &= \alpha^{-1}(\mathcal{G}(6i-2) \cup \mathcal{G}(6i) \cup \mathcal{G}(6i+2)) \\
\mathcal{G}(6i+2) &= \alpha^{-1}(\mathcal{G}(-10) \cup \mathcal{G}(-8) \cup \mathcal{G}(-6)) \\
\mathcal{G}(6i-2) &= \alpha^{-1}(\mathcal{G}(-6i-8) \cup \mathcal{G}(-6i-6) \cup \mathcal{G}(-6i-4)) \\
\mathcal{G}(6i) &= \alpha^{-1}(\mathcal{G}(-3i-9) \cup \mathcal{G}(-3i-7) \cup \mathcal{G}(-3i-5)) \qquad (3.23) \\
\mathcal{G}(2) &= \alpha^{-1}(\mathcal{G}(3i-1) \cup \mathcal{G}(3i+1) \cup \mathcal{G}(3i+3)) \\
\mathcal{G}(3i-1) &= \alpha^{-1}(\mathcal{G}(-3i-3) \cup \mathcal{G}(-3i-1)) \\
\mathcal{G}(3i+3) &= \alpha^{-1}(\mathcal{G}(3i-5) \cup \mathcal{G}(3i-3)) \\
\mathcal{G}(3i+1) &= \alpha^{-1}(\mathcal{G}(-4) \cup \mathcal{G}(-2)).
\end{aligned}$$

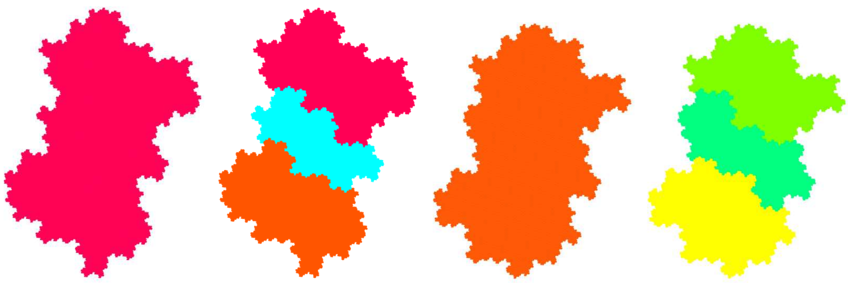


FIGURE 3.9: $\mathcal{G}(0) = \alpha^{-1}(\mathcal{G}(0) \cup \mathcal{G}(2) \cup \mathcal{G}(4))$ (left), $\mathcal{G}(4) = \alpha^{-1}(\mathcal{G}(6i-2) \cup \mathcal{G}(6i) \cup \mathcal{G}(6i+2))$ (right).

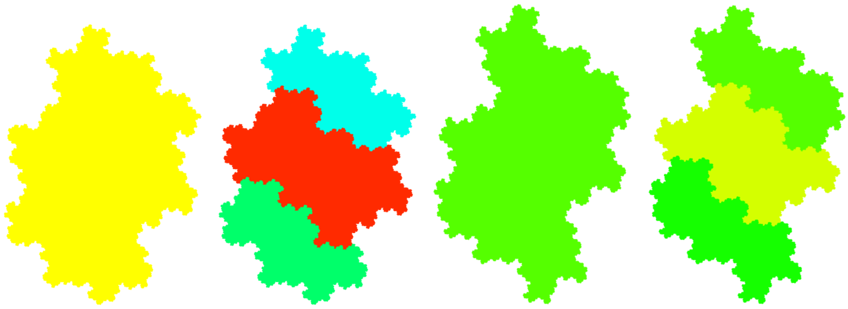


FIGURE 3.10: $\mathcal{G}(6i+2) = \alpha^{-1}(\mathcal{G}(-10) \cup \mathcal{G}(-8) \cup \mathcal{G}(-6))$ (left),
$\mathcal{G}(6i-2) = \alpha^{-1}(\mathcal{G}(-6i-8) \cup \mathcal{G}(-6i-6) \cup \mathcal{G}(-6i-4))$ (right).

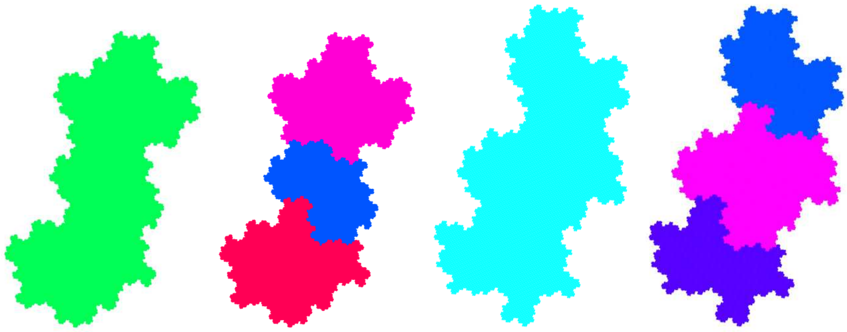


FIGURE 3.11: $\mathcal{G}(6i) = \alpha^{-1}(\mathcal{G}(-3i-9) \cup \mathcal{G}(-3i-7) \cup \mathcal{G}(-3i-5))$ (left),
$\mathcal{G}(2) = \alpha^{-1}(\mathcal{G}(3i-1) \cup \mathcal{G}(3i+1) \cup \mathcal{G}(3i+3))$ (right).

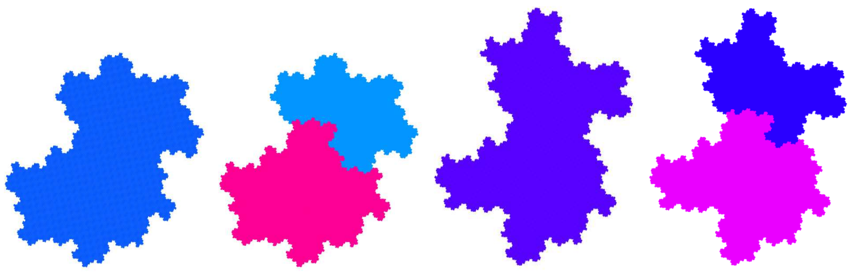


FIGURE 3.12: $\mathcal{G}(3i-1) = \alpha^{-1}(\mathcal{G}(-3i-3) \cup \mathcal{G}(-3i-1))$ (left),
$\mathcal{G}(3i+3) = \alpha^{-1}(\mathcal{G}(3i-5) \cup \mathcal{G}(3i-3))$ (right).

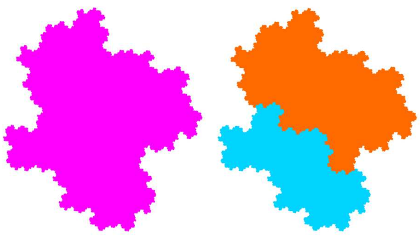


FIGURE 3.13: $\mathcal{G}(3i+1) = \alpha^{-1}(\mathcal{G}(-4) \cup \mathcal{G}(-2))$.

The subdivision of each of the 9 tiles is illustrated in Figures 3.9, 3.10, 3.11, 3.12 and 3.13. Note that now we have a total of 24 new tiles, yet 9 of them are again the same as the original ones. The color choice is consistent with the previous iteration.

In Figure 3.14 we see that the union of the 24 tiles forms again an enlarged version of $\mathcal{G}(0)$. All the tiles have different shapes because they all correspond to a different slice of $\mathcal{F}(\alpha, \mathcal{D})$. In each step, the slices are translated via the lattice Λ, so they form a lattice tiling of the plane. Moreover, we can place each of the slices at the corresponding height to obtain an approximation of $\mathcal{F}(\alpha, \mathcal{D})$ depicted in $\mathbb{R}^3$, and that is exactly what we have in Pictures 3.2 and 3.3.

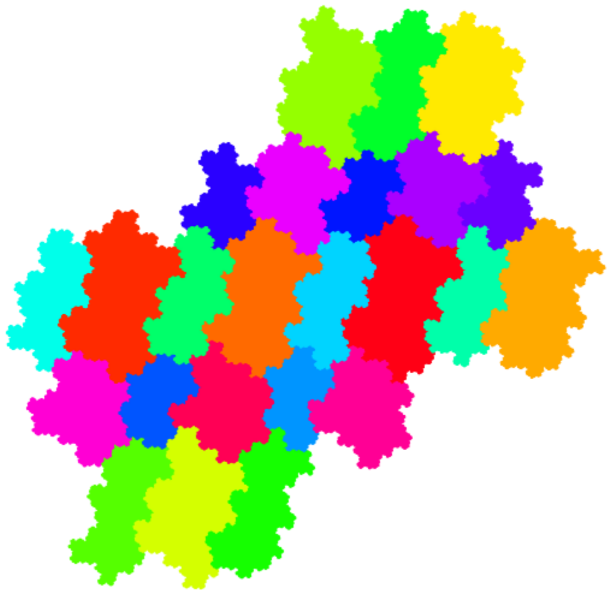


FIGURE 3.14: Patch of tiling with 24 tiles.

In Figure 3.15 we have a patch of the tiling with 155 tiles. For this image, we chose a different pattern for the coloring of the pieces: for each $z \in \Lambda$, consider its $(1+i)$-adic expansion $\sum_{j=0}^{\infty}(1+i)^j c_j = [c_0, c_1, c_2, \ldots]_{1+i}$. Note in particular that, for the points we are considering, $c_0 = 0$. We paint each tile $\mathcal{G}(z)$ red whenever $c_1c_2 = 00$, blue whenever $c_1c_2 = 01$, yellow whenever $c_1c_2 = 10$ and green whenever $c_1c_2 = 11$.

The iteration process can be iterated until the tiles cover the whole plane. From equation (3.13) we can deduce that

$$\mathbb{C} = \bigcup_{z\in\Lambda} \mathcal{G}(z). \tag{3.24}$$

3.4 Generalizations

As a final section of this chapter, we link algebraic number systems with shift radix systems and we express them in matrix form.

Shift radix systems.

We proceed to rewrite (3.21) using shift radix systems (SRS), in a way that does not need the condition of the intersection with Λ, making it faster to compute. We now follow [78, Proposition 6.15].

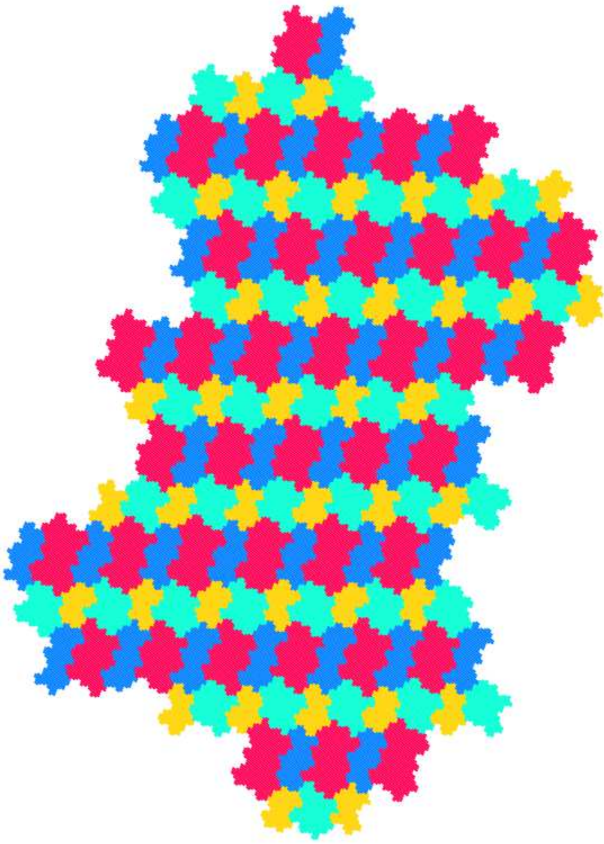


FIGURE 3.15: Patch of tiling with 155 tiles.

Consider an expanding algebraic number α, the minimal polynomial $P_\alpha = a_n X^n + \cdots + a_1 X + a_0$. Recall that $\mathbb{Q}(\alpha)$ is an n-dimensional $\mathbb{Q}$-vector space; it has a special basis which is defined in terms of P_α. It is known as the *Brunotte basis* and is given by $\{w_0, w_1, \dots, w_{n-1}\}$ where

$$\begin{aligned} w_0 &= a_n \\ w_1 &= a_n\alpha + a_{n-1} \\ &\vdots \\ w_{n-1} &= a_n\alpha^{n-1} + a_{n-1}\alpha^{n-2} \cdots + a_1 \end{aligned} \tag{3.25}$$

which satisfies $w_i = \alpha w_{i-1} + a_{n-i}$ for $1 \le i \le n$.

The $\mathbb{Z}$-module $\Lambda = \mathbb{Z}[\alpha] \cap \alpha^{-1}\mathbb{Z}[\alpha^{-1}]$ of $\mathbb{Q}(\alpha)$ is generated by the Brunotte basis over $\mathbb{Z}$. In other words, consider the map

$$\iota_\alpha : \mathbb{Q}^n \to \mathbb{Q}(\alpha), \qquad (z_0, \dots, z_{n-1}) \mapsto \operatorname{sgn}(a_0) \sum_{i=0}^{n-1} z_i w_i. \tag{3.26}$$

Then it holds that a complex number belongs to the lattice Λ if and only if it can be expressed in the Brunotte basis using only integer coefficients. That is, $z \in \mathbb{Z}^n$

if and only if $\iota_\alpha(z) \in \Lambda$. Hence we can select points from Λ by restricting a map to an integer domain. This becomes much easier to compute. For that, consider the vector $r = (\frac{a_n}{a_0}, \dots, \frac{a_1}{a_0})$ and the $n \times n$ matrix

$$M_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & \cdots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{pmatrix}.$$

The matrix M_r represents the multiplication by α^{-1} with respect to the Brunotte basis of $\mathbb{Q}(\alpha)$, that is,

$$\alpha^{-1}\iota_\alpha(z) = \iota_\alpha(M_r z)$$

for all $z \in \mathbb{Q}^n$.

Consider the SRS map

$$\tau_r : \mathbb{Z}^n \to \mathbb{Z}^n, \qquad (z_0, \dots, z_{n-1}) \mapsto (z_1, \dots, z_{n-1}, -\lfloor r \cdot z \rfloor),$$

where "$\cdot$" here denotes the scalar product in $\mathbb{R}^n$. Then it holds that

$$\tau_r(z) = M_r z + (0, \dots, 0, r \cdot z - \lfloor r \cdot z \rfloor).$$

We define the set

$$\mathcal{T}_r(z) := \operatorname*{Lim}_{k\to\infty} \alpha^{-k}(\tau_\alpha^{-k}(z)),$$

known as an SRS tile. Suppose that the set of digits considered for the base α is $\mathcal{D} = \{0, 1, \dots, |a_0| - 1\}$. It is important that $\mathcal{D}$ is of this form to consider intersective tiles as SRS tiles. Recall that $\varphi_\infty(x) = (\tau_1(x), \dots, \tau_{r+s}(x))$ where τ_j $(1 \le j \le r+s)$ are the Galois embeddings of α. Consider the map ι_α from (3.26). If $\mathcal{G}(z)$ is the slice of $\mathcal{F}(\alpha, \mathcal{D})$ at height z embedded in $\mathbb{R}^n$, then

$$\mathcal{G}(\iota_\alpha(z)) = \varphi_\infty \circ \iota_\alpha(\mathcal{T}_r(z)).$$

Rewriting $\mathcal{G}(z)$ in terms of SRS has one very useful application. If we create a picture of $\mathcal{G}(z)$ by approximating some of its points via an algorithm and we are not aware of SRS, the way to proceed would be to compute all points of the tile $\mathcal{F}(\alpha, \mathcal{D})$ and afterward select those that correspond to the height z. Alternatively, one could select at each step the k-th preimage of the map T_α and take its intersection with Λ, as in (3.20). In both cases, a lot of points would have to be computed and only a few of them would actually end up belonging to the desired slice. This is very expensive and ends up taking a long time to come up with a nice enough image. With the use of SRS tiles, however, this process can be sped up drastically. This is the algorithm that we used to compute all the pictures of slices of this thesis.

Matrix form.

For a given expanding algebraic number α of degree n, a number system in base α can be thought of in terms of an expanding matrix $A_\alpha \in \mathbb{Q}^{n\times n}$ whose characteristic polynomial is P_α. The eigenvalues of A_α are the Galois conjugates of α, all of which are assumed to have modulus greater than 1. We know that $\mathbb{Q}(\alpha)$ is an n-dimensional $\mathbb{Q}$-vector space with canonical basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$. Multiplying by α is a linear transformation. Define A_α as the matrix associated with the multiplication by α on this basis.

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. There is a canonical vector space isomorphism

$$\Psi : \mathbb{Q}(\alpha) \to \mathbb{Q}^n, \qquad \alpha^{j-1} \mapsto e_j \qquad (1 \le j \le n).$$

The following diagram commutes:

$$\begin{tikzcd}
\mathbb{Q}(\alpha) \arrow[r, "\alpha\cdot"] \arrow[d, "\Psi"'] & \mathbb{Q}(\alpha) \arrow[d, "\Psi"] \\
\mathbb{Q}^n \arrow[r, "A_\alpha\cdot"] & \mathbb{Q}^n
\end{tikzcd}$$

Recall that we established an isomorphism between $\mathbb{K}_\infty$ and $\mathbb{R}^r \times \mathbb{C}^s \simeq \mathbb{R}^n$ in Proposition 3.2.4. In this setting, $\mathbb{Q}(\alpha)$ acts on $\mathbb{R}^r \times \mathbb{C}^s$ by means of the Galois embeddings, satisfying $\alpha \cdot (x_1, \ldots, x_{r+s}) = (\alpha_1 x_1, \ldots, \alpha_{r+s} x_{r+s})$.

Given a digit set $\mathcal{D} \subset \mathbb{Q}(\alpha)$ (here we do not impose any restriction on $\mathcal{D}$), let $F_1 \subset \mathbb{R}^n$ be the attractor of the set equation

$$A_\alpha F_1 = \bigcup_{d\in\mathcal{D}} (F_1 + \Psi(d)),$$

and let $F_2 \subset \mathbb{R}^r \times \mathbb{C}^s \simeq \mathbb{R}^n$ be the attractor of the set equation

$$\alpha F_2 = \bigcup_{d\in\mathcal{D}} (F_2 + \varphi_\infty(d)), \tag{3.27}$$

Note that we are not considering any p-adic limit whatsoever, so in general F_1 and F_2 are not self affine in the rational case. Consider the map $\Psi \circ \varphi_\infty^{-1} : \varphi_\infty(\mathbb{Q}(\alpha)) \to \mathbb{R}^n$, which is well defined because φ_∞ is clearly injective. Then $F_1 = \Psi \circ \varphi_\infty^{-1}(F_2)$.

When α is an algebraic integer, $F_2 = \mathcal{F}(\alpha, \mathcal{D})$, hence self-affine tiles for algebraic integers can be defined in terms of matrices, and in this case $A_\alpha \in \mathbb{Z}^{n\times n}$, so F_2 is an integral self-affine tile. Consider the example $\alpha = -2 + i$ and $\mathcal{D} = \{0, 1, 2, 3, 4\}$. We get that α is a root of $P_\alpha(X) = X^2 + 4X + 5$, and the companion matrix of P_α is given by

$$A_\alpha = \begin{pmatrix} 0 & -5 \\ 1 & -4 \end{pmatrix}.$$

The eigenvalues of A_α are α and $\bar{\alpha}$. There is an isomorphism

$$\Psi : \mathbb{Q}(\alpha) \to \mathbb{Q}^2, \qquad 1 \mapsto \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \qquad \alpha \mapsto \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right).$$

Then $\Psi(\mathcal{D}) = \left\{ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right) \right\}$. For this example, φ_∞ is the canonical embedding of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$. Then we obtain $\Psi(\mathcal{F}(\alpha, \mathcal{D})) = \mathcal{F}(A_\alpha, \Psi(\mathcal{D}))$.

How is this connection established for non-integer α? The set F_2 from (3.27) is only the projection of $\mathcal{F}(\alpha, \mathcal{D})$ into $\mathbb{R}^n$. We know we can map the archimedean part of $\mathcal{F}(\alpha, \mathcal{D})$ into $\mathbb{R}^n$ via the map $\Psi \circ \varphi_\infty^{-1}$, but how does the non-archimedean component manifest in the matrix setting?

Given a series $\sum_{j=k}^{\infty} \alpha^{-j} d_{-j}$, we can consider its limit in $\mathbb{K}_\infty$ and its limit in $K_{\mathfrak{b}}$. The series

$$\Psi(x) = \sum_{j=k}^{\infty} A_\alpha^{-j} \Psi(d_{-j})$$

is convergent under the usual Euclidean metric in $\mathbb{R}^n$. We need a non-archimedean metric in which this series also converges and that commutes with the $\mathfrak{b}$-adic valuation $\nu_{\mathfrak{b}}$ from Definition 3.2.5. In Chapter 5, we will construct a valuation ν_B such that $\nu_{\mathfrak{b}}(x) = \nu_B(\Psi(x))$. More generally, we will consider a representation space for number systems in terms of a general expanding rational matrix A.

Chapter 4

Standard and non-standard digit systems

The results presented in this chapter are contained in the paper *Rational Self-affine Tiles for Standard and Nonstandard Digit Systems* (Panoramas et Synthéses by Société Mathématique de France, accepted in 2023, [73]). We make a contribution to the theory of self-affine tiles that was established in the early 1990s by Bandt [14], Kenyon [43], Gröchenig and Haas [35], as well as Lagarias and Wang [51, 53, 50], and has gained a lot of attention in the past decades.

We refresh the reader with the definitions of digit systems and self-affine tiles. Let $A \in \mathbb{R}^{n\times n}$ be an expanding matrix (*i.e.*, all its eigenvalues lie outside the unit circle) with integer determinant, and let $\mathcal{D} \subset \mathbb{R}^n$ be a *digit set* with $|\mathcal{D}| = |\det A|$. Then we call the pair $(A, \mathcal{D})$ a *digit system*. By Hutchinson [39], there exists a unique non-empty compact subset $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ of $\mathbb{R}^n$ that satisfies the set equation

$$A\mathcal{F} = \bigcup_{d\in\mathcal{D}} (\mathcal{F} + d). \tag{4.1}$$

If $\mathcal{F}$ has positive Lebesgue measure, it is called a *self-affine tile*. Of special interest are the so-called *integral self-affine tiles* (see [51]), which are obtained when the matrix and the digits have integer coefficients. By Bandt [14], the Lebesgue measure of an integral self-affine tile $\mathcal{F}$ is certainly positive if $(A, \mathcal{D})$ is a *standard digit system*, that is when $\mathcal{D}$ is a complete set of residue class representatives of $\mathbb{Z}^n/A\mathbb{Z}^n$. Lagarias and Wang [51, 53] regarded the matter of $(A, \mathcal{D})$ being a *non-standard* digit system for a given matrix $A \in \mathbb{Z}^{n\times n}$. It is a highly non-trivial problem to characterize all digit sets $\mathcal{D}$ for which $\mathcal{F}(A, \mathcal{D})$ has positive Lebesgue measure (cf. [5, 54]).

A beautiful example in $\mathbb{R}^2$ found by Lagarias and Wang is given by $\mathcal{F}(A, \mathcal{D})$ where $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right\}$. It is depicted in Figure 4.1, where it becomes apparent that this tile has infinitely many connected components. The self-affinity of $\mathcal{F}(A, \mathcal{D})$ is illustrated in Figure 4.2. A tiling of $\mathbb{R}^2$ given by $\mathcal{F}(A, \mathcal{D})$ is exhibited in Figure 4.3.

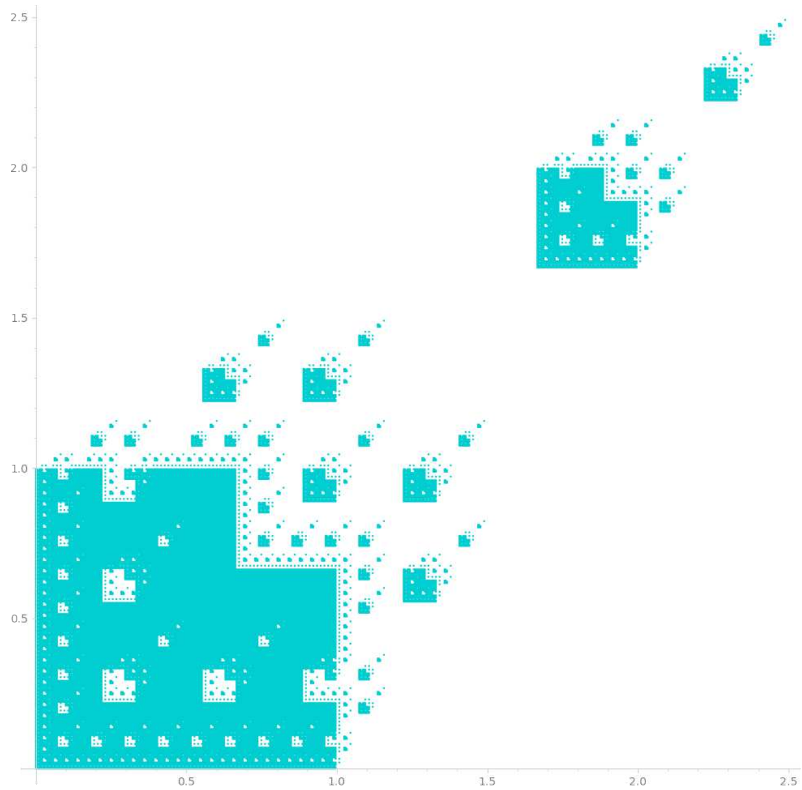


FIGURE 4.1: The tile $\mathcal{F}(A, \mathcal{D})$ in $\mathbb{R}^2$ found by Lagarias and Wang for $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

Rational self-affine tiles are introduced in [78]. They constitute a generalization of self-affine tiles to rational matrices with non integer determinant. In the examples considered up to now, the digit set $\mathcal{D}$ was always a complete set of residues modulo the base. This was a very strong assumption and all the results presented in the previous chapters heavily relied on it. If we allow the digit set $\mathcal{D}$ not to be a residue set, we lose the group structure when considering certain digit expansions. This makes it harder to define tilings.

We provide results in the spirit of Lagarias and Wang in the rational setting. Given $A \in \mathbb{Q}^{n\times n}$ an expanding matrix, we introduce a space $\mathbb{K}_A := \mathbb{R}^n \times \mathbb{Z}^n((A^{-1}))$, where $\mathbb{Z}^n((A^{-1}))$ is a valuation ring of certain Laurent series of powers of A^{-1} with coefficients in $\mathbb{Z}^n$. The ring $\mathbb{Z}^n((A^{-1}))$ is a solenoid, and in the one-dimensional case, it is isomorphic to the ring $\mathbb{Q}_b$ of b-adic numbers for some $b \in \mathbb{N}$ (as studied in Chapter 2). We establish a suitable "diagonal" embedding φ that maps the elements of $\mathbb{Z}^n[A]$ into $\mathbb{K}_A$ in a natural way. This allows us to define the *rational self-affine tile* $\mathcal{F} = \mathcal{F}(A, \mathcal{D}) \subset \mathbb{K}_A$, through the set equation

$$A\mathcal{F} = \bigcup_{d\in\mathcal{D}} (\mathcal{F} + \varphi(d)).$$

We want to study tilings of $\mathbb{K}_A$ induced by $\mathcal{F}$, so we require rational self-affine

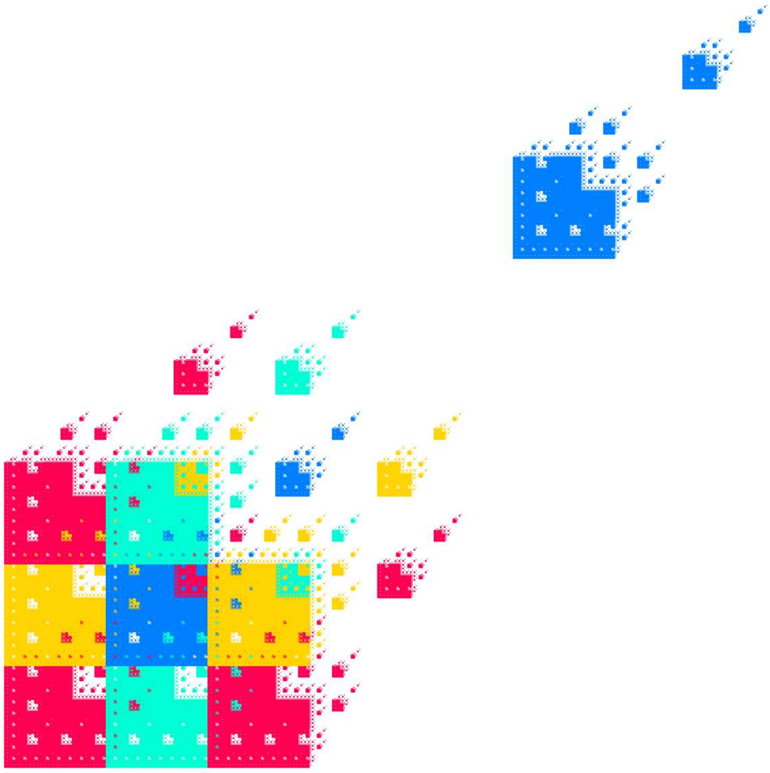


FIGURE 4.2: Subdivision of $\mathcal{F}(A,\mathcal{D})$ into nine subtiles.

tiles to have positive measure, and we give a criterion in terms of the digits to guarantee this. When dealing with matrices, computing quotients is not always so simple, and we make use of some machinery of linear algebra (like the Frobenius normal form) to solve some of these issues. We prove some topological properties of rational self-affine tiles, as well as the existence of a tiling and multiple tiling. For that, we present a careful analysis of the character group of $\mathbb{K}_A$.

4.1 Digit systems for rational matrices

Let $A \in \mathbb{Q}^{n\times n}$ be an expanding matrix and let

$$\mathbb{Z}^n[A] := \bigcup_{k=1}^{\infty} \left(\mathbb{Z}^n + A\mathbb{Z}^n + \cdots + A^{k-1}\mathbb{Z}^n\right)$$

be the smallest A-invariant $\mathbb{Z}$-module containing $\mathbb{Z}^n$. We first define a *digit system* $(A,\mathcal{D})$ where A acts as a *base* and $\mathcal{D} \subset \mathbb{Z}^n[A]$ is some finite *digit set*, and explore its properties (we refer to [40] for more on rational matrix digit systems). We will always assume $|\mathcal{D}| = |\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]|$, which turns out to be a natural size for a digit set. This leads to the following definition.

Definition 4.1.1 (Digit system)**.** *Let $A \in \mathbb{Q}^{n\times n}$ be an expanding matrix and let $\mathcal{D} \subset \mathbb{Z}^n[A]$ be such that $|\mathcal{D}| = |\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]|$. Then we say that $(A,\mathcal{D})$ constitutes a*

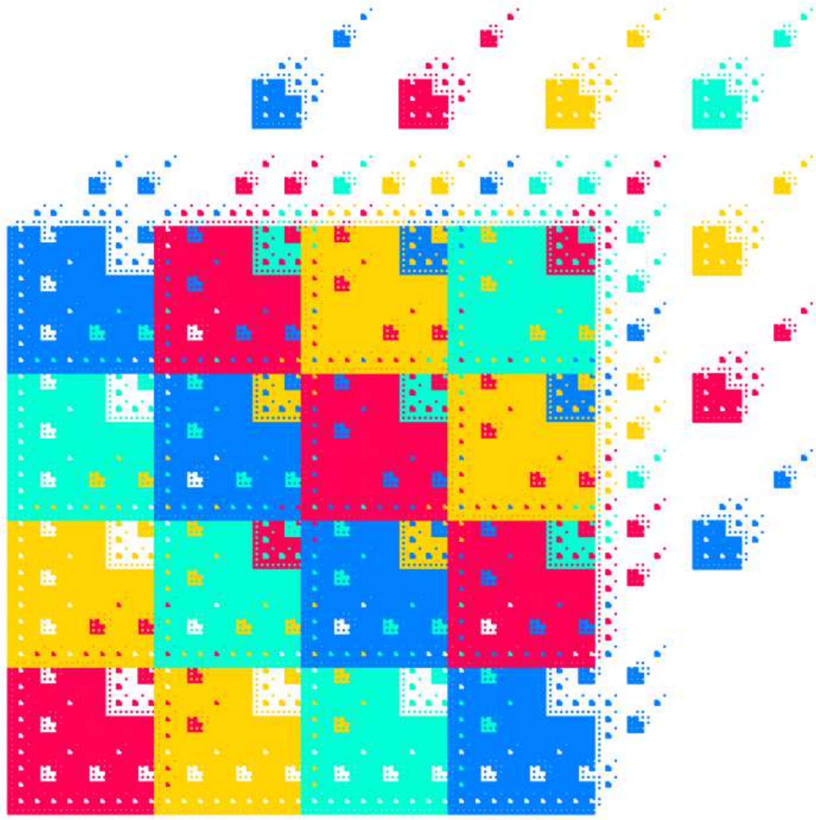


FIGURE 4.3: A tiling of the plane by $\mathcal{F}(A,\mathcal{D})$ for $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

digit system, where A is the base *and $\mathcal{D}$ is the digit set. When $\mathcal{D}$ is a complete set of residue class representatives of $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$, we say that $(A,\mathcal{D})$ is a standard digit system (following [51, p. 163]). Otherwise, we say that $(A,\mathcal{D})$ is a non-standard digit system.*

In connection with digit systems, finite expansions are desirable. For digit systems $(A,\mathcal{D})$, the *finiteness property*, stating that every vector $x \in \mathbb{Z}^n[A]$ has a finite expansion of the form $x = A^k d_k + \cdots + Ad_1 + d_0$ has been studied extensively (see [40] and the references given there). The requirement that $\mathcal{D}$ is a complete system of residue class representatives of $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$ is a necessary but in general not sufficient, condition for $(A,\mathcal{D})$ to have the finiteness property. Eventually periodic expansions have also been investigated.

When the characteristic polynomial is reducible.

Computing the size of a digit set for a given expanding matrix $A \in \mathbb{Q}^{n\times n}$ amounts to computing the order of the quotient group $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$, and this is not always straightforward. From here onward, set

$$a := |\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]|, \qquad b := |\mathbb{Z}^n[A^{-1}]/A^{-1}\mathbb{Z}^n[A^{-1}]|. \tag{4.2}$$

We show how to make use of the Frobenius normal form of A to compute a and b, and we prove that $|\det A| = \frac{a}{b}$, which will be crucial later. Let $A \in \mathbb{Q}^{n\times n}$ with characteristic polynomial χ_A be given. Let $\mathbb{Q}[t]$ be the polynomial ring in one indeterminate t with coefficients in $\mathbb{Q}$. Consider the space $\mathbb{Q}^n$ regarded as a finitely generated $\mathbb{Q}[t]$-module with the action of t given by multiplication by A, that is, if $v \in \mathbb{Q}^n$ then $t \cdot v := Av$, and the action can be linearly extended to all elements in $\mathbb{Q}[t]$. According to the structure theorem for finitely generated modules over principal ideal domains (see [26, Chapter 12, Theorem 6]), there exists an isomorphism of the form

$$\mathbb{Q}^n \simeq \bigoplus_{i=1}^{k} \mathbb{Q}[t]/(p_i),$$

where $p_i \in \mathbb{Q}[t]$ are the so-called *invariant factors* of $\mathbb{Q}^n$, with the divisibility properties $p_1 \mid p_2 \mid \ldots \mid p_k \mid \chi_A$. The polynomials p_i are assumed to be monic, and with this assumption they are unique. In fact, χ_A is the product of the invariant factors of A. This implies that A is similar to a block diagonal matrix $F = \operatorname{diag}(C_1, \ldots . C_k)$, where C_i is the companion matrix of p_i $(1 \leqslant i \leqslant k)$. F is the well-known *Frobenius normal from* of A, also called *rational canonical form*, see [26, Section 12.2]. Using this notation, we get the following result which shows how to compute the value of a.

Proposition 4.1.2. *Let $A \in \mathbb{Q}^{n\times n}$ be given, let $p_i \in \mathbb{Q}[t]$ $(1 \leqslant i \leqslant k)$ be the corresponding invariant factors, and consider the integer polynomials $q_i = c_i p_i \in \mathbb{Z}[t]$, where each $c_i \in \mathbb{Z}$ is chosen so that q_i has coprime coefficients. Let $q_i^* \in \mathbb{Z}[t]$ be the reciprocal polynomial of q_i, namely $q_i^*(t) := t^{\deg(q_i)} q_i(t^{-1})$. Then*

$$a = \prod_{i=1}^{k} |q_i(0)|, \qquad b = \prod_{i=1}^{k} |q_i^*(0)|, \tag{4.3}$$

and $|\det A| = \frac{a}{b}$.

Proof. Let $F = \operatorname{diag}(C_1, \ldots, C_k)$ be the Frobenius normal form of A. Then it is clear that

$$a = \prod_{i=1}^{k} |\mathbb{Z}^{\deg(q_i)}[C_i]/C_i\mathbb{Z}^{\deg(q_i)}[C_i]|. \tag{4.4}$$

Let $C \in \mathbb{Q}^{m\times m}$ be the companion matrix of some polynomial $p \in \mathbb{Q}[t]$ and let $q = cp \in \mathbb{Z}[t]$, where $c \in \mathbb{Z}$ is chosen so that q has coprime coefficients. We claim that

$$\mathbb{Z}^m[C] \simeq \mathbb{Z}[t]/(q). \tag{4.5}$$

To prove this, let $v \in \mathbb{Z}^m[C]$ be given; then it can be expressed as $v = \sum_{j=0}^{L} C^j v_j$ with $v_j \in \mathbb{Z}^m$, $L \geqslant 0$. For $1 \leqslant j \leqslant m$, denote by $\mathbf{e}_j \in \mathbb{Q}^m$ the j-th canonical basis vector. Clearly, each v_j can be expressed in the canonical basis with integer coefficients. Since C is a companion matrix, it follows that $C\mathbf{e}_j = \mathbf{e}_{j+1}$ for $1 \leqslant j < m$, so all this yields the result that v is of the form

$$v = \sum_{j=0}^{\ell} b_j C^j \mathbf{e}_1, \tag{4.6}$$

for some $\ell \in \mathbb{N}$ minimal and $b_0, \dots, b_\ell \in \mathbb{Z}$. If $\ell \geqslant m$, we will show that we can take $b_m, \dots, b_\ell \in \{0, \dots, |q^*(0)| - 1\}$ (here, $q^* \in \mathbb{Z}[t]$ is the reciprocal polynomial of q, so $q^*(0)$ corresponds to the leading coefficient of q). In fact, note that $q(C) = 0$ because C is the companion matrix of q. Suppose that b_ℓ is not in $\{0, \dots, |q^*(0)| - 1\}$; then one can add or subtract $C^{\ell-m}q(C) = 0$ in order to obtain another expression on the right side of (4.6), without altering the value of v. This can be done the appropriate number of times, so we can assume w.l.o.g. that $b_\ell \in \{0, \dots, |q^*(0)| - 1\}$. Repeating this for $\ell - 1, \ell - 2, \dots, m$, we arrive at $b_m, \dots, b_\ell \in \{0, \dots, |q^*(0)| - 1\}$, and the representation (4.6) with this property and ℓ minimal is unique.

By [75, Lemma 4.1] each polynomial $r \in \mathbb{Z}[t]/(q)$ can be expressed uniquely as $r = r' + \sum_{j=m}^{\ell} r_j t^j \mod q$, with $r' \in \mathbb{Z}[x]$, $\deg(r') < m$, $\ell \in \mathbb{N}$ and $r_m, \dots, r_\ell \in \{0, \dots, |q^*(0)| - 1\}$. Using this, one easily checks that

$$h : \mathbb{Z}^m[C] \to \mathbb{Z}[t]/(q), \qquad \sum_{i=0}^{\ell} b_i C^i \mathbf{e}_1 \mapsto \sum_{i=0}^{\ell} b_i t^i \tag{4.7}$$

is an isomorphism, and the claim in (4.5) is proved.

Because $t\,h(G) = h(CG)$ for any $G \in \mathbb{Z}^m[C]$, this isomorphism implies that

$$\mathbb{Z}^m[C]/C\mathbb{Z}^m[C] \simeq \mathbb{Z}[t]/(q, t). \tag{4.8}$$

It is easy to check that $|\mathbb{Z}[t]/(q, t)| = |q(0)|$ (see [75, p. 1460]) and, hence, we have that

$$|\mathbb{Z}^m[C]/C\mathbb{Z}^m[C]| = |q(0)|. \tag{4.9}$$

Applying (4.9) for $q = q_i$ $(1 \leqslant i \leqslant k)$ in (4.4), the left equation of (4.3) follows. The right equation of (4.3) is proved in the same way by replacing A by A^{-1}. The assertion $\det A = \frac{a}{b}$ follows from (4.3) (recall the definition of q_i and the fact that each p_i is monic), because

$$|\det A| = \prod_{i=1}^{k} |\det C_i| = \prod_{i=1}^{k} |p_i(0)| = \prod_{i=1}^{k} \frac{|q_i(0)|}{|q_i^*(0)|} = \frac{a}{b}.$$

□

An example of the computation of a and b.

Let

$$A = \begin{pmatrix} \frac{7}{3} & \frac{3}{2} & -\frac{5}{6} & -1 \\ -\frac{10}{3} & -\frac{17}{6} & \frac{5}{2} & \frac{10}{3} \\ 3 & \frac{5}{2} & -\frac{1}{2} & -3 \\ -\frac{4}{3} & -\frac{13}{6} & \frac{7}{6} & \frac{8}{3} \end{pmatrix}$$

We have $\det(A) = \frac{40}{9}$. The characteristic polynomial of A is

$$\chi_A(X) = (X - \tfrac{4}{3})^2(X^2 + X + \tfrac{5}{2}),$$

which is clearly reducible, so the ring $\mathbb{Z}^n[A]$ is not isomorphic to a number field. The matrix A has three distinct eigenvalues: $\frac{4}{3}$ (with degree 2), $\frac{-1+3i}{2}$ (with degree 1), and $\frac{-1-3i}{2}$ (with degree 1). Hence, A is an expanding matrix. We won't go into the details of how to compute the invariant factors of A (algorithms can be found in [26, Section 12.2]). It holds that the invariant factors of A are

$$\begin{aligned} p_1(X) &= X - \tfrac{4}{3} \\ p_2(X) &= (X - \tfrac{4}{3})(X^2 + X + \tfrac{5}{2}) = X^3 - \tfrac{1}{3}X^2 + \tfrac{7}{6}X - \tfrac{10}{3}. \end{aligned}$$

This yields that the Frobenius normal form of A is given by

$$F = \begin{pmatrix} \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{10}{3} \\ 0 & 1 & 0 & -\frac{7}{6} \\ 0 & 0 & 1 & \frac{1}{3} \end{pmatrix}.$$

(One way to verify that this is correct is by checking that A and F are similar). The integer polynomials q_i and their reciprocals are then given by

$$\begin{aligned} q_1(X) &= 3X - 4 \\ q_1^*(X) &= 4X - 3 \\ q_2(X) &= 6X^3 - 2X^2 + 7X - 20 \\ q_2^*(X) &= -20X^3 + 7X^2 - 2X + 6. \end{aligned} \tag{4.10}$$

Then $a = |q_1(0)||q_2(0)| = 4 \cdot 20 = 80$ and $b = |q_1^*(0)||q_2^*(0)| = 3 \cdot 6 = 18$. Then

$$\frac{a}{b} = \frac{80}{18} = \frac{40}{9} = \det(A).$$

This is an example where a and b are not coprime, which also shows that the determinant of A does not give enough information about the cardinality of $\mathcal{D}$.

We show how to construct a residue set $\mathcal{D}$ for $\mathbb{Z}^4[A]/A\mathbb{Z}^4[A]$. We know that $\mathcal{D}$ must have 80 elements. The Frobenius normal form has two blocks C_1 and C_2, corresponding to the companion matrices of each of the invariant factors p_1 and p_2.

We can then decompose F as a direct sum

$$F = C_1 \oplus C_2 = \left(\tfrac{4}{3}\right) \oplus \begin{pmatrix} 0 & 0 & \frac{10}{3} \\ 1 & 0 & -\frac{7}{6} \\ 0 & 1 & \frac{1}{3} \end{pmatrix}.$$

The Frobenius normal form is similar to A, which yields

$$\mathbb{Z}^4[A]/A\mathbb{Z}^4[A] \simeq \mathbb{Z}^4[F]/F\mathbb{Z}^4[F] \simeq (\mathbb{Z}[\tfrac{4}{3}]/\tfrac{4}{3}\mathbb{Z}[\tfrac{4}{3}]) \oplus (\mathbb{Z}^3[C_2]/C_2\mathbb{Z}^3[C_2]).$$

A residue set for $\mathbb{Z}[\frac{4}{3}]/\frac{4}{3}\mathbb{Z}[\frac{4}{3}]$ is given by $\mathcal{D}_1 = \{0,1,2,3\}$. From (4.8) we obtain that $\mathbb{Z}^3[C_2]/C_2\mathbb{Z}^3[C_2] \simeq \mathbb{Z}[t]/(q_2,t)$, and using that $|q(0)| = 20$ we get that a residue set for $\mathbb{Z}[t]/(q_2,t)$ is given by $\{0,1,\dots,19\}$. Using the inverse of the isomorphism h from (4.7), we can map these points back and obtain that a residue set for $\mathbb{Z}^3[C_2]/C_2\mathbb{Z}^3[C_2]$ given by $\mathcal{D}_2 = \{(j,0,0) \,:\, 1 \le j \le 19\}$. Hence, a digit set is given by

$$\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2 = \{(i,j,0,0) \,:\, 1 \le i \le 4,\, 1 \le j \le 19\}.$$

It follows that $|\mathcal{D}| = 80$ and $(A,\mathcal{D})$ is a standard digit system.

4.2 The representation space

Let $A \in \mathbb{Q}^{n\times n}$ be expanding. For convenience in the notation, from here onward we set

$$B := A^{-1}.$$

Consider the ring

$$\mathbb{Z}^n(B) = \bigcup_{k\geqslant 1}(B^{-k}\mathbb{Z}^n + B^{-k+1}\mathbb{Z}^n + \dots + B^k\mathbb{Z}^n)$$

(note that $\mathbb{Z}^n(A) = \mathbb{Z}^n(B)$). We define on $\mathbb{Z}^n(B)$ the B-*adic valuation* $\nu_B : \mathbb{Z}^n(B) \to \mathbb{Z} \cup \{\infty\}$ as

$$\nu_B(y) := \begin{cases} \min\{k \in \mathbb{Z} \,:\, y \in B^k\mathbb{Z}^n[B] \setminus B^{k+1}\mathbb{Z}^n[B]\}, & y \neq 0, \\ \infty, & y = 0. \end{cases} \tag{4.11}$$

On $\mathbb{Z}^n(B)$ the B-*adic metric* is defined by

$$\mathbf{d}_B(y,y') := b^{-\nu_B(y-y')}, \tag{4.12}$$

for b as in (4.2) and $y, y' \in \mathbb{Z}^n(B)$, with the convention that $b^{-\infty} = 0$.

Definition 4.2.1 (B-adic series)**.** *We define the space $\mathbb{Z}^n((B))$ of B-adic series as the completion of $\mathbb{Z}^n(B)$ with respect to the metric $\mathbf{d}_B$.*

We extend the metric $\mathbf{d}_B$ to the completion $\mathbb{Z}^n((B))$, and hence we extend the B-adic valuation ν_B to $\mathbb{Z}^n((B))$ so that it satisfies (4.12). Then, every non-zero $y \in \mathbb{Z}^n((B))$ can be expressed as Laurent series

$$y = \sum_{j=\nu_B(y)}^{\infty} B^j y_j, \qquad y_j \in \mathbb{Z}^n, \tag{4.13}$$

of powers of B with coefficients in $\mathbb{Z}^n$, which converges with respect to the metric $\mathbf{d}_B$. Then $\nu_B(y)$ is the smallest index such that y has an expansion (4.13) with $y_{\nu_B(y)} \neq 0$. We denote by $\mathbb{Z}^n[[B]]$ the subring of $\mathbb{Z}^n((B))$ consisting of points $y \in \mathbb{Z}^n((B))$ with $\nu_B(y) \geqslant 0$, the ring of power series in B with coefficients in $\mathbb{Z}^n$. The B-adic metric satisfies the *ultrametric inequality*, namely

$$\mathbf{d}_B(y, y') \leqslant \max\{\mathbf{d}_B(y, y''), \mathbf{d}_B(y'', y')\}$$

for every $y, y', y'' \in \mathbb{Z}^n((B))$. This metric turns $\mathbb{Z}^n((B))$ into a complete separable space, which is also a locally compact topological group. Thus there is a Haar measure μ_B on $\mathbb{Z}^n((B))$ which is normalized in a way that $\mu_B(\mathbb{Z}^n[[B]]) = 1$, and we call it the *$B$-adic measure.* If $M \subset \mathbb{Z}^n((B))$ is a measurable set, then

$$\mu_B(A^k M) = b^k \mu_B(M). \tag{4.14}$$

Definition 4.2.2 (The representation space)**.** *Given an expanding matrix $A \in \mathbb{Q}^{n\times n}$, define the representation space $\mathbb{K}_A$ as*

$$\mathbb{K}_A := \mathbb{R}^n \times \mathbb{Z}^n((B)).$$

We endow the space $\mathbb{K}_A$ with the following structures:

1. *It inherits the structure of an additive group from its cartesian factors.*

2. *Consider the group given by*

$$\mathbb{Z}[A] := \bigcup_{k\geqslant 1} (\mathbb{Z}A^{-k} + \mathbb{Z}A^{-k+1} + \cdots + \mathbb{Z}A^k).$$

   Then $\mathbb{Z}[A]$ acts on $\mathbb{K}_A$ by multiplication, i.e., $G \cdot (x, y) = (Gx, Gy)$ if $G \in \mathbb{Z}[A]$ and $(x, y) \in \mathbb{K}_A$.

3. *We define the metric*

$$\boldsymbol{d}((x, y), (x', y')) := \max\{\|x - x'\|, \mathbf{d}_B(y, y')\},$$

for $(x, y), (x', y') \in \mathbb{K}_A$, *where* $\|\cdot\|$ *denotes the Euclidean norm in* $\mathbb{R}^n$ *and* $\mathbf{d}_B$ *is the* B*-adic metric in* $\mathbb{Z}^n((B))$. *This turns* $\mathbb{K}_A$ *into a locally compact topological group. It is easy to check that for every closed ball* $\mathbf{B}_r(x, y)$ *of radius* $r > 0$ *and center* $(x, y) \in \mathbb{K}_A$, *there is a decomposition*

$$\mathbf{B}_r(x, y) = \mathbf{B}_r(x) \times \mathbf{B}_r(y),$$

into closed balls on each respective space. This characterizes the topology of $\mathbb{K}_A$.

4. *We define a measure* μ *in* $\mathbb{K}_A$ *as the product measure*

$$\mu := \lambda \times \mu_B,$$

where λ *is the Lebesgue measure in* $\mathbb{R}^n$ *and* μ_B *the* B*-adic measure in* $\mathbb{Z}^n((B))$. *Then* μ *is the Haar measure on* $\mathbb{K}_A$ *satisfying* $\mu([0, 1] \times \mathbb{Z}^n[[B]]) = 1$.

Remark 4.2.3. *When* $A \in \mathbb{Z}^{n\times n}$ *is an integer matrix, the space* $\mathbb{Z}^n((B))$ *is trivial and does not play a role, and hence* $\mathbb{K}_A = \mathbb{R}^n$. *However,* $\mathbb{K}_A = \mathbb{R}^n$ *may also happen in the non-integer case: for example, if* $A = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 3 \end{pmatrix}$ *we have* $b = 1$. *Suppose* $b = 1$, *i.e., that* $\mathbb{Z}^n[B]/B\mathbb{Z}^n[B]$ *is trivial. Then by Lemma 4.1.2* $\det A = a$ *is an integer. The results presented in Section 4.4 are proven by Lagarias and Wang in [53] for real expanding matrices with integer determinant. For this reason, in all that follows we assume* $b \geqslant 2$.

Lemma 4.2.4. *If* $M \subset \mathbb{K}_A$ *is a measurable set, then* $\mu(AM) = a\,\mu(M)$.

Proof. Consider a measurable subset of $\mathbb{K}_A$ of the form $M_1 \times M_2$, where $M_1 \subset \mathbb{R}^n$ and $M_2 \subset \mathbb{Z}^n((B))$ are both measurable sets. We have seen in Proposition 4.1.2 that $\det A = \frac{a}{b}$. Then

$$\mu(A(M_1 \times M_2)) = \lambda(AM_1)\,\mu_B(AM_2) = \frac{a}{b}\,\lambda(M_1)\,b\,\mu_B(M_2) = a\,\mu(M_1 \times M_2).$$

Since $\mu = \lambda \times \mu_B$ is a product measure, the σ-algebra of μ-measurable sets is generated by sets of the form $M_1 \times M_2$. Therefore, if $M \subset \mathbb{K}_A$ is measurable we have $\mu(AM) = a\,\mu(M)$. □

4.3 Rational self-affine tiles

We proceed to introduce the set $\mathcal{F}(A, \mathcal{D})$ associated with the digit system $(A, \mathcal{D})$, which can be regarded as the set of "fractional parts" and reflects features of its structure. In Section 4.4 we will study topological properties of $\mathcal{F}$.

We need the following lemma, which is in the spirit of Lind [58].

Lemma 4.3.1. *Let* $A \in \mathbb{Q}^{n\times n}$ *be expanding and assume* $b \geqslant 2$, *with* b *as in* (4.2). *Then there exists a metric* $\boldsymbol{\ell}$ *on* $\mathbb{K}_A$ *with respect to which the action of* $B = A^{-1}$ *is a* *contraction. In particular, there exists* $0 \leqslant \kappa < 1$ *such that*

$$\boldsymbol{\ell}(B \cdot (x,y), B \cdot (x',y')) \leqslant \kappa\, \boldsymbol{\ell}((x,y),(x',y')) \qquad ((x,y),(x',y') \in \mathbb{K}_A). \tag{4.15}$$

Proof. Let $Spec(A)$ denote the set of eigenvalues of A. Since A is expanding, there exists $\rho \in \mathbb{R}$ such that $1 < \rho < \min\{|\eta| : \eta \in Spec(A)\}$. For $x \in \mathbb{R}^n$, define

$$\|x\|' := \sum_{k=0}^{\infty} \rho^k \, \|B^k x\|. \tag{4.16}$$

Since all the eigenvalues of $\rho B = \rho A^{-1}$ are strictly smaller than 1 in modulus, the series on the right-hand side of (4.16) converges and $\|\cdot\|'$ becomes a norm in $\mathbb{R}^n$ that satisfies

$$\|B\,x\|' = \frac{1}{\rho} \sum_{k=1}^{\infty} \rho^k \, \|B^k\, x\| \leqslant \frac{1}{\rho}\, \|x\|'.$$

Also, for all $y, y' \in \mathbb{Z}^n((B))$, it follows from the definition of the B-adic metric that $\mathbf{d}_B(By, By') = \frac{1}{b}\, \mathbf{d}_B(y,y')$. Let $(x,y),(x',y') \in \mathbb{K}_A$. Define on $\mathbb{K}_A$ the metric $\boldsymbol{\ell}$ given by

$$\boldsymbol{\ell}((x,y),(x',y')) := \max\{\|x - x'\|', \mathbf{d}_B(y,y')\}.$$

Then (4.15) follows with $\kappa := \max\{\frac{1}{\rho}, \frac{1}{b}\} < 1$. $\square$

Note that $\mathbf{d}$ and $\boldsymbol{\ell}$ are equivalent, because $\|\cdot\|'$ is equivalent to $\|\cdot\|$.

We now introduce a suitable way to embed our digit system into the representation space $\mathbb{K}_A$. Define the *diagonal embedding* φ as

$$\varphi : \mathbb{Z}^n(B) \to \mathbb{K}_A, \qquad x \mapsto (x,x).$$

Definition 4.3.2 (Rational self-affine tile)**.** *Let* $(A, \mathcal{D})$ *be a digit system. Define* $\mathcal{F} = \mathcal{F}(A,\mathcal{D}) \subset \mathbb{K}_A$ *as the unique non-empty compact set satisfying the set equation*

$$A\mathcal{F} = \bigcup_{d \in \mathcal{D}} (\mathcal{F} + \varphi(d)). \tag{4.17}$$

If $\mu(\mathcal{F}) > 0$, *then* $\mathcal{F}$ *is called a rational self-affine tile.*

Because A is expanding, Lemma 4.3.1 implies that the mapping

$$\mathbb{K}_A \to \mathbb{K}_A, \qquad (x,y) \mapsto A^{-1}((x,y) + \varphi(d))$$

is a contraction for each $d \in \mathcal{D}$. Let $\mathcal{H}(\mathbb{K}_A)$ be the family of non-empty compact subsets of $\mathbb{K}_A$, and consider the map

$$\Psi : \mathcal{H}(\mathbb{K}_A) \to \mathcal{H}(\mathbb{K}_A), \qquad X \mapsto \bigcup_{d \in \mathcal{D}} A^{-1}(X + \varphi(d)). \tag{4.18}$$

By Hutchinson [39], there is a unique non-empty compact set which is a fixed point of Ψ, hence $\mathcal{F}$ is well defined. This set is the attractor of an iterated function system, meaning that it is the Hausdorff limit of the sequence of compact sets $\{\Psi^k(X)\}_{k\geq 1}$, for any compact set X.

Every point of $\mathcal{F}$ can be expressed in base A with digits in $\varphi(\mathcal{D})$ using only negative powers of the base. In fact, $\mathcal{F}$ is given explicitly by

$$\mathcal{F} = \Big\{ \sum_{j=1}^{\infty} A^{-j}\varphi(d_j) \,:\, d_j \in \mathcal{D} \Big\}. \tag{4.19}$$

Indeed, it is easy to see that $\mathcal{F}$ is non-empty, bounded, and satisfies (4.17). The fact that $\mathcal{F}$ is closed follows by a Cantor diagonal argument.

Suppose $\mathcal{F}$ has positive measure. In order to define a tiling, we want the union $\bigcup_{d\in\mathcal{D}}(\mathcal{F}+\varphi(d))$ to be essentially disjoint (that is, disjoint up to a μ-measure zero set); since multiplication by A on $\mathbb{K}_A$ enlarges the measure by a factor of a, then it is necessary for $\mathcal{D}$ to have exactly a elements. In the next section, we show that if $(A,\mathcal{D})$ is a standard digit system, then $\mathcal{F}(A,\mathcal{D})$ has positive measure.

Remark 4.3.3. *Without loss of generality, we will always assume that $0 \in \mathcal{D}$. This can be done because replacing $\mathcal{D}$ by $\mathcal{D}-v$, where $v \in \mathbb{Z}^n[A]$ is a constant vector, means that $\mathcal{F}(A,\mathcal{D}-v)$ is a translation of $\mathcal{F}(A,\mathcal{D})$, hence it is equivalent when we study the existence of tilings.*

Remark 4.3.4. *When $n = 1$ and $A = \frac{a}{b}$ where a and b are coprime integers, we obtain $\mathbb{Z}^n((B)) \simeq \mathbb{Q}_b$, where $\mathbb{Q}_b$ is the ring of b-adic numbers, and $\mathbb{Z}^n[[B]] \simeq \mathbb{Z}_b$, where $\mathbb{Z}_b$ is the ring of b-adic integers.*

4.4 Topological results on rational self-affine tiles

Let $(A,\mathcal{D})$ be a digit system, and let $\mathbb{K}_A$ be the representation space with metric $\mathbf{d}$ and Haar measure μ as before. In this section, we give some equivalent topological and combinatorial conditions for the set $\mathcal{F}(A,\mathcal{D})$.

We introduce some definitions before stating the results. To denote blocks of digits, let

$$\mathcal{D}_k := \big\{ d_0 + Ad_1 + \cdots + A^{k-1}d_{k-1} \,:\, d_0,\ldots,d_{k-1} \in \mathcal{D} \big\} \qquad \text{and} \qquad \mathcal{D}_\infty := \bigcup_{k\geqslant 1} \mathcal{D}_k. \tag{4.20}$$

From the set equation (4.17), we deduce the *iterated set equation*

$$A^k\mathcal{F} = \bigcup_{d\in\mathcal{D}_k} (\mathcal{F}+\varphi(d)). \tag{4.21}$$

Definition 4.4.1 (Uniform discreteness)**.** *We say that a set $M \subset \mathbb{K}_A$ is uniformly discrete if there exists $r > 0$ such that every open ball of radius r in $\mathbb{K}_A$ contains at most one point of M.*

Our first result is a criterion for $\mathcal{F}(A, \mathcal{D})$ to have positive measure formulated in terms of $\mathcal{D}$. It is an extension of [53, Theorem 1.1] and [43, Theorem 10] to the case of rational self-affine tiles.

Theorem 4.4.2. *Let $(A, \mathcal{D})$ be a digit system, and let $\mathcal{F} = \mathcal{F}(A, \mathcal{D}) \subset \mathbb{K}_A$. Then $\mathcal{F}$ has positive measure if and only if for every $k \geqslant 1$, all a^k expansions in $\mathcal{D}_k$ are distinct, and $\varphi(\mathcal{D}_\infty)$ is a uniformly discrete subset of $\mathbb{K}_A$.*

Proof. We mention that in [53, p. 32 – 34] a similar proof is provided in the setting of self-affine tiles in $\mathbb{R}^n$. Assume first that $\varphi(\mathcal{D}_\infty)$ is a uniformly discrete set and that all the elements in $\mathcal{D}_k$ are distinct for every $k \geqslant 1$. Recall the metric $\boldsymbol{\ell}$ and the constant $0 \leqslant \kappa < 1$ defined in Lemma 4.3.1. Then $\boldsymbol{\ell}(B \cdot (x, y), 0) \leqslant \kappa \boldsymbol{\ell}((x, y), 0)$ for every $(x, y) \in \mathbb{K}_A$. Consider the closed ball $\mathbf{B}'_r(0) := \{(x, y) \in \mathbb{K}_A : \boldsymbol{\ell}((x, y), 0) \leqslant r\}$, and let $(x, y) \in \mathbf{B}'_r(0)$. Let Ψ be the map defined in (4.18); then by Hutchinson [39], $\mathcal{F}$ is the Hausdorff limit of the sequence $\{\Psi^k(\mathbf{B}'_r(0))\}_{k \geqslant 1}$. Suppose that $r \geqslant \frac{\kappa}{1-\kappa} \max_{d \in \mathcal{D}}\{\boldsymbol{\ell}(\varphi(d), 0)\}$; then $\Psi(\mathbf{B}'_r(0)) \subset \mathbf{B}'_r(0)$. Consequently, by Lebesgue's dominated convergence theorem, we get

$$\mu(\mathcal{F}) = \lim_{k \to \infty} \mu(\Psi^k(\mathbf{B}'_r(0))).$$

It suffices to find a set of positive measure contained in every $\Psi^k(\mathbf{B}'_r(0))$. Since $\varphi(\mathcal{D}_\infty)$ is uniformly discrete, there exists $\delta > 0$ such that for every $(x, y) \neq (x', y')$ in $\varphi(\mathcal{D}_\infty)$ it holds that $\boldsymbol{\ell}((x, y), (x', y')) > \delta$. For $0 < \varepsilon < \min\{\frac{\delta}{2}, r\}$, consider the closed ball $\mathbf{B}'_\varepsilon(0)$. By hypothesis, $\mathcal{D}_k$ has a^k distinct elements for all k, and all sets of the form $\mathbf{B}'_\varepsilon(0) + \varphi(d)$ for $d \in \mathcal{D}_k$ are pairwise disjoint. Therefore $\mu(\Psi^k(\mathbf{B}'_\varepsilon(0))) = \mu(\mathbf{B}'_\varepsilon(0))$ for every k, and hence

$$\mu(\mathcal{F}) \geqslant \lim_{k \to \infty} \mu(\Psi^k(\mathbf{B}'_\varepsilon(0))) = \mu(\mathbf{B}'_\varepsilon(0)) > 0.$$

For the converse, assume $\mu(\mathcal{F}) > 0$. Then

$$a^k \mu(\mathcal{F}) = \mu(A^k \mathcal{F}) = \mu\Big(\bigcup_{d \in \mathcal{D}_k} (\mathcal{F} + \varphi(d))\Big) \leqslant \sum_{d \in \mathcal{D}_k} \mu(\mathcal{F} + \varphi(d)) \leqslant a^k \mu(\mathcal{F}),$$

so all the terms are equal. Hence, $|\mathcal{D}_k| = a^k$ and the union is essentially disjoint, meaning that for $d \neq d'$ in $\mathcal{D}_k$, we get

$$\mu((\mathcal{F} + \varphi(d)) \cap (\mathcal{F} + \varphi(d'))) = 0. \tag{4.22}$$

It remains to show that $\varphi(\mathcal{D}_\infty)$ is uniformly discrete. Suppose that this was not the case, then we can find a sequence $\{(d_l, d'_l)\}_{l \geqslant 1}$ where, d_l and d'_l are distinct elements of some $\mathcal{D}_{k_l}$ for each $l \geqslant 1$, and such that

$$\lim_{l\to\infty} \mathbf{d}(\varphi(d_l), \varphi(d_l')) = 0.$$

We claim that there is (x, y) sufficiently close to 0 such that $\mu((\mathcal{F} + (x, y)) \cap \mathcal{F}) > 0$. If $\mu(\mathcal{F}) > 0$, then by Federer [28, page 156, Corollary 2.9.9], there exists a Lebesgue point $(x^*, y^*) \in \mathcal{F}$ such that, if $\chi_{\mathcal{F}}$ is the characteristic function of $\mathcal{F}$, then

$$\lim_{r\searrow 0} \frac{1}{\mu(\mathbf{B}_r(x^*, y^*))} \int_{\mathbf{B}_r(x^*,y^*)} \chi_{\mathcal{F}}(x, y)\, d\mu = \chi_{\mathcal{F}}(x^*, y^*) = 1. \tag{4.23}$$

Given $\varepsilon > 0$, this implies the existence of a sufficiently small r for which

$$\mu(\mathbf{B}_r(x^*, y^*) \cap \mathcal{F}) \geqslant (1 - \varepsilon)\, \mu(\mathbf{B}_r(x^*, y^*)). \tag{4.24}$$

Here, $\mathbf{B}_r(x^*, y^*)$ denotes the closed ball of center (x^*, y^*) and radius r with respect to the metric $\mathbf{d}$. Let $0 < \varepsilon' < r$, and consider $(x, y) \in \mathbb{K}_A$ such that $\mathbf{d}((x, y), 0) < \varepsilon' < r$. Then

$$\begin{aligned} \mu(\mathbf{B}_r(x^*, y^*) \cap (\mathcal{F} + (x, y))) &\geqslant \mu(\mathbf{B}_{r-\varepsilon'}((x^*, y^*) + (x, y)) \cap (\mathcal{F} + (x, y))) \\ &\geqslant (1 - \varepsilon)\mu(\mathbf{B}_{r-\varepsilon'}(x^*, y^*)). \end{aligned} \tag{4.25}$$

Recall that b satisfies (4.14). Note that $\mathbf{B}_r(y^*) = \mathbf{B}_{b^{\lfloor \log_b r \rfloor}}(y^*)$, and define $K := \lfloor \log_b(r) \rfloor - \lfloor \log_b(r - \varepsilon') \rfloor > 0$. Then $y \in \mathbf{B}_r(y^*)$ if and only if $B^K y \in \mathbf{B}_{r-\varepsilon'}(B^K y^*)$. Hence, since μ is the product measure $\mu = \lambda \times \mu_B$, we get

$$\mu(\mathbf{B}_{r-\varepsilon'}(x^*, y^*)) = \lambda\Big(\frac{r - \varepsilon'}{r}\mathbf{B}_r(x^*)\Big)\, \mu_B(B^K \mathbf{B}_r(y^*)) = \Big(\frac{r - \varepsilon'}{r}\Big)^n b^{-K} \mu(\mathbf{B}_r(x^*, y^*)),$$

and thus for the appropriate value of $\varepsilon'' > 0$ it follows from (4.25) that

$$\mu(\mathbf{B}_r(x^*, y^*) \cap (\mathcal{F} + (x, y))) > (1 - \varepsilon'')\mu(\mathbf{B}_r(x^*, y^*)). \tag{4.26}$$

By inclusion-exclusion and combining (4.24) and (4.26), we get

$$\mu((\mathcal{F} + (x, y)) \cap \mathcal{F}) > (1 - \varepsilon - \varepsilon'')\mu(\mathbf{B}_r(x^*, y^*)) > 0$$

for (x, y) sufficiently close to 0. This implies that, for large enough l,

$$\mu((\mathcal{F} + \varphi(d_l)) \cap (\mathcal{F} + \varphi(d_l')) = \mu(\mathcal{F} \cap (\mathcal{F} + \varphi(d_l' - d_l)) > 0,$$

which is a contradiction. □

Corollary 4.4.3. *If $(A, \mathcal{D})$ is a standard digit system, then $\mathcal{F}(A, \mathcal{D})$ has positive measure.*

The second result of this section gives some topological equivalences for $\mathcal{F}$ being a rational self-affine tile. It is in the spirit of [53, Theorem 1.1].

Theorem 4.4.4. *Let $(A, \mathcal{D})$ be a digit system and let $\mathcal{F} = \mathcal{F}(A, \mathcal{D}) \subset \mathbb{K}_A$. The following assertions are equivalent:*

(i) $\mathcal{F}$ has positive measure.

(ii) $\mathcal{F}$ has non-empty interior.

(iii) $\mathcal{F}$ is the closure of its interior, and its boundary $\partial\mathcal{F}$ has measure zero.

Proof. $(iii) \Rightarrow (ii) \Rightarrow (i)$ is trivial.

$(ii) \Rightarrow (iii)$ follows from the same arguments as the proof found in [53, p. 30 – 32]. Suppose $\mathcal{F}^\circ \neq \varnothing$. The inclusion $\overline{\mathcal{F}^\circ} \subset \mathcal{F}$ holds because $\mathcal{F}$ is closed. To prove the converse inclusion, recall that $\mathcal{F}$ satisfies the set equation (4.17), and so $\bigcup_{d\in\mathcal{D}} A^{-1}(\mathcal{F}^\circ + \varphi(d))$ is an open set contained in $\mathcal{F}$ and hence in $\mathcal{F}^\circ$. Taking closure yields

$$\bigcup_{d\in\mathcal{D}} A^{-1}(\overline{\mathcal{F}^\circ} + \varphi(d)) \subset \overline{\mathcal{F}^\circ},$$

hence $\Psi(\overline{\mathcal{F}^\circ}) \subset \overline{\mathcal{F}^\circ}$ where Ψ is the map in (4.18), and $\Psi^k(\overline{\mathcal{F}^\circ})$ converges to the attractor $\mathcal{F}$. Thus, every point of $\mathcal{F}$ is a limit point of a sequence in $\Psi^k(\overline{\mathcal{F}^\circ})$, and so $\mathcal{F} \subset \overline{\mathcal{F}^\circ}$.

Next, we prove that the boundary has measure zero. Suppose $\mathcal{F}$ contains an open set. Then we can choose k large enough so that there is an open ball in $A^k\mathcal{F}$ containing the set $\mathcal{F} + \varphi(d)$ for some d in $\mathcal{D}_k$. Since we get an open ball covered by compact sets, they must necessarily overlap. Therefore, the boundary of $\mathcal{F} + \varphi(d)$ is contained in $\bigcup_{d'\in\mathcal{D}_k\setminus\{d\}} \mathcal{F} + \varphi(d')$. We have shown in (4.22) that for $d \neq d'$ in $\mathcal{D}_k$, the sets $\mathcal{F} + \varphi(d)$ and $\mathcal{F} + \varphi(d')$ are essentially disjoint. This is equivalent to having disjoint interiors, so we get

$$\partial(\mathcal{F} + \varphi(d)) \subset \bigcup_{d'\in\mathcal{D}_k\setminus\{d\}} \partial(\mathcal{F} + \varphi(d')).$$

Hence

$$\mu(\partial(\mathcal{F} + \varphi(d))) \leqslant \sum_{d'\in\mathcal{D}_k\setminus\{d\}} \mu(\partial(\mathcal{F} + \varphi(d)) \cap \partial(\mathcal{F} + \varphi(d'))) = 0$$

and therefore $\mu(\partial\mathcal{F}) = 0$.

$(i) \Rightarrow (ii)$: Because $\mathcal{F}$ has positive measure, it has a Lebesgue point (x^*, y^*) satisfying (4.24). This implies that we can consider a sequence $\varepsilon_k \searrow 0$ together with a sequence of radii $r_k \searrow 0$ such that, for every $l > 0$,

$$\mu(A^l(\mathbf{B}_{r_k}(x^*, y^*) \cap \mathcal{F})) \geqslant (1 - \varepsilon_k)\, \mu(A^l\mathbf{B}_{r_k}(x^*, y^*)). \tag{4.27}$$

Claim 1: *For every index k there exists a large enough $l_k > 0$ and $(u^{(k)}, v^{(k)}) \in \mathbb{K}_A$ such that $\mathbf{B}_1(u^{(k)}, v^{(k)}) \subset A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)$ with*

$$\mu(\mathbf{B}_1(u^{(k)}, v^{(k)}) \cap A^{l_k}\mathcal{F}) \geqslant (1 - C\,\varepsilon_k)\,\mu(\mathbf{B}_1(u^{(k)}, v^{(k)})),$$

where (x^*, y^*) is a Lebesgue point of $\mathcal{F}$ and $C > 0$ is a constant depending only on the space $\mathbb{K}_A$.

To prove the claim, we draw on ideas from [53, p. 35]. With a slight abuse of notation, we will use $\mathbf{B}_r(\cdot)$ to denote closed balls of radius r in each respective space. Fix k, and note that, since A is expanding, $A^l\mathbf{B}_{r_k}(x^*) \subset \mathbb{R}^n$ is an ellipsoid whose shortest axis' length goes to infinity as l goes to infinity. Consider $l_k > 0$ large enough so that $b^{-l_k} \leqslant r_k$, and such that the ellipsoid $A^{l_k}\mathbf{B}_{r_k}(x^*)$ has a shortest axis greater than 4. Define $E_k := \{x \in A^{l_k}\mathbf{B}_{r_k}(x^*) : d(x, \partial(A^{l_k}\mathbf{B}_{r_k}(x^*))) \geqslant 1\}$ as the set of points of $A^{l_k}\mathbf{B}_{r_k}(x^*)$ whose (Euclidean) distance from the boundary is at least 1. Consider the set $2E_k - x^*$, obtained by doubling E_k and centering around x^*. Then $E_k \subsetneq A^{l_k}\mathbf{B}_{r_k}(x^*) \subsetneq 2E_k - x^*$ and $\lambda(A^{l_k}\mathbf{B}_{r_k}(x^*)) \leqslant \lambda(2E_k - x^*) = 2^n\lambda(E_k)$.

Consider the compact subset of $\mathbb{K}_A$ given by $\mathcal{U}_k := E_k \times A^{l_k}\mathbf{B}_{r_k}(y^*)$, which is trivially covered by the collection of unit balls $\mathcal{G} := \{\mathbf{B}_1(u, v) : (u, v) \in \mathcal{U}_k\}$. Let $\{\mathbf{B}_1(u_1, v_1), \ldots, \mathbf{B}_1(u_s, v_s)\}$ be a maximal disjoint subcollection of $\mathcal{G}$. Then $\mathcal{U}_k \subset \bigcup_{j=1}^{s} \mathbf{B}_2(u_j, v_j)$ because of the following reason: let $(x, y) \in \mathcal{U}_k$. If $\mathbf{B}_1(x, y) \notin \mathcal{G}$, then by maximality there exists $j \in \{1, \ldots, s\}$ with $\mathbf{B}_1(x, y) \cap \mathbf{B}_1(u_j, v_j) \neq \varnothing$. Take $(x', y') \in \mathbf{B}_1(x, y) \cap \mathbf{B}_1(u_j, v_j)$. Then

$$\mathbf{d}((x, y), (u_j, v_j)) \leqslant \mathbf{d}((x, y), (x', y')) + \mathbf{d}((x', y'), (u_j, v_j)) \leqslant 2.$$

This yields

$$\begin{aligned} \mu(A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)) &= \lambda(A^{l_k}\mathbf{B}_{r_k}(x^*))\,\mu_B(A^{l_k}\mathbf{B}_{r_k}(y^*)) \\ &\leqslant 2^n\lambda(E_k)\,\mu_B(A^{l_k}\mathbf{B}_{r_k}(y^*)) \\ &= 2^n\mu(\mathcal{U}_k) \leqslant 2^n \sum_{j=1}^{s} \mu(\mathbf{B}_2(u_j, v_j)). \end{aligned} \tag{4.28}$$

Note that $\lambda(\mathbf{B}_2(u_j)) = 2^n\lambda(\mathbf{B}_1(u_j))$ while $\mu_B(\mathbf{B}_2(v_j)) = b^{\lfloor \log_b 2 \rfloor}\mu_B(\mathbf{B}_1(v_j))$; hence

$$\sum_{j=1}^{s} \mu(\mathbf{B}_2(u_j, v_j)) = 2^n b^{\lfloor \log_b 2 \rfloor}\mu\Big(\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j)\Big)$$

because the unit balls are disjoint. Combining this with (4.28), we have

$$\mu(A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)) \leqslant C\,\mu\Big(\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j)\Big) \tag{4.29}$$

for $C := 4^n b^{\lfloor \log_b 2 \rfloor}$. We show next that all the balls $\mathbf{B}_1(u_j, v_j) = \mathbf{B}_1(u_j) \times \mathbf{B}_1(v_j)$ are contained in $A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)$. Fix $j \in \{1, \dots, s\}$. For the real part, $u_j \in E_k$, meaning it is a point of $A^{l_k}\mathbf{B}_{r_k}(x^*)$ which is at distance at least one from its boundary, hence $\mathbf{B}_1(u_j) \subset A^{l_k}\mathbf{B}_{r_k}(x^*)$. For the B-adic part, consider $y \in \mathbf{B}_1(v_j)$ and recall that $A^{-l_k}v_j \in \mathbf{B}_{r_k}(y^*)$. From the ultrametric inequality it follows

$$\begin{aligned}\mathbf{d}_B(A^{-l_k}y, y^*) &\leqslant \max\{\mathbf{d}_B(A^{-l_k}y, A^{-l_k}v_j), \mathbf{d}_B(A^{-l_k}v_j, y^*)\} \\ &= \max\{b^{-l_k}\mathbf{d}_B(y, v_j), r_k\} \\ &\leqslant \max\{b^{-l_k}, r_k\} = r_k,\end{aligned} \tag{4.30}$$

since we assumed $b^{-l_k} \leqslant r_k$. Therefore, $A^{-l_k}y \in \mathbf{B}_{r_k}(y^*)$ for every $y \in \mathbf{B}_1(v_j)$, and so $\mathbf{B}_1(v_j) \subset A^{l_k}\mathbf{B}_{r_k}(y^*)$. From (4.27) it follows that

$$\mu(A^{l_k}\mathbf{B}_{r_k}(x^*, y^*) \setminus (A^{l_k}\mathbf{B}_{r_k}(x^*, y^*) \cap A^{l_k}\mathcal{F})) \leqslant \varepsilon_k \mu(A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)). \tag{4.31}$$

Equations (4.29) and (4.31) and the fact that $\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j) \subset A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)$ imply

$$\mu\Big(\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j) \setminus \Big(\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j) \cap A^{l_k}\mathcal{F}\Big)\Big) \leqslant \varepsilon_k\, C\, \mu\Big(\bigsqcup_{j=1}^{s} \mathbf{B}_1(u_j, v_j)\Big).$$

Since the balls $\mathbf{B}_1(u_j, v_j)$ are pairwise disjoint and contained in $A^{l_k}\mathbf{B}_{r_k}(x^*, y^*)$, then for at least one $j_k \in \{1, \dots, s\}$ it holds that

$$\mu(\mathbf{B}_1(u_{j_k}, v_{j_k}) \cap A^{l_k}\mathcal{F}) \leqslant (1 - \varepsilon_k\, C)\mu(\mathbf{B}_1(u_{j_k}, v_{j_k})),$$

which yields Claim 1 with $(u^{(k)}, v^{(k)}) = (u_{j_k}, v_{j_k})$.

Back to the main proof, Claim 1 together with the iterated set equation (4.21) implies that, for every k, there exists $l_k > 0$ and $(u^{(k)}, v^{(k)}) \in \mathbb{K}_A$ such that

$$\mu\Big(\mathbf{B}_1(u^{(k)}, v^{(k)}) \cap \Big(\bigcup_{d \in \mathcal{D}_{l_k}} \mathcal{F} + \varphi(d)\Big)\Big) \leqslant (1 - \varepsilon_k\, C)\mu(\mathbf{B}_1(u^{(k)}, v^{(k)})). \tag{4.32}$$

Define the finite sets

$$\mathcal{V}_k := \{\varphi(d) - (u^{(k)}, v^{(k)}) \,:\, d \in \mathcal{D}_{l_k},\ (\mathcal{F} + \varphi(d) - (u^{(k)}, v^{(k)})) \cap \mathcal{F} \neq \varnothing\}.$$

Then shifting the arguments inside the measures in (4.32) by $-(u^{(k)}, v^{(k)})$ and restricting to translates contained in $\mathcal{V}_k$ yields

$$\mu(\mathbf{B}_1(0) \cap (\mathcal{F} + \mathcal{V}_k)) \leqslant (1 - \varepsilon_k\, C)\mu(\mathbf{B}_1(0)).$$

Note that $\mathcal{F} \cap \mathbf{B}_1(e) \neq \varnothing$ for every $e \in \mathcal{V}_k$. Thus, because $\mathcal{F}$ is bounded, all $\mathcal{V}_k \subset \mathbf{B}_R(0)$ for a sufficiently large constant R. Recall that $\varphi(\mathcal{D}_\infty)$ is a uniformly discrete set by Theorem 4.4.2, and $\mathcal{D}_{l_k} \subset \mathcal{D}_\infty$, hence there exists $\delta > 0$ such that $\mathbf{d}(e, e') \geqslant \delta$ for every $e, e' \in \mathcal{V}_k$ for every k. This implies that the sequence of cardinalities $\{|\mathcal{V}_k|\}_{k \geqslant 1}$

is bounded. Therefore, $\{\mathcal{V}_k\}_{k\geqslant 1}$ has a convergent subsequence $\{\mathcal{V}_{k_j}\}_{j\geqslant 1}$ whose limit, denoted by $\mathcal{V}$, is a finite set. Then

$$\begin{aligned}\mu(\mathbf{B}_1(0)\cap(\mathcal{F}+\mathcal{V})) &\geqslant \liminf_{j\to\infty}\mu(\mathbf{B}_1(0)\cap(\mathcal{F}+\mathcal{V}_{k_j})\\ &\geqslant \liminf_{j\to\infty}(1-C\,\varepsilon_{k_j})\mu(\mathbf{B}_1(0))=\mu(\mathbf{B}_1(0)).\end{aligned}\tag{4.33}$$

Because T is closed this implies that $(\mathcal{F}+\mathcal{V})\cap\mathbf{B}_1(0)=\mathbf{B}_1(0)$. Thus $\mathcal{F}+\mathcal{V}$ is a finite union of translates of the compact set $\mathcal{F}$ containing inner points. Baire's theorem implies that $\mathcal{F}$ has non-empty interior. □

Example.

Let $A=\frac{4}{3}$ and $\mathcal{D}=\{0,1,8,9\}$. Note that $(\frac{4}{3},\mathcal{D})$ is not a standard digit system; however, $\mathcal{D}$ can be decomposed as $\mathcal{D}=\{0,1\}+4\,\{0,2\}$, and every $d\in\mathcal{D}$ can be uniquely expressed as $d=e+4\,e'$ with $e\in\{0,1\}$ and $e'\in\{0,2\}$. This is what Lagarias and Wang called a *product form digit set.* Consider the representation space $\mathbb{R}\times\mathbb{Q}_3$. We will show that set $\mathcal{F}(\frac{4}{3},\mathcal{D})\subset\mathbb{R}\times\mathbb{Q}_3$ has positive measure. Let

$$E_1:=\Big\{\sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(e_j)\ :\ e_j\in\{0,1\}\Big\}$$

and

$$E_2:=\Big\{\sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(e'_j)\ :\ e'_j\in\{0,2\}\Big\}.$$

The unique decomposition of the elements of $\mathcal{D}$ yields that $\mathcal{F}(\frac{4}{3},\mathcal{D})=E_1+4\,E_2$. We have

$$\begin{aligned}4\,E_2&=\Big\{3\sum_{j=0}^{\infty}(\tfrac{4}{3})^{-j}\varphi(e'_{j+1})\ :\ e'_j\in\{0,2\}\Big\}\\ &=\Big\{3\left(\varphi(e'_1)+\sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(e'_{j+1})\right)\ :\ e'_j\in\{0,2\}\Big\}\\ &=3\,\varphi(\{0,2\})+\Big\{\sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(3\,e'_{j+1})\ :\ e'_j\in\{0,2\}\Big\}\\ &=\varphi(\{0,6\})+\Big\{\sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(e''_j)\ :\ e''_j\in\{0,6\}\Big\}\end{aligned}$$

Define

$$E_3=\Big\{\sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(e''_j)\ :\ e''_j\in\{0,6\}\Big\},$$

then $\mathcal{F}(\frac{4}{3},\mathcal{D})=E_1+E_3+\varphi(\{0,6\})$. Since $\{0,1\}+\{0,6\}=\{0,1,6,7\}$ and every $d\in\{0,1,6,7\}$ has a unique decomposition $d=e+e''$ with $e\in\{0,1\}$ and $e''\in\{0,6\}$, we get that $\mathcal{F}(\frac{4}{3},\{0,1,6,7\})=E_1+E_3$ and, since $(\frac{4}{3},\{0,1,6,7\})$ is a standard digit

FIGURE 4.4: The tile $\mathcal{F}$ related to the digit system with base $\frac{4}{3}$ and digits $\{0,1,8,9\}$.

system, the set $\mathcal{F}(\frac{4}{3},\{0,1,6,7\})$ has positive measure. Therefore, $\mathcal{F}(\frac{4}{3},\mathcal{D})$ has positive measure.

Figure 4.4 (left) depicts $\mathcal{F}(\frac{4}{3},\mathcal{D})$ in the plane. The picture is obtained by embedding $\mathbb{Q}_3$ in $\mathbb{R}$ using the map

$$\gamma : \mathbb{Q}_3 \to \mathbb{R}, \qquad \sum_{i=k}^{\infty} c_i 3^i \mapsto \sum_{i=k}^{\infty} c_i 3^{-i}, \qquad (c_i \in \{0,1,2\}), \tag{4.34}$$

and for any $(x,y) \in \mathbb{R} \times \mathbb{Q}_3$ we compute points $(x, \gamma(y)) \in \mathbb{R}^2$. On the right-hand side of Figure 4.4 we see the subdivision of $\mathcal{F}(\frac{4}{3},\mathcal{D})$ corresponding to the digits 0 (red), 1 (green), 8 (yellow) and 9 (blue). The different subtiles appear to have different shapes, but this is just a consequence of the way the embedding γ was defined.

4.5 A tiling theorem

In what follows, we will restrict ourselves to the case where $\mathcal{F}$ has positive measure. We have referred to $\mathcal{F}$ in this case as a tile, because we will show that there exists a *tiling* of the space $\mathbb{K}_A$ by translates of $\mathcal{F}$.

Definition 4.5.1 (Tiling, self-replicating tiling, multiple tiling)**.** *Assume $\mu(\mathcal{F}) > 0$. Let $\mathcal{S} \subset \mathbb{K}_A$ and consider the collection $\{\mathcal{F}+s : s \in \mathcal{S}\}$, which we denote as $\mathcal{F}+\mathcal{S}$ with a slight abuse of notation.*

1. *$\mathcal{F}+\mathcal{S}$ is said to be a tiling of $\mathbb{K}_A$ if it is a covering of $\mathbb{K}_A$ such that, for any $s \neq s'$ in $\mathcal{S}$, it holds that $\mu((\mathcal{F}+s) \cap (\mathcal{F}+s')) = 0$, or, equivalently, if $\mathcal{F}+s$ and $\mathcal{F}+s'$ have disjoint interiors. $\mathcal{S}$ is called a tiling set for $\mathcal{F}$. We say that $\mathcal{F}+\mathcal{S}$ tiles $\mathbb{K}_A$.*

2. *$\mathcal{F}+\mathcal{S}$ is said to be a self-replicating tiling if there exists an expanding linear map T on $\mathbb{K}_A$ such that, for each $s \in \mathcal{S}$, there exists a finite subset $J(s) \subset \mathcal{S}$ with*

$$T(\mathcal{F}+s) = \bigcup_{s' \in J(s)} (\mathcal{F}+s').$$

3. *$\mathcal{F}+\mathcal{S}$ is said to be a multiple tiling of $\mathbb{K}_A$, if there exists $k \in \mathbb{N}$ such that μ-almost every point of $\mathbb{K}_A$ is contained in exactly k distinct sets of the form $\mathcal{F}+s$ with $s \in \mathcal{S}$.*

It follows that a self-replicating tiling is completely determined by the set of tiles that touch the origin. We call a self-replicating tiling *atomic* if the origin touches exactly one tile. We want to study the nature of the tilings of $\mathbb{K}_A$ obtained using self-affine tiles. For any $k \geqslant 1$, consider the difference sets

$$\mathcal{D}_k - \mathcal{D}_k = \{d - d' \,:\, d, d' \in \mathcal{D}_k\},$$

and define

$$\Delta := \bigcup_{k=1}^{\infty} \varphi(\mathcal{D}_k - \mathcal{D}_k).$$

Theorem 4.5.2. *Suppose that $\mathcal{F}$ contains an open set. Then:*

(*i*) *There exists a set of translations $\mathcal{S} \subset \Delta$ such that $\mathcal{F}+\mathcal{S}$ tiles $\mathbb{K}_A$. Furthermore, there exists a translate $\mathcal{S}'$ of $\mathcal{S}$ such that $\mathcal{F}+\mathcal{S}'$ is an atomic self-replicating tiling of $\mathbb{K}_A$, and the expanding linear map associated to it is of the form $T = A^k$ for some sufficiently large k.*

(*ii*) *If Δ is a group, then $\mathcal{F}+\Delta$ is a tiling.*

Proof. (*i*) The proof is in the spirit of the one in [53, Theorem 1.2]. Suppose first that $0 \in \mathcal{F}^\circ$. Recall the definitions of $\mathcal{D}_k$ and $\mathcal{D}_\infty$ in (4.20) and the iterated set equation (4.21). Since A is expanding, we have

$$\mathbb{K}_A = \bigcup_{k\geqslant 1} A^k\mathcal{F} = \bigcup_{k\geqslant 1} \bigcup_{d\in\mathcal{D}_k} \mathcal{F}+\varphi(d) = \bigcup_{d\in\mathcal{D}_\infty} \mathcal{F}+\varphi(d),$$

hence $\mathcal{F}+\varphi(\mathcal{D}_\infty)$ is a covering of $\mathbb{K}_A$; also, (4.22) implies that two different translates of $\mathcal{F}$ are measure disjoint. Therefore, $\mathcal{S} = \varphi(\mathcal{D}_\infty)$ is a tiling set for $\mathcal{F}$.

Let $s \in \varphi(\mathcal{D}_\infty)$. Note that

$$A(\mathcal{F}+s) = A\mathcal{F} + As = \bigcup_{d\in\mathcal{D}} (\mathcal{F}+\varphi(d)+As),$$

and define $J(s) := \{\varphi(d)+As \,:\, d \in \mathcal{D}\}$. Since $A\mathcal{D}_k + \mathcal{D} = \mathcal{D}_{k+1}$, we have $A\varphi(\mathcal{D}_\infty) + \varphi(\mathcal{D}) \subset \varphi(\mathcal{D}_\infty)$, therefore $J(s) \subset \varphi(\mathcal{D}_\infty)$ We found an atomic self-replicating tiling with matrix A.

Suppose now that 0 is in the boundary of $\mathcal{F}$. Let $d^* \in \mathcal{D}_k$ and $\mathcal{D}' := \mathcal{D}_k - d^*$. Denote by Id the identity matrix. Consider $\mathcal{F}(A^k, \mathcal{D}')$ the self-affine tile with matrix

A^k and set of digits $\mathcal{D}'$. Then

$$\begin{aligned}
\mathcal{F}(A^k, \mathcal{D}') &= \sum_{j=1}^{\infty} A^{-jk} \varphi(\mathcal{D}_k - d^*) \\
&= \sum_{j=1}^{\infty} A^{-jk} ((A^{k-1} + \ldots + A + Id)\varphi(\mathcal{D}) - \varphi(d^*)) \\
&= \sum_{j=1}^{\infty} A^{-j} \varphi(\mathcal{D}) - \Big(\sum_{j=1}^{\infty} A^{-jk} \Big) \varphi(d^*) \\
&= \mathcal{F} - \Big(\sum_{j=1}^{\infty} A^{-jk} \Big) \varphi(d^*)
\end{aligned} \tag{4.35}$$

Suppose that d^* satisfies the property that $\mathcal{F}(A^k, \mathcal{D}')$ contains 0 in its interior. Then, because $0 \in \mathcal{D}'$, the proof above shows that $\mathcal{F}(A^k, \mathcal{D}')$ gives an atomic self-replicating tiling of $\mathbb{K}_A$ with matrix A^k and translation set

$$\varphi(\mathcal{D}'_\infty) = \Big\{ \sum_{j=0}^{\infty} A^{kj} \varphi(d_j) \,:\, d_j \in \mathcal{D}' \Big\}.$$

But the tile $\mathcal{F}$ is a translated copy of $\mathcal{F}(A^k, \mathcal{D}')$, meaning that, in that case, $\mathcal{F}$ would also induce a self-replicating tiling with matrix A^k and translation set $\varphi(\mathcal{D}'_\infty - \sum_{j=1}^{\infty} A^{-jk} d^*)$. So, it remains to find $d^* \in \mathcal{D}_k$ such that $\mathcal{F}(A^k, \mathcal{D}')$ has 0 as an interior point, which is equivalent to $\Big(\sum_{j=1}^{\infty} A^{-jk} \Big) \varphi(d^*)$ being an interior point of $\mathcal{F}$.

Consider the set of finite sums

$$\mathcal{L} := \Big\{ \sum_{j=1}^{k} A^{-j} \varphi(d_j) \,:\, k \geqslant 1, d_j \in \mathcal{D} \Big\},$$

which is dense in $\mathcal{F}$. Since by hypothesis $\mathcal{F}^\circ$ is non-empty, we can find some point $x^* = \sum_{j=1}^{k_0} A^{-j} \varphi(d_j) \in \mathcal{F}^\circ \cap \mathcal{L}$. Let $\varepsilon > 0$ such that the ball $\mathbf{B}_\varepsilon(x^*)$ is contained in $\mathcal{F}^\circ$. Take $k \geqslant k_0$ and set

$$d^* := A^k x^* = \sum_{j=1}^{k_0} A^{k-j} d_j \in \mathcal{D}_k$$

Now,

$$\begin{aligned}\mathbf{d}\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)\varphi(d^*), \varphi(x^*)\Big) &= \max\Big\{\Big\|\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)d^* - x^*\Big\|, \mathbf{d}_B\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)d^*, x^*\Big)\Big\}\\ &= \max\Big\{\Big\|\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)A^k x^* - x^*\Big\|, \mathbf{d}_B\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)A^k x^*, x^*\Big)\Big\}\\ &= \max\Big\{\Big\|\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)A^k x^* - x^*\Big\|, \mathbf{d}_B\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)A^k x^* - x^*, 0\Big)\Big\}.\end{aligned} \tag{4.36}$$

Since A is expanding, we get the equality

$$\sum_{j=1}^{\infty} A^{-jk} = A^{-k}(Id - A^{-k})^{-1},$$

which implies

$$\begin{aligned}\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)A^k x^* - x^* &= (Id - A^{-k})^{-1}x^* - x^*\\ &= (Id - A^{-k})^{-1}(x^* - (Id - A^{-k})x^*)\\ &= A^{-k}(Id - A^{-k})^{-1}x^*\\ &= \Big(\sum_{j=1}^{\infty} A^{-jk}\Big)x^*.\end{aligned} \tag{4.37}$$

Therefore, from (4.36) follows

$$\mathbf{d}\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)\varphi(d^*), \varphi(x^*)\Big) = \max\Big\{\Big\|\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)x^*\Big\|, \mathbf{d}_B\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)x^*, 0\Big)\Big\}.$$

In the Euclidean component, we get

$$\Big\|\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)x^*\Big\| \leqslant \|x^*\|\Big(\sum_{j=1}^{\infty} \|A^{-k}\|^j\Big)$$

where $\|A^{-k}\|$ denotes the Euclidean operator norm. Since A is expanding, the series converges because $\|A^{-k}\| \leqslant \rho(A)^{-k} < 1$, where $\rho(A)$ is the spectral radius of A. Moreover, $\|A^{-k}\| \to 0$ as $k \to \infty$ and thus $\sum_{j=1}^{\infty} \|A^{-k}\|^j \to 0$ as $k \to \infty$.

In the B-adic component, we get

$$\nu_B\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)x^*\Big) = \nu_B\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)\Big(\sum_{j=1}^{k_0} A^{-j}d_j\Big)\Big) \geq k+1,$$

and hence

$$\mathbf{d}_B\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)x^*, 0\Big) \leq b^{-k-1},$$

which also tends to 0 as $k \to \infty$.

This means that we can choose k large enough so that

$$\mathbf{d}\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)\varphi(d^*), \varphi(x^*)\Big) < \frac{\varepsilon}{2}$$

and so

$$\mathbf{B}_{\varepsilon/2}\Big(\Big(\sum_{j=1}^{\infty} A^{-jk}\Big)\varphi(d^*)\Big) \subset \mathbf{B}_{\varepsilon}(x^*) \subset \mathcal{F}^{\circ}.$$

(ii) Assume that Δ is a group. Since by (i) there is a subset $\mathcal{S} \subset \Delta$ for which $\mathcal{F} + \mathcal{S}$ tiles $\mathbb{K}_A$, $\mathcal{F} + \Delta$ is a covering of $\mathbb{K}_A$. Given any $s \neq s'$ in Δ, then it suffices to show

$$0 = \mu((\mathcal{F} + s) \cap (\mathcal{F} + s')) = \mu(\mathcal{F} \cap (\mathcal{F} + s' - s)).$$

Because Δ is a group, then $s' - s \in \Delta$. That means that there exists $k \geqslant 1$ and $d \neq d'$ in $\mathcal{D}_k$ such that $s' - s = \varphi(d - d')$, so the assertion is equivalent to

$$\mu((\mathcal{F} + \varphi(d)) \cap (\mathcal{F} + \varphi(d'))) = 0,$$

which holds by (4.22). $\square$

4.6 Characters

In order to prove the multiple tiling theorem, we use some results on the characters of $\mathbb{K}_A$ and of a certain torus $\mathbb{T}$. For more on the topic of character theory on locally compact abelian groups, we refer the reader to [70, Chapter 4].

Definition 4.6.1 (Character)**.** *A character χ on an abelian group G is a continuous function $\chi : G \to \mathbb{S}^1$ such that $\chi(x+y) = \chi(x)\chi(y)$ for all $x, y \in G$.*

Lemma 4.6.2. *The set of all characters of G constitutes a group, called the Pontryagin dual of G, denoted by $\widehat{G}$. It satisfies the following properties:*

1. *The Pontryagin dual of $\widehat{G}$ is isomorphic to G.*
2. *The Pontryagin dual of the product $G_1 \times G_2$ is isomorphic to $\widehat{G}_1 \times \widehat{G}_2$ and the characters are of the form $\chi = \chi_1 \cdot \chi_2$ with $\chi_1 \in \widehat{G}_1$ and $\chi_2 \in \widehat{G}_2$.*

3. *Given a subgroup $H \subset G$, define the annihilator of H on G as*

$$Ann(H) := \{\chi \in \widehat{G} \,:\, \chi(H) = 1\}.$$

Then $\widehat{(G/H)} \simeq Ann(H)$ and $\widehat{H} \simeq \widehat{G}/Ann(H)$.

Proof. See [70, Chapter 4]. □

Characters of $\mathbb{Q}_b$ and more.

We are going to start our study of the characters in the simpler setting of $n = 1$. In this case, an expanding rational matrix is a rational number $\frac{a}{b}$ with modulus greater than one. Let $\frac{a}{b} \in \mathbb{Q}$ with a and b coprime integers, $b \geq 2$. Rational base number systems were introduced and studied in Section 2.6, where we introduced b-adic numbers. We defined the b-adic valuation ν_b and the b-adic absolute value $|\cdot|_b$ in Definition 2.6.1. The space $\mathbb{K}_{\frac{a}{b}} = \mathbb{R} \times \mathbb{Q}_b$ was the representation space. We begin by showing that the Pontryagin dual of $\mathbb{Q}_b$ is isomorphic to $\mathbb{Q}_b$.

Definition 4.6.3 (b-adic integer and fractional parts)**.** *Let $z \in \mathbb{Q}_b$ with b-adic expansion $z = \sum_{j=\nu_b(z)}^{\infty} z_j b^j$ with $z_j \in \{0, \ldots, b-1\}$. Define the b-adic integer part $\lfloor \cdot \rfloor_b$ and the b-adic fractional part $\{\cdot\}_b$ of z, respectively, as*

$$\lfloor z \rfloor_b := \sum_{j=0}^{\infty} z_j b^j \in \mathbb{Z}_b, \qquad \{z\}_b := \sum_{j=\nu_b(z)}^{-1} z_j b^j \in \mathbb{Z}[\tfrac{1}{b}].$$

Lemma 4.6.4. *For any $z \in \mathbb{Q}_b$ define the map*

$$\chi_z : \mathbb{Q}_b \to S^1, \qquad y \mapsto \exp(2\pi i \{zy\}_b). \tag{4.38}$$

Then χ_z is continuous and multiplicative, that is, $\chi_z(y + y') = \chi_z(y)\chi_z(y')$.

Proof. Fix $z = \sum_{j=\nu_b(z)}^{\infty} b^j z_j \in \mathbb{Q}_b$. For the multiplicativity, we prove first the following claim: given $\omega, \omega' \in \mathbb{Q}_b$, it holds that

$$\{\omega\}_b + \{\omega'\}_b - \{\omega + \omega'\}_b \in \mathbb{Z}, \tag{4.39}$$

By definition of the fractional part, $\{\omega\}_b + \{\omega'\}_b - \{\omega + \omega'\}_b \in \mathbb{Z}[b]$. Also,

$$\begin{aligned}\{\omega\}_b + \{\omega'\}_b - \{\omega + \omega'\}_b &= (\{\omega\}_b - \omega) + (\{\omega'\}_b - \omega') + (\omega + \omega' - \{\omega + \omega'\}_b) \\ &= -\lfloor \omega \rfloor_b - \lfloor \omega' \rfloor_b + \lfloor \omega + \omega' \rfloor_b \in \mathbb{Z}[\tfrac{1}{b}]\end{aligned}$$

Since $\mathbb{Z}[b] \cap \mathbb{Z}[\frac{1}{b}] = \mathbb{Z}$, this yields the claim. Now, for any y, y' we have $\{z(y + y')\}_b = \{zy\}_b + \{zy'\}_b \mod \mathbb{Z}$, and this proves the multiplicativity.

For the continuity, let $y, y' \in \mathbb{Q}_b$ such that $|y - y'|_b \leq b^{\nu_b(z)}$. Then, for every $j \geq \nu_b(z)$, it holds that $|b^j(y - y')|_b \leq 1$, and so $b^j(y - y') \in \mathbb{Z}_b$ which implies

$\{b^j(y-y')\}_b = 0$. Then $\{z(y-y')\}_b = 0$ and hence, by multiplicativity, $\chi_z(y) = \chi_z(y')$. $\square$

Lemma 4.6.5. *The characters over $\mathbb{Q}_b$ are of the form*

$$\chi_z : \mathbb{Q}_b \to S^1, \qquad \chi_z(y) = \exp(2\pi i\{zy\}_b)$$

for $z \in \mathbb{Q}_b$. Moreover, there is a group isomorphism between $\mathbb{Q}_b$ and $\widehat{\mathbb{Q}}_b$ given by $z \mapsto \chi_z$.

Proof. Consider the map $\mathbb{Q}_b \to \widehat{\mathbb{Q}}_b$, $z \mapsto \chi_z$. We first prove that the group operations are compatible on both sets, that is, $\chi_{z+z'}(y) = \chi_z(y)\chi_{z'}(y)$ for every $y = \sum_{j=\nu_b(y)}^{\infty} b^j y_j \in \mathbb{Q}_b$. It is enough to show that $S_{z+z'}(y) = S_z(y) + S_{z'}(y) \mod \mathbb{Z}$. This is done analogously to the proof of Proposition 4.6.4.

Next, we show $\chi_z \neq \chi_0$ for $z \neq 0$, which implies injectivity. Let $z = \sum_{j=\nu_b(z)}^{\infty} b^j z_j$ with $z_{\nu_b(z)} \in \{1, \ldots, b-1\}$, and let $y = b^{-\nu_b(z)-1}$. Then

$$\{yz\}_b = \left\{ \sum_{j=-1}^{\infty} b^j z_{j+\nu_b(z)+1} \right\}_b = \frac{z_{\nu_b(z)}}{b} \notin \mathbb{Z}.$$

For the rest of the proof, we refer the reader to Hewitt and Ross [37, p. 400]. $\square$

Proposition 4.6.6. *The characters of $\mathbb{K}_{\frac{a}{b}} = \mathbb{R} \times \mathbb{Q}_b$ are of the form*

$$\chi_{r,z} : \mathbb{K}_{\frac{a}{b}} \to S^1, \qquad \chi_{r,z}(x,y) = \chi_r(x)\chi_z(y) = \exp(2\pi i x r)\exp(2\pi i\{zy\}_b)$$

for $(r,z) \in \mathbb{K}_{\frac{a}{b}}$. Moreover, there is a group isomorphism between $\mathbb{K}_{\frac{a}{b}}$ and $\widehat{\mathbb{K}}_{\frac{a}{b}}$ given by $(r,z) \mapsto \chi_{r,z}$.

Proof. In view of Lemma 4.6.2 we have the isomorphism $\widehat{\mathbb{K}}_{\frac{a}{b}} \simeq \widehat{\mathbb{R}} \times \widehat{\mathbb{Q}}_b$. Since it is known that $\widehat{\mathbb{R}} \simeq \mathbb{R}$, it remains to show that $\widehat{\mathbb{Q}}_b \simeq \mathbb{Q}_b$. This is a direct consequence of Lemma 4.6.5. $\square$

We have shown in Proposition 2.6.2 that the set $\varphi(\mathbb{Z}[\frac{a}{b}])$ is a lattice in $\mathbb{K}_{\frac{a}{b}}$, where φ is the diagonal embedding. Since a and b and coprime, it is not hard to check that $\varphi(\mathbb{Z}[\frac{a}{b}]) = \varphi(\mathbb{Z}[\frac{1}{b}])$. We write it as $\varphi(\mathbb{Z}[\frac{1}{b}])$ now so that the statements do not involve a.

Lemma 4.6.7. *The set $\mathbb{T} = [0,1] \times \mathbb{Z}_b$ is a fundamental domain for the lattice $\varphi(\mathbb{Z}[\frac{1}{b}])$, that is, $\mathbb{T} + \varphi(\mathbb{Z}[\frac{1}{b}])$ is a tiling of $\mathbb{K}_{\frac{a}{b}}$.*

Proof. Let $(x,y) \in \mathbb{K}_{\frac{a}{b}}$. Denote by $\lfloor \cdot \rfloor$ and $\{\cdot\}$ the (real) integer part and fractional part, respectively, and let $z := \{y\}_b + \lfloor x - \{y\}_b \rfloor \in \mathbb{Z}[\frac{1}{b}]$. One easily checks that

$$x = z + \{x - \{y\}_b\},$$

where $\{x - \{y\}_b\} \in [0,1]$, and

$$y = z + \lfloor y \rfloor_b - \lfloor x - \{y\}_b \rfloor$$

where $\lfloor y \rfloor_b - \lfloor x - \{y\}_b \rfloor \in \mathbb{Z}_b$. Hence, we have found $z \in \mathbb{Z}[\frac{1}{b}]$ such that $(x,y) \in \mathbb{T} + \varphi(z)$, therefore $\mathbb{T} + \varphi(\mathbb{Z}[\frac{1}{b}])$ is a covering of $\mathbb{K}_{\frac{a}{b}}$. We show next that this value of z is unique for almost every $(x,y) \in \mathbb{K}_{\frac{a}{b}}$.

First note that $\mathbb{Z}[\frac{1}{b}] \cap \mathbb{Z}_b = \mathbb{Z}$, hence if $z' \in \mathbb{Z}[\frac{1}{b}]$ satisfies that $y \in z' + \mathbb{Z}_b$, then $z' = \{y\}_b + c$ with $c \in \mathbb{Z}$. Suppose that $x \in z' + [0,1]$, then there exists $u \in [0,1]$ such that $x - \{y\}_b = c + u$. If $x - \{y\}_b \notin \mathbb{Z}$, then the only possible value of c is $\lfloor x - \{y\}_b \rfloor$ and in that case, $z' = z$. Since x is an arbitrary real number and $\{y\}_b \in \mathbb{Q}$, the set of points $(x,y) \in \mathbb{K}_{\frac{a}{b}}$ such that $x - \{y\}_b \in \mathbb{Z}$ has μ-measure zero. $\square$

The next step is to find the characters of $\mathbb{T} = [0,1] \times \mathbb{Z}_b$. This involves the quotient group $\mathbb{Q}_b/\mathbb{Z}_b$; when $b = p$ is a prime number, it is called the Prüfer group and is also denoted $\mathbb{Z}(p^\infty)$.

Proposition 4.6.8. *The characters over $\mathbb{T} = [0,1] \times \mathbb{Z}_b$ are of the form*

$$\chi_{r,z} : \mathbb{K}_{\frac{a}{b}} \to S^1, \qquad \chi_{r,z}(x,y) = \chi_r(x)\chi_z(y) = \exp(2\pi i x r)\exp(2\pi i \{yz\}_b)$$

for $r \in \mathbb{Z}$, $z \in \mathbb{Q}_b/\mathbb{Z}_b$. Moreover, there is a group isomorphism between $\mathbb{Z} \times (\mathbb{Q}_b/\mathbb{Z}_b)$ and $\widehat{\mathbb{T}}$ given by $(r,z) \mapsto \chi_{r,z}$.

Proof. It is known that the dual of $[0,1]$ is $\mathbb{Z}$, hence the characters of $[0,1]$ are of the form χ_r with $r \in \mathbb{Z}$. We prove next that $\widehat{\mathbb{Z}_b} \simeq \mathbb{Q}_b/\mathbb{Z}_b$. In view of Proposition (4.6.6) and Lemma (4.6.2), it suffices to show that $\mathrm{Ann}(\mathbb{Z}_b) \simeq \mathbb{Z}_b$, that is, given χ_z a character of $\mathbb{Q}_b$, we show $\chi_z(y) = 1$ for every $y \in \mathbb{Z}_b$ if and only if $z \in \mathbb{Z}_b$.

Let $z \in \mathbb{Z}_b$ and $y \in \mathbb{Z}_b$. Then $\{zy\}_b = 0$. Therefore, $\chi_z(y) = 1$. Consider now $z = \sum_{j=\nu_b(z)}^{\infty} b^j z_j \in \mathbb{Q}_b \setminus \mathbb{Z}_b$; then $\nu_b(z) \leq -1$. Let $y = 1 \in \mathbb{Z}_b$. Then $\{zy\}_b = \sum_{j=\nu_b(z)}^{-1} b^j z_j \notin \mathbb{Z}$. $\square$

Lattices.

In order to arrive at the study of the characters of $\mathbb{K}_A$ for an n-dimensional matrix A, we state some lemmas regarding lattices. The module $\mathbb{Z}^n[A]$ plays a principal role in the study of tilings by rational self-affine tiles. We will prove first that, embedded into the representation space $\mathbb{K}_A$, this module becomes a lattice, and we show later that it is a translation set for a multiple tiling given by copies of $\mathcal{F}$. First, we formalize the notion of lattice in our setting.

Definition 4.6.9 (Lattice)**.** *A subset Λ of $\mathbb{K}_A$ is a* lattice *if it satisfies the three following conditions:*

1. *Λ is a group.*

2. *Λ is uniformly discrete, meaning there exists $r > 0$ such that every open ball of radius r in $\mathbb{K}_A$ contains at most one point of Λ.*

3. *Λ is relatively dense, meaning there exists $R > 0$ such that every closed ball of radius R in $\mathbb{K}_A$ contains at least one point of Λ.*

We show next that $\varphi(\mathbb{Z}^n[A])$ satisfies these properties. We state a lemma first.

Lemma 4.6.10. *There exists an integer $K \geqslant 1$ such that*

$$\mathbb{Z}^n \cap B\mathbb{Z}^n[B] = \mathbb{Z}^n \cap \left(B\mathbb{Z}^n + B^2\mathbb{Z}^n + \cdots + B^K\mathbb{Z}^n\right). \tag{4.40}$$

Proof. For $k \geqslant 1$, define the lattices

$$\mathcal{L}_k[B] := \sum_{j=1}^{k} B^j \mathbb{Z}^n.$$

Since $\mathcal{L}_k[B]$ contains $B\mathbb{Z}^n$ and B is invertible, the lattice $\mathcal{L}_k[B]$ has full rank. Consider a non-zero integer m_k such that $m_k\mathcal{L}_k[B] \subset \mathbb{Z}^n$. Then $m_k\mathcal{L}_k[B]$ has finite index in $\mathbb{Z}^n$. From this fact, one deduces that the intersection $\mathbb{Z}^n \cap \mathcal{L}_k[B]$ has finite index in $\mathbb{Z}^n$ for every $k \geqslant 1$. Therefore, the chain of nested lattices

$$(\mathbb{Z}^n \cap \mathcal{L}_1[B]) \subset (\mathbb{Z}^n \cap \mathcal{L}_2[B]) \subset \cdots \subset (\mathbb{Z}^n \cap B\mathbb{Z}^n[B]) \subset \mathbb{Z}^n$$

must eventually stabilize after some $K \geqslant 1$. □

Proposition 4.6.11. *The set $\varphi(\mathbb{Z}^n[A])$ is a lattice in $\mathbb{K}_A$.*

Proof. The fact that $\varphi(\mathbb{Z}^n[A])$ is a group follows from the additive group structure of $\mathbb{Z}^n[A]$ because φ is a group homomorphism.

To prove the uniform discreteness of $\varphi(\mathbb{Z}^n[A])$, we claim that there exists $0 < r \leqslant 1$ such that $\mathbf{d}(\varphi(z), 0) \geqslant r$ for every non-zero $z \in \mathbb{Z}^n[A]$. If $\mathbf{d}(\varphi(z), 0) \geqslant 1$ then we are done. Suppose on the contrary that $\mathbf{d}(\varphi(z), 0) < 1$. Since $z \in \mathbb{Z}^n[A]$, we can write it as

$$z = \sum_{j=0}^{k} A^j z_j, \qquad z_j \in \mathbb{Z}^n, \tag{4.41}$$

with $z_k \neq 0$, and there is a minimal index k with this property. If $z_k \notin B\mathbb{Z}^n[B]$, then $B^k z \in \mathbb{Z}^n[B] \setminus B\mathbb{Z}^n[B]$, so one has $\mathbf{d}_B(z, 0) = b^k \mathbf{d}_B(B^k z, 0) \geqslant 1$, in contradiction to $\mathbf{d}(\varphi(z), 0) < 1$. Thus, $z_k \in \mathbb{Z}^n \cap B\mathbb{Z}^n[B]$. By Lemma 4.6.10, there are vectors w_1, w_2, $\dots$, $w_K \in \mathbb{Z}^n$ such that

$$z_k = Bw_1 + B^2 w_2 + \cdots + B^K w_K.$$

We claim that, in such case, one must have $k \le K$. Suppose that $k > K$. Then

$$z = A^k z_k + \sum_{j=0}^{k-1} A^j z_j = \sum_{j=0}^{k-K-1} A^j z_j + \sum_{j=k-K}^{k-1} (z_j + w_{k-j})A^j = \sum_{j=0}^{k-1} A^j z_j', \qquad z_j' \in \mathbb{Z}^n.$$

However, that would contradict the minimality of k. Therefore, $k \le K$. Now, let m denote the least common multiple of the denominators of the entries of A. Since z is a sum of integer vectors multiplied by A^k, $0 \leqslant k \leqslant K$, the non-zero entries of z are at least $1/m^K$ in absolute value. Hence, $\mathbf{d}(\varphi(z), 0) \geqslant \|z\| \geqslant 1/m^K = r$.

We now turn to the proof of relative denseness. Let $(x, y) \in \mathbb{K}_A$ be arbitrary. Choose $x' \in \mathbb{Z}^n$ to be the closest integer vector to x, so that $\|x - x'\| < 1$, and choose $y' \in \mathbb{Z}^n[A]$ such that $\mathbf{d}_B(y, y') \leqslant 1$ (this holds by taking $y' = \{y\}_B$, see Definition 4.6.14 below). Hence, $\mathbf{d}((x, y), (x', y')) \leqslant 1$. Choose $y'' \in \mathbb{Z}^n$ to be the closest integer vector to $x' - y' \in \mathbb{R}^n$. Let $z := y' + y'' \in \mathbb{Z}^n[A]$. Then $\|x' - z\| = \|x' - y' - y''\| < 1$. Moreover, $\mathbf{d}_B(z, y') = \mathbf{d}_B(y'', 0) \leqslant 1$ because $y'' \in \mathbb{Z}^n$. Therefore, $\mathbf{d}((x', y'), \varphi(z)) \leqslant 1$. This yields $\mathbf{d}((x, y), \varphi(z)) \leqslant 2$, which proves the Lemma. □

The next step is to define a space $\mathbb{Z}^n((B^*))$ that will be crucial later when we study the characters of $\mathbb{K}_A$. Prior to that, we prove the following Lemma.

Lemma 4.6.12. *The group $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B]$ is a lattice in $\mathbb{R}^n$.*

Proof. We show that $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B] \subset \mathbb{Z}^n + A\mathbb{Z}^n + \cdots + A^K\mathbb{Z}^n$ for some $K \geqslant 1$. Let $z \in \mathbb{Z}^n[A] \cap \mathbb{Z}^n[B]$. Since $z \in \mathbb{Z}^n[A]$, write $z = \sum_{j=0}^{k} A^j z_j$, with $z_j \in \mathbb{Z}^n$, $z_k \neq 0$ and k minimal. If $k = 0$, then $z \in \mathbb{Z}^n$. Assume $k \geqslant 1$. Since $z \in \mathbb{Z}^n[B]$, from solving for z_k it follows that $z_k \in \mathbb{Z}^n \cap B\mathbb{Z}^n[B]$. By Lemma 4.6.10, one can find vectors $w_1, w_2, \ldots, w_K \in \mathbb{Z}^n$ such that

$$z_k = Bw_1 + B^2 w_2 + \cdots + B^K w_K, \tag{4.42}$$

and by proceeding like in the proof of Proposition 4.6.11, one shows $k \leqslant K$. Therefore the inclusion follows. This implies that $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B]$ is contained in a lattice, and also it trivially contains the lattice $\mathbb{Z}^n$. Since it is a group, it is itself a lattice. □

For an arbitrary lattice $\Lambda \subset \mathbb{R}^n$, define its *dual lattice* by

$$\Lambda^* := \{x \in \mathbb{Z}^n \;:\; \langle x, z\rangle \in \mathbb{Z} \text{ for every } z \in \Lambda\},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^n$. Denote by A^* the transpose of A, and let Λ and Γ be full rank lattices such that

$$\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B] \subset \Lambda \qquad \text{and} \qquad \Gamma[A^*] \cap \Gamma[B^*] \subset \Lambda^* \subset \mathbb{Z}^n. \tag{4.43}$$

This is possible because $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B]$ is a lattice by Lemma 4.6.12, and the proof that $\Gamma[A^*] \cap \Gamma[B^*]$ is also a lattice is analogous.

Definition 4.6.13 (B-adic and B^*-adic expansions)**.** *Let $\mathcal{E} \subset \mathbb{Z}^n$ be a complete set of residue classes of $\mathbb{Z}^n[B]/B\mathbb{Z}^n[B]$ with $0 \in \mathcal{E}$. Then every $y \in \mathbb{Z}^n((B))$ has a unique expansion of the form*

$$y = \sum_{j=\nu_B(y)}^{\infty} B^j y_j, \qquad y_j \in \mathcal{E}, \tag{4.44}$$

which we call the B-adic expansion of y with coefficients in $\mathcal{E}$. Recall that $\nu_B(0) = \infty$, so the B-adic expansion of 0 is the empty sum.

Let B^* denote the transpose of B. Consider the full rank integer lattice Γ satisfying (4.43) and let $\mathcal{E}^* \subset \Gamma \subset \mathbb{Z}^n$ be a complete set of residue class representatives of $\Gamma[B^*]/B^*\Gamma[B^*]$ with $0 \in \mathcal{E}^*$. Consider the space $\mathbb{Z}^n((B^*))$ defined analogously to $\mathbb{Z}^n((B))$. Then every $s \in \mathbb{Z}^n((B^*))$ has a unique expansion of the form

$$s = \sum_{j=\nu^*(s)}^{\infty} {B^*}^j s_j, \qquad s_j \in \mathcal{E}^*, \tag{4.45}$$

where ν^* is the valuation in $\mathbb{Z}^n((B^*))$ defined in the same way as ν_B. We call this the B^*-adic expansion of s with coefficients in $\mathcal{E}^*$.

Definition 4.6.14 (B-adic and B^*-adic fractional and integer part)**.** *Given $y \in \mathbb{Z}^n((B))$ with B-adic expansion 4.44, we define the B-adic fractional part and the B-adic integer part of y, respectively, as*

$$\{y\}_B := \sum_{j=\nu_B(y)}^{-1} B^j y_j, \qquad \lfloor y \rfloor_B := \sum_{j=0}^{\infty} B^j y_j.$$

Given $s \in \mathbb{Z}^n((B^*))$ with B^*-adic expansion 4.45, we define the B^*-adic fractional part and the B^*-adic integer part of s, respectively, as

$$\{s\}_B^* := \sum_{j=\nu^*(s)}^{-1} {B^*}^j s_j, \qquad \lfloor s \rfloor_B^* := \sum_{j=0}^{\infty} {B^*}^j s_j.$$

From here onwards, whenever we have a B-adic series (resp. B^*-adic series), we assume the coefficients to lie in $\mathcal{E}$ (resp. $\mathcal{E}^*$).

Remark 4.6.15. *Recall that $b = |\mathcal{E}|$. We claim that, if $b = 1$, then the multiple tiling theorem holds. Note that, in this case,* $\det A = a$ *is an integer. Indeed, an analogous version of Theorem 4.7.3 is proven by Lagarias and Wang in [51] for integer matrices. However, they show in [51, Lemma 2.1] that this results also holds for self-affine tiles associated to expanding real matrices $A \in \mathbb{R}^{n\times n}$ with integer determinant, as long as there exists an A-invariant lattice in $\mathbb{R}^n$ containing the difference set $\mathcal{D}-\mathcal{D}$. If $\mathcal{F}(A, \mathcal{D})$*

is a rational self-affine tile and $b = 1$, then $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B]$ is a lattice by Lemma 4.6.12, and it is A-invariant because $\mathbb{Z}^n[B]/B\mathbb{Z}^n[B]$ is trivial and hence $A\mathbb{Z}^n[B] \subset \mathbb{Z}^n[B]$. Consider $c \in \mathbb{Z} \setminus \{0\}$ such that $c\mathcal{D} \subset \mathbb{Z}^n$; then $\mathcal{D} - \mathcal{D} \subset \frac{1}{c}(\mathbb{Z}^n[A] \cap \mathbb{Z}^n[B])$, which is an A-invariant lattice. Therefore, the assumption that $b \geqslant 2$ made in Remark 4.2.3 also applies to this section.

Remark 4.6.16. *We can assume w.l.o.g. that $\mathcal{E}$ has a subset $\{c_1, \dots, c_n\}$ such that the lattice $\Theta := \langle c_1, \dots, c_n \rangle_{\mathbb{Z}}$ has full rank in $\mathbb{R}^n$. To show this, suppose first that $b \geqslant n$. Take a matrix $R \in \mathbb{Z}^{n\times n}$ whose columns are distinct elements $\{c_1, \dots, c_n\}$ of $\mathcal{E}$. Consider the integer matrix $N(t) := tB - R$, where $t \in \mathbb{N}$ is chosen so that $tB \in \mathbb{Z}^{n\times n}$. Its determinant is $\det(N(t)) = \det(B)\det(t\cdot Id - AR) = \det(B)\,\chi_{AR}(t)$, where $\chi_{AR}(t) \in \mathbb{Q}[t]$ is the characteristic polynomial of AR. For all but finitely many $t \in \mathbb{N}$, it holds that $\chi_{AR}(t) \neq 0$. Hence, we can choose t in a way that the column vectors $\{\widetilde{c_1}, \dots, \widetilde{c_n}\}$ of $N(t)$ are linearly independent, and hence they span a full rank integer lattice. Note that $c_j - \widetilde{c_j} \in B\mathbb{Z}^n$ for $j = 1, \dots, n$, so we can replace each c_j by $\widetilde{c_j}$, and this produces a new residue set $\widetilde{\mathcal{E}}$ with the required property.*

Suppose now that $1 < b < n$. Choose $k \geqslant 1$ so that $b^k \geqslant n$. Note that a complete set of residues for $\mathbb{Z}^n[B^k]/B^k\mathbb{Z}^n[B^k]$ is given by $\mathcal{E} + B\mathcal{E} + \cdots + B^{k-1}\mathcal{E}$, so $|\mathbb{Z}^n[B^k]/B^k\mathbb{Z}^n[B^k]| = b^k \geqslant n$. Suppose that we consider the digit system $(A^k, \mathcal{D}_k)$ with $\mathcal{D}_k$ as in (4.20); then $\mathcal{E}_k$ satisfies the assumption of the previous paragraph. Note that from the iterated set equation (4.21) it follows that $\mathcal{F}(A^k, \mathcal{D}_k) = \mathcal{F}(A, \mathcal{D})$. Hence, whenever $|\mathcal{E}| < n$ we can work with $(A^k, \mathcal{D}_k)$ instead of $(A, \mathcal{D})$ and with $\mathcal{E} + B\mathcal{E} + \cdots + B^{k-1}\mathcal{E}$ instead of $\mathcal{E}$.

It is well known that the characters on $\mathbb{R}^n$ are given by

$$\chi_r : \mathbb{R}^n \to S^1, \qquad x \mapsto \exp(2\pi i \langle x, r\rangle),$$

where $r \in \mathbb{R}^n$, and $\widehat{\mathbb{R}}^n \simeq \mathbb{R}^n$ via the isomorphism $r \mapsto \chi_r$.

For any $s = \sum_{j=\nu^*(s)}^{\infty} B^{*j} s_j \in \mathbb{Z}^n((B^*))$ with $s_j \in \mathcal{E}^*$, define the map

$$\chi_s : \mathbb{Z}^n((B)) \to S^1, \qquad y \mapsto \exp(2\pi i S_s(y)), \tag{4.46}$$

where

$$S_s(y) := \sum_{j=\nu^*(s)}^{\infty} \langle \{B^j y\}_B, s_j\rangle. \tag{4.47}$$

The map is well defined because $\{B^j y\}_B = 0$ for all but finitely many indices j. We show next that this map is indeed a character.

Proposition 4.6.17. *For every $s \in \mathbb{Z}^n((B^*))$, the map χ_s defined in (4.46) is continuous and multiplicative, that is, $\chi_s(y + y') = \chi_s(y)\chi_s(y')$.*

Proof. Fix $s = \sum_{j=\nu^*(s)}^{\infty} B^{*j} s_j \in \mathbb{Z}^n((B^*))$ with $s_j \in \mathcal{E}^*$. For the multiplicativity, if suffices to show that $S_s(y + y') = S_s(y) + S_s(y') \mod \mathbb{Z}$. We prove first the following claim: given $\omega, \omega' \in \mathbb{Z}^n((B))$, it holds that

$$\{\omega\}_B + \{\omega'\}_B - \{\omega + \omega'\}_B \in \Lambda, \tag{4.48}$$

where Λ is the lattice satisfying (4.43). By definition of the B-adic fractional part, $\{\omega\}_B + \{\omega'\}_B - \{\omega + \omega'\}_B \in \mathbb{Z}^n[A]$. Also,

$$\begin{aligned}\{\omega\}_B + \{\omega'\}_B - \{\omega + \omega'\}_B &= (\{\omega\}_B - \omega) + (\{\omega'\}_B - \omega') + (\omega + \omega' - \{\omega + \omega'\}_B) \\ &= -\lfloor \omega \rfloor_B - \lfloor \omega' \rfloor_B + \lfloor \omega + \omega' \rfloor_B \in \mathbb{Z}^n[A^{-1}].\end{aligned}$$

Since $\mathbb{Z}^n[A] \cap \mathbb{Z}^n[A^{-1}] \subset \Lambda$ by definition of Λ, this yields the claim. Now, for any y, y' we have

$$S_s(y) + S_s(y') - S_s(y + y') = \sum_{j=\nu^*(s)}^{\infty} \langle \{B^j y\}_B + \{B^j y'\}_B - \{B^j (y + y')\}_B, s_j \rangle \tag{4.49}$$

where $\{B^j y\}_B + \{B^j y'\}_B - \{B^j (y + y')\}_B \in \Lambda$ by (4.48). The summands in (4.49) are non-zero only for a finite number of j's. For every index j, we have $s_j \in \mathcal{E}^* \subset \Gamma \subset \Lambda^*$, thus by definition of dual lattice,

$$\langle \{B^j y\}_B + \{B^j y'\}_B - \{B^j (y + y')\}_B, s_j \rangle \in \mathbb{Z}$$

and the multiplicativity of χ_s is established.

For the continuity, let $y, y' \in \mathbb{Z}^n((B))$ such that $\mathbf{d}_B(y, y') \leqslant b^{\nu^*(s)}$. Then, for every $j \geqslant \nu^*(s)$, it holds that $\mathbf{d}_B(B^j y, B^j y') \leqslant 1$, and so $B^j(y - y') \in \mathbb{Z}^n[[B]]$ which implies $\{B^j(y - y')\}_B = 0$. Then $S_s(y - y') = 0$ and hence, by multiplicativity, $\chi_s(y) = \chi_s(y')$. Thus χ_s is locally constant and, hence, continuous. $\square$

We will show that the Pontryagin dual of $\mathbb{K}_A$ is isomorphic to $\mathbb{R}^n \times \mathbb{Z}^n((B^*))$. To do so, we prove some lemmas first.

Lemma 4.6.18. *Let*

$$y = \sum_{k=\nu_B(y)}^{\infty} B^k y_k \in \mathbb{Z}^n((B))$$

with $y_k \in \mathcal{E}$ *and*

$$s = \sum_{j=\nu^*(s)}^{\infty} B^{*j} s_j \in \mathbb{Z}^n((B^*))$$

with $s_j \in \mathcal{E}^*$ *be given. Then*

$$S_s(y) = \sum_{j=\nu^*(s)}^{\infty} \langle \{B^j y\}_B, s_j \rangle = \sum_{k=\nu_B(y)}^{\infty} \langle y_k, \{B^{*k} s\}_B^* \rangle.$$

Proof. From direct calculation, we obtain

$$\begin{aligned}
S_s(y) &= \sum_{j=\nu^*(s)}^{\infty} \left\langle \{B^j y\}_B, s_j \right\rangle = \sum_{j=\nu^*(s)}^{\infty} \left\langle \sum_{k=\nu_B(y)}^{-j-1} B^{j+k} y_k, s_j \right\rangle \\
&= \sum_{j=\nu^*(s)}^{\infty} \sum_{k=\nu_B(y)}^{-j-1} \left\langle y_k, B^{*j+k} s_j \right\rangle = \sum_{k=\nu_B(y)}^{\infty} \sum_{j=\nu^*(s)}^{-k-1} \left\langle y_k, B^{*j+k} s_j \right\rangle \\
&= \sum_{k=\nu_B(y)}^{\infty} \left\langle y_k, \sum_{j=\nu^*(s)}^{-k-1} B^{*j+k} s_j \right\rangle = \sum_{k=\nu_B(y)}^{\infty} \left\langle y_k, \{B^{*k} s\}_B^* \right\rangle.
\end{aligned}$$

□

Our next step is to establish a Pontryagin duality between $\mathbb{Z}^n((B))$ and $\mathbb{Z}^n((B^*))$. For that purpose, we express both sets in terms of projective limits. For more on the topic we refer the reader to [72]. For each $k \in \mathbb{N}$, consider the quotients

$$\mathcal{E}_k := \mathbb{Z}^n[B]/B^k \mathbb{Z}^n[B].$$

Clearly, $\mathcal{E}_k \subset \mathcal{E}_{k+1}$ for every k, so we can define the canonical projections

$$\pi_k : \mathcal{E}_{k+1} \to \mathcal{E}_k, \qquad x \mapsto x \mod B^k.$$

Therefore, we have a projective system

$$\cdots \longrightarrow \mathcal{E}_{k+1} \xrightarrow{\pi_k} \mathcal{E}_k \xrightarrow{\pi_{k-1}} \mathcal{E}_{k-1} \longrightarrow \ldots \xrightarrow{\pi_1} \mathcal{E}_1,$$

which entitles the existence of the projective limit

$$\varprojlim_{k\in\mathbb{N}} \mathcal{E}_k = \{(M_k)_{k\in\mathbb{N}} \,:\, M_k \in \mathcal{E}_k \text{ and } \pi_k(M_{k+1}) = M_k \text{ for every } k\},$$

and it holds that

$$\mathbb{Z}^n((B)) \simeq \varinjlim_{j\in\mathbb{N}} \varprojlim_{k\in\mathbb{N}} B^{-j}\mathcal{E}_k. \tag{4.50}$$

Analogously, for $k \in \mathbb{N}$ consider

$$\mathcal{E}_k^* := \Gamma[B^*]/B^{*k}\Gamma[B^*].$$

Then

$$\mathbb{Z}^n((B^*)) \simeq \varinjlim_{j\in\mathbb{N}} \varprojlim_{k\in\mathbb{N}} B^{*-j}\mathcal{E}_k^*.$$

Proposition 4.6.19. *The characters of $\mathbb{K}_A$ are of the form*

$$\chi_{r,s} : \mathbb{K}_A \to S^1, \qquad \chi_{r,s}(x,y) = \chi_r(x)\chi_s(y) = \exp(2\pi i\langle x, r\rangle)\exp(2\pi i S_s(y)),$$

for $r \in \mathbb{R}^n$*,* $s \in \mathbb{Z}^n((B^*))$*, with* $S_s(y)$ *as in (4.47). Moreover, there is a group morphism given by* $(r, s) \mapsto \chi_{r,s}$*.*

Proof. In view of Lemma 4.6.2 we have the isomorphism $\widehat{\mathbb{K}_A} \simeq \widehat{\mathbb{R}^n} \times \widehat{\mathbb{Z}^n((B))}$. It is known that there is an isomorphism $\mathbb{R}^n \simeq \widehat{\mathbb{R}^n}$ given by $r \mapsto \chi_r$. Consider the map $\mathbb{Z}^n((B^*)) \to \widehat{\mathbb{Z}^n((B))}$, $s \mapsto \chi_s$; we show that it is an isomorphism. We first prove that the group operations are compatible on both sets, that is, $\chi_{s+s'}(y) = \chi_s(y)\chi_{s'}(y)$ for every $y = \sum_{j=\nu_B(y)}^{\infty} B^j y_j \in \mathbb{Z}^n((B))$ with $y_j \in \mathcal{E}$. It is enough to show that $S_{s+s'}(y) = S_s(y) + S_{s'}(y) \mod \mathbb{Z}$. Applying Lemma 4.6.18, we get

$$S_s(y) + S_{s'}(y) - S_{s+s'}(y) = \sum_{j=\nu_B(y)}^{\infty} \langle y_j, \{B^{*j}s\}_B^* + \{B^{*j}s'\}_B^* - \{B^{*j}(s+s')\}_B^* \rangle.$$

Proceeding in analogy to the proof of Proposition 4.6.17 and using the definition of B^*-adic fractional part, we can see that, for every index $j \geqslant \nu_B(y)$,

$$\{B^{*j}s\}_B^* + \{B^{*j}s'\}_B^* - \{B^{*j}(s+s')\}_B^* \subset \Gamma[B^*] \cap \Gamma[B^{*-1}] \subset \Lambda^* \subset \mathbb{Z}^n,$$

with Λ^* as in (4.43). Since $y_j \in \mathcal{E} \subset \mathbb{Z}^n$ for every $j \geqslant \nu_B(y)$ and is finite only for a finite number of indices, this yields the first part of the proof.

Next, we show the injectivity. In view of the first part of the proof, it suffices to show that $\chi_s \neq 1$ for $s \neq 0$. Let $s = \sum_{j=\nu^*(s)}^{\infty} B^{*j} s_j \in \mathbb{Z}^n((B^*)) \setminus \{0\}$ with $s_j \in \mathcal{E}^*$, and consider a point of the form $B^{-l}c \in \mathbb{Z}^n((B))$ for $0 \neq c \in \mathcal{E}$ and $l \in \mathbb{N}$. Note that $\{B^{j-l}c\}_B = 0$ whenever $j \geqslant l$. Therefore, applying again Lemma 4.6.18 we get

$$S_s(B^{-l}c) = \sum_{j=\nu^*(s)}^{l-1} \langle B^{j-l}c, s_j \rangle = \sum_{j=\nu^*(s)}^{l-1} \langle c, B^{*j-l}s_j \rangle = \Big\langle c, \sum_{j=\nu^*(s)}^{l-1} B^{*j-l}s_j \Big\rangle.$$

Suppose $\chi_s = 1$, then $S_s(B^{-l}c) \in \mathbb{Z}$ for every $l \in \mathbb{N}$ and every $c \in \mathcal{E}$. Recall that in Remark 4.6.16 we assumed w.l.o.g. that $\mathcal{E}$ has a subset $\{c_1, \ldots, c_n\}$ such that $\Theta := \langle c_1, \ldots, c_n \rangle_{\mathbb{Z}}$ is a full rank lattice in $\mathbb{R}^n$. Thus $S_s(B^{-l}\Theta) \subset \mathbb{Z}$ and hence, by the definition of dual lattice, $\sum_{j=\nu^*(s)}^{l-1} B^{*l-j}s_j \in \Theta^*$ holds for all $l \in \mathbb{N}$. Since $s \neq 0$, we have that

$$\sum_{j=\nu^*(s)}^{l-1} B^{*l-j}s_j \in (B^{*l-\nu^*(s)}\Gamma + \cdots + B^*\Gamma) \setminus (B^{*l-\nu^*(s)-1}\Gamma + \cdots + B^*\Gamma).$$

Then $(B^{*l-\nu^*(s)}\Gamma + \cdots + B^*\Gamma)_{l \geqslant 1}$ is a strictly nested sequence of lattices in $\mathbb{Q}^n$. Given l, consider the entries of all the vectors of $B^{*l-\nu^*(s)}\Gamma + \cdots + B^*\Gamma$ expressed as irreducible fractions, and define m_l to be the maximum of the denominators of these fractions. It is clear that $m_l \leqslant m_{l+1}$. However, note that $(m_l)_{l \geqslant 1}$ does not stabilize: this would imply that, for some lattice Λ_l, there are infinitely many steps in which we can add a point and form a strictly larger lattice, while not increasing the bound on the denominators, which is not possible. This means that Θ^* contains points with entries having arbitrarily large denominators, which is a contradiction.

For the surjectivity, consider a character $\chi \in \widehat{\mathbb{Z}^n((B))}$. By classical arguments following [70, p. 139], we obtain from (4.50) that

$$\widehat{\mathbb{Z}^n((B))} \simeq \varprojlim_{j\in\mathbb{N}} \varinjlim_{k\in\mathbb{N}} \widehat{B^{-j}\mathcal{E}_k}. \tag{4.51}$$

Since $B^{-j}\mathcal{E}_k = B^{-j}\mathbb{Z}^n[B]/B^{k-j}\mathbb{Z}^n[B]$ is a finite group of cardinality b^k, so is its dual. Consider the group

$$G_{j,k} := B^{*j-k}\Gamma[B^*]/B^{*j}\Gamma[B^*],$$

and note that $|G_{j,k}| = b^k$ because Γ is a full rank lattice (and hence, isomorphic to $\mathbb{Z}^n$). Given $s \in G_{j,k}$, we can regard it as an element in $B^{*j-k}\mathcal{E}^* + \cdots + B^{*j-1}\mathcal{E}^*$ and consider the character χ_s as in (4.46). Note that $\chi_s \in \mathrm{Ann}(B^{k-j}\mathbb{Z}^n[B])$ (see Lemma 4.6.2); hence, χ_s is a character of $B^{-j}\mathcal{E}_k$. Also, for $s \neq 0$ in $G_{j,k}$, there is $y \in B^{-j}\mathcal{E}_k$ such that $S_s(y) \neq 0$: in fact, there exists l with $j-k \leqslant l \leqslant j-1$ with $s_l \neq 0$; since $\mathcal{E}$ spans a full rank lattice, find $c \in \mathcal{E}$ such that $\langle c, s_l\rangle \neq 0$ and take $y = B^{-l}c$. Hence all the characters χ_s for $s \in G_{j,k}$ are distinct over $B^{-j}\mathcal{E}_k$, and since $|B^{-j}\mathcal{E}_k| = b^k = |G_{j,k}|$, this implies that $\widehat{B^{-j}\mathcal{E}_k} \simeq G_{j,k}$, so (4.51) yields

$$\widehat{\mathbb{Z}^n((B))} \simeq \varprojlim_{j\in\mathbb{N}} \varinjlim_{k\in\mathbb{N}} G_{j,k}$$

It is not hard to establish an isomorphism $G_{j,k} \simeq B^{*-j}\mathcal{E}^*_k$, and so

$$\chi \in \varprojlim_{j\in\mathbb{N}} \varinjlim_{k\in\mathbb{N}} \{\chi_s \;:\; s \in G_{j,k}\} \simeq \varprojlim_{j\in\mathbb{N}} \varinjlim_{k\in\mathbb{N}} B^{*-j}\mathcal{E}^*_k \simeq \mathbb{Z}^n((B^*)),$$

therefore every character χ is of the form $\chi = \chi_s$ for some $s \in \mathbb{Z}^n((B^*))$. □

4.7 Multiple tiling theorem

The main result of this section states that, whenever $\mathcal{F}(A, \mathcal{D})$ is a tile, it gives a multiple tiling of $\mathbb{K}_A$. In this final section, we will prove that rational self-affine tiles give a multiple tiling of the representation space. We will make use of the character theory we have developed in the previous section.

Recall that the set $\varphi(\mathbb{Z}^n[A])$ is a lattice by Lemma 4.6.11, hence the torus $\mathbb{T} := \mathbb{K}_A/\varphi(\mathbb{Z}^n[A])$ is well defined and compact. We endow it with the normalized quotient measure $\bar{\mu}$, which is the Haar measure on $\mathbb{T}$. Denote the multiplication by A on $\mathbb{T}$ as

$$\tau_A : \mathbb{T} \to \mathbb{T}, \qquad (x, y) \mapsto A(x, y) \mod \varphi(\mathbb{Z}^n[A]),$$

which is well defined because $\varphi(\mathbb{Z}^n[A])$ is A-invariant. We will prove that this map is ergodic by using the following lemma.

Lemma 4.7.1. *If G is a compact abelian group with normalized Haar measure and $\tau : G \to G$ is a surjective continuous endomorphism of G, then τ is ergodic if and only if the trivial character $\chi = 1$ is the only character of G that satisfies $\chi \circ \tau^k = \chi$ for some $k \geqslant 1$.*

Proof. See [81, Theorem 1.10.1]. □

We are in position to prove the following result.

Lemma 4.7.2. *The map τ_A is ergodic.*

Proof. Because A is an invertible matrix, τ_A is a continuous surjective homomorphism. We first prove that it is measure preserving. Note that $\varphi(\mathbb{Z}^n[A])$ is a sublattice of index a of $A^{-1}\varphi(\mathbb{Z}^n[A])$. This implies that, for any measurable set $E \subset \mathbb{T}$,

$$\begin{aligned}
\bar{\mu}(\tau_A^{-1}(E)) &= \bar{\mu}(\{(x,y) \in \mathbb{T} : A(x,y) \mod \varphi(\mathbb{Z}^n[A]) \in E\}) \\
&= \bar{\mu}(\{(x,y) \in \mathbb{T} : (x,y) \mod A^{-1}\varphi(\mathbb{Z}^n[A]) \in A^{-1}E\}) \\
&= a\,\bar{\mu}(\{(x,y) \in \mathbb{T} : (x,y) \mod \varphi(\mathbb{Z}^n[A]) \in A^{-1}E\}) \\
&= a\,\bar{\mu}(A^{-1}E) = \bar{\mu}(E).
\end{aligned}$$

By Proposition 4.6.19, the characters of $\mathbb{K}_A$ are of the form $\chi_{r,s}$ for $r \in \mathbb{R}^n$, $s \in \mathbb{Z}^n((B^*))$. Since $\mathbb{T} = \mathbb{K}_A/\varphi(\mathbb{Z}^n[A])$, by Lemma 4.6.2 there is an isomorphism $\widehat{\mathbb{T}} \simeq \mathrm{Ann}(\varphi(\mathbb{Z}^n[A]))$; this means that a character of $\mathbb{T}$ is of the form $\chi_{r,s}$ and satisfies $\chi_{r,s}(\varphi(z)) = 1$ for every $z \in \mathbb{Z}^n[A]$. Suppose there exists a character $\chi_{r,s} \in \widehat{\mathbb{T}}$ satisfying $\chi_{r,s} \circ \tau_A^k = \chi_{r,s}$ for some $k \geqslant 1$. In view of Lemma 4.7.1, it suffices to show that $\chi_{r,s}$ is constantly equal to 1. We have that, for every $(x,y) \in \mathbb{T}$,

$$\chi_{r,s} \circ \tau_A^k(x,y) = \chi_{r,s}(x,y),$$

meaning

$$\exp(2\pi i(\langle r, A^k x\rangle + S_s(A^k y))) = \exp(2\pi i(\langle r, x\rangle + S_s(y)))$$

with S_s defined in (4.47). Letting $y = 0$ implies

$$\langle r, A^k x\rangle = \langle r, x\rangle \mod \mathbb{Z}$$

and so, for every $x \in [0,1]^n$,

$$\langle {A^*}^k r - r, x\rangle \in \mathbb{Z},$$

which can only be true if ${A^*}^k r - r = 0$. Suppose $r \neq 0$; this implies that ${A^*}^k$ has 1 as an eigenvalue, and so therefore A has 1 as an eigenvalue, which contradicts the fact that A is expanding. Hence, $r = 0$.

Next, we prove that $\chi_{(0,s)} \in \widehat{\mathbb{T}}$ implies $s = 0$. For every $z \in \mathbb{Z}^n[A]$ we have $1 = \chi_{(0,s)}(\varphi(z)) = \chi_0(z)\chi_s(z) = \chi_s(z)$. Consider points of the form $z = A^l c = B^{-l}c$ with $l \geqslant 1$, $c \in \mathcal{E}$. Then $S_s(B^{-l}c) = \langle c, \{B^{*-l}s\}_B^*\rangle \in \mathbb{Z}$. Recall that $\mathcal{E}$ contains a subset that spans a full rank lattice Θ. By the definition of dual lattice, this implies that $\{B^{*-l}s\}_B^* \in \Theta^*$ for every $l \geqslant 1$. Suppose $s \neq 0$, then the leading B^*-adic coefficient satisfies $s_{\nu^*(s)} \in \Gamma \setminus B^*\Gamma$, because the coefficients live in $\mathcal{E}^* \subset \Gamma$ which is defined to be a residue set for $\Gamma[B^*]/B^*\Gamma[B^*]$. Moreover, we have that

$$\{B^{*-l}s\}_B^* \in (B^{*-l}\Gamma + \cdots + B^{*-1}\Gamma + \Gamma) \setminus (B^{*-l+1}\Gamma + \cdots + B^{*-1}\Gamma + \Gamma).$$

Define inductively $\Lambda_0 := \Gamma$ and Λ_{l+1} the lattice spanned by Λ_l and $\{B^{*-l}s\}_B^*$. Then $(\Lambda_l)_{l\geqslant 1}$ is a strictly nested sequence of lattices in $\mathbb{Q}^n$, all of which are contained in Θ^*. Given l, consider the entries of all the vectors of Λ_l expressed as irreducible fractions, and define m_l to be the maximum of the denominators of these fractions. It is clear that $m_l \leqslant m_{l+1}$. However, note that $(m_l)_{l\geqslant 1}$ does not stabilize: this would imply that, for some lattice Λ_l, there are infinitely many steps in which we can add a point and form a strictly larger lattice, while not increasing the bound on the denominators, which is not possible. Hence the sequence of denominators $(m_l)_{l\geqslant 1}$ tends to infinity. This contradicts the uniform discreteness of the lattice Θ^*. □

We arrive at our final result. The proof is based on the one of [51, Theorem 1.1]. We recall the reader of the definition of multiple tiling given in Definition 4.5.1.

Theorem 4.7.3. *Let $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ be a rational self-affine tile. Then $\mathcal{F} + \varphi(\mathbb{Z}^n[A])$ is a multiple tiling of $\mathbb{K}_A$.*

Proof. Let $\bar{\mu}$ be the normalized Haar measure on the torus $\mathbb{T} = \mathbb{K}_A/\varphi(\mathbb{Z}^n[A])$. Consider the canonical projection $\pi : \mathbb{K}_A \to \mathbb{T}$, and define the function $\Phi : \mathbb{T} \to \mathbb{Z}_{\geqslant 0}$ as

$$\Phi(x, y) := |\pi^{-1}(x, y) \cap \mathcal{F}|,$$

where $|\cdot|$ in this context denotes the cardinality (not to be confused with the absolute value). Then Φ counts the points on $\mathcal{F}$ that are congruent to (x, y) modulo $\varphi(\mathbb{Z}^n[A])$. Since $\mathcal{F}$ is compact, Φ is finite everywhere, hence it is well defined. Also, Φ is positive in a set of positive measure since $\mu(\mathcal{F}) > 0$. If we prove that $\Phi(x, y)$ is equal almost everywhere to some $k \in \mathbb{N}$, this implies the statement of the theorem: it means that almost every point of $\mathbb{K}_A$ gets covered by exactly k translates of $\mathcal{F}$ when translating via the set $\varphi(\mathbb{Z}^n[A])$. Note that Φ is constant almost everywhere if and only if there exists k such that every $S \subset \mathbb{T}$ satisfies

$$\int_S \Phi(x, y)\, d\bar{\mu}(x, y) = k\, \bar{\mu}(S). \tag{4.52}$$

To obtain this, we show first that Φ satisfies

$$\Phi(x, y) = \frac{1}{a} \sum_{(x',y') \in \tau_A^{-1}(x,y)} \Phi(x', y'), \tag{4.53}$$

where τ_A is the multiplication by A on the torus.

Using the set equation (4.17) of $\mathcal{F}$ and the fact that $\varphi(\mathcal{D}) \subset \varphi(\mathbb{Z}^n[A])$, we have

$$\begin{aligned} |\pi^{-1}(x, y) \cap A\mathcal{F}| &= \Big|\pi^{-1}(x, y) \bigcap \Big(\bigcup_{d \in \mathcal{D}} (\mathcal{F} + \varphi(d)) \Big)\Big| \\ &= \Big| \bigcup_{d \in \mathcal{D}} (\pi^{-1}(x, y) - \varphi(d)) \cap \mathcal{F} \Big| \\ &= \sum_{d \in \mathcal{D}} |\pi^{-1}(x, y) \cap \mathcal{F}| = a\,\Phi(x, y). \end{aligned} \tag{4.54}$$

Recall that the index of the sublattice $A\varphi(\mathbb{Z}^n[A])$ of $\varphi(\mathbb{Z}^n[A])$ is a, which implies that $|\tau_A^{-1}(x, y)| = a$, and so

$$|\pi^{-1}(x, y) \cap A\mathcal{F}| = \frac{1}{a} \sum_{(x',y') \in \tau_A^{-1}(x,y)} |\pi^{-1}(\tau_A(x', y')) \cap A\mathcal{F}|. \tag{4.55}$$

Also, for any $(u, v) \in \mathbb{K}_A$, it holds that $|((u, v) + \varphi(\mathbb{Z}^n[A])) \cap A\mathcal{F}| = a|((u, v) + A\varphi(\mathbb{Z}^n[A])) \cap A\mathcal{F}|$, and hence, for each $(x', y') \in \tau_A^{-1}(x, y)$,

$$|\pi^{-1}(\tau_A(x', y')) \cap A\mathcal{F}| = a\,|A\pi^{-1}(x', y') \cap A\mathcal{F}| = a\,|\pi^{-1}(x', y') \cap \mathcal{F}| = a\,\Phi(x', y'). \tag{4.56}$$

Thus from (4.55) and (4.56) follows that

$$|\pi^{-1}(x, y) \cap A\mathcal{F}| = \sum_{(x',y') \in \tau_A^{-1}(x,y)} \Phi(x', y'). \tag{4.57}$$

Combining (4.54) and (4.57) yields (4.53). Applying (4.53) and doing the change of variables $(x, y) \mapsto \tau_A(x, y)$ we get

$$\begin{aligned} \int_S \Phi(x, y) d\bar{\mu}(x, y) &= \int_S \frac{1}{a} \sum_{(x',y') \in \tau_A^{-1}(x,y)} \Phi(x', y')\, d\bar{\mu}(x, y) \\ &= \int_{\tau_A^{-1}(S)} \frac{1}{a} \sum_{\tau_A(x',y') = \tau_A(x,y)} \Phi(x', y')\, a\, d\bar{\mu}(x, y) \\ &= \int_{\tau_A^{-1}(S)} \Phi(x, y)\, d\bar{\mu}(x, y). \end{aligned} \tag{4.58}$$

Because the map τ_A is ergodic by Lemma 4.7.2, iterating (4.58) $j \geqslant 1$ times and afterwards applying the ergodic theorem (see [27, Theorem 3.20]), yields

$$\begin{aligned}
\int_S \Phi(x,y)d\bar{\mu}(x,y) &= \int_{\tau_A^{-j}(S)} \Phi(x,y)d\bar{\mu}(x,y) \\
&= \int_{\mathbb{T}} 1_s(\tau_A^j(x,y))\Phi(x,y)d\bar{\mu}(x,y) \\
&= \int_{\mathbb{T}} \Big(\frac{1}{N}\sum_{j=1}^{N} 1_s(\tau_A^j(x,y))\Big)\Phi(x,y)d\bar{\mu}(x,y) \\
&\xrightarrow[N\to\infty]{} \bar{\mu}(S)\int_{\mathbb{T}} \Phi(x,y)d\bar{\mu}(x,y) = k\,\bar{\mu}(S),
\end{aligned} \tag{4.59}$$

and this concludes the proof. $\square$

Chapter 5

N-continued fraction sequences

This chapter contains results published in the article *Generalizations of Sturmian sequences associated with N-continued fraction algorithms* (Journal of Number Theory, 2023, [56]).

In this chapter, we will introduce a family of sequences over the alphabet $\{0, 1\}$ related to the so-called *N-continued fraction algorithms* that were introduced by Burger *et al.* [21]. Given a positive integer N and $x \in [0, 1] \setminus \mathbb{Q}$, an *$N$-continued fraction expansion* of x is defined analogously to the classical continued fraction expansion, but with the numerators being all equal to N. Inspired by Sturmian sequences, we introduce the *N-continued fraction sequences* $\omega(x, N)$ and $\widehat{\omega}(x, N)$, which are related to the N-continued fraction expansion of x. They are infinite words over a two-letter alphabet obtained as the limit of a directive sequence of certain substitutions, hence they are *S-adic sequences*. When $N = 1$, we are in the case of the classical continued fraction algorithm, and obtain the well-known Sturmian sequences. We show that $\omega(x, N)$ and $\widehat{\omega}(x, N)$ are C-balanced for some explicit values of C and compute their factor complexity function. We also obtain uniform word frequencies and deduce unique ergodicity of the associated subshifts. Finally, we provide a Farey-like map for N-continued fraction expansions, which provides an additive version of N-continued fractions, for which we prove ergodicity and give the invariant measure explicitly.

5.1 Preliminaries

5.1.1 N-continued fraction expansions

Let N be a positive integer. As a variation on the regular continued fraction algorithm, Burger *et al.* introduced N-continued fraction expansions in [21]. Given a real number $x \in [0, 1]$, an *N-continued fraction expansion* (or NCF expansion, for short) of x is an expansion of the form

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \ddots}},$$

with *N-continued fraction digits* $d_n \geq 1$. If $N > 1$, it turns out that there exist infinitely many different NCF expansions of x (see [25]). Though, when we impose that $d_n \geq N$, we find a unique infinite expansion for all irrational numbers and exactly two finite expansions for rational numbers. These expansions are called the greedy NCF expansions.

For $N = 1$, we find back the regular continued fraction algorithm. For the case $N \geq 2$, these continued fractions share some properties with the regular ones but there are also differences worth mentioning. For example, in contrast to the regular continued fraction expansions, any number has infinitely many different NCF expansions, see [7, 25]. Also the behavior of quadratic irrationals seems to be very different. For regular continued fractions, we know that any quadratic irrational has a purely or eventually periodic expansion. In [21] it is proven that for every quadratic irrational number there exist infinitely many eventually periodic NCF expansions with period-length 1. On the other hand, for a fixed $N \geq 2$ it seems that there are quadratic irrational numbers with aperiodic NCF expansions, see [25]. Another difference is that for Lebesgue almost all $x \in [0,1]$ the regular continued fraction expansion has arbitrarily large digits, but for NCF expansions we can find for every x an NCF expansion such that the digits are bounded, see [49].

We look at the *greedy NCF expansion* obtained from the map T_N. Fix $N \geq 1$ and define $T_N : [0,1] \to [0,1]$ as

$$T_N(x) = \begin{cases} \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor & x \neq 0, \\ 0 & x = 0, \end{cases} \tag{5.1}$$

see Figure 5.1 for examples. Set $d_1(x) = \left\lfloor \frac{N}{x} \right\rfloor$ and $d_n(x) = d_1(T_N^{n-1}(x))$ whenever $T_N^{n-1}(x) \neq 0$. For x we find

$$\begin{aligned} x &= \frac{N}{d_1(x) + T_N(x)} \\ &= \frac{N}{d_1(x) + \cfrac{N}{d_2(x) + T_N^2(x)}} \\ &= \frac{N}{d_1(x) + \cfrac{N}{d_2(x) + \ddots}}. \end{aligned}$$

This continued fraction expansion is finite if and only if $x \in \mathbb{Q}$. We only want to consider expansions with infinitely many digits, hence from here onward we assume $x \in [0,1] \setminus \mathbb{Q}$. Following the notation of [21], we denote $d_n = d_n(x)$, and write $x = [0; d_1, d_2, \ldots]_N$ for the greedy expansion of x. Note that this greedy expansion is the unique NCF expansion of x whose digits are all greater than or equal to N. On the other hand, each sequence $(d_n)_{n \geq 1}$ with $d_n \geq N$ occurs as a greedy expansion of

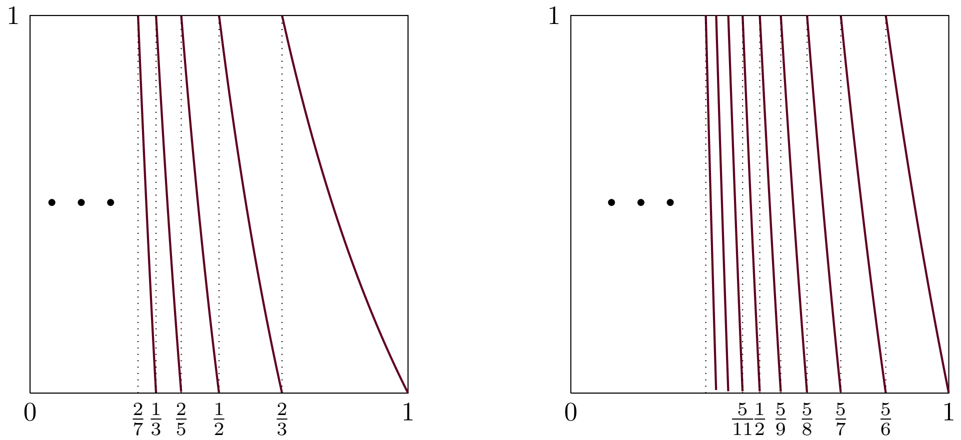


FIGURE 5.1: The map T_N for $N = 2$ on the left and $N = 5$ on the right.

some irrational $x \in [0, 1] \setminus \mathbb{Q}$.

5.1.2 Substitutions and S-adic sequences

Consider a finite alphabet $\mathcal{A}$ and let $\mathcal{A}^*$ be the free monoid generated by $\mathcal{A}$ equipped with the operation of concatenation, that is, $\mathcal{A}^*$ consists of all the *(finite) words* $w_0 \cdots w_{n-1}$ with $n \in \mathbb{N}$ and letters $w_0, \ldots, w_{n-1} \in \mathcal{A}$. The choice $n = 0$ corresponds to the empty word which is denoted by ε. A word $u \in \mathcal{A}^*$ is called a *factor* of $v \in \mathcal{A}^*$ if $v \in \mathcal{A}^* u \mathcal{A}^*$ and we denote it by $u \subset v$. We call $u \in \mathcal{A}^*$ a *prefix* of v if $v \in u\mathcal{A}^*$, and a *suffix* of v if $v \in \mathcal{A}^* u$. We also define $\mathcal{A}^{\mathbb{N}}$ as the space of *(right) infinite words* or *sequences* $w_0 w_1 \cdots$ with $w_0, w_1, \ldots \in \mathcal{A}$. We endow $\mathcal{A}^{\mathbb{N}}$ with the product topology of the discrete topology on each copy of $\mathcal{A}$. A word $u \in \mathcal{A}^*$ is a factor of $\omega \in \mathcal{A}^{\mathbb{N}}$ if $\omega \in \mathcal{A}^* u \mathcal{A}^{\mathbb{N}}$ and a prefix of ω if $\omega \in u\mathcal{A}^{\mathbb{N}}$.

For every $u \in \mathcal{A}^*$, denote by $|u|$ the length (*i.e.*, the number of letters) of u. For every $a \in \mathcal{A}$, denote by $|u|_a$ the number of occurrences of the letter a in the word u, and, for a word v, denote by $|u|_v$ the number of occurrences of v in u. Given $\omega \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$, we write $\omega = \omega_0 \omega_1 \omega_2 \cdots$ where $\omega_i \in \mathcal{A}$. Given $a \in \mathcal{A}$ and $d \in \mathbb{N}$, we write $a^d = \underbrace{a \cdots a}_{d \text{ times}}$.

We define the *abelianization map* as

$$\mathbf{l} : \mathcal{A}^* \to \mathbb{N}^{|\mathcal{A}|}, \qquad u \mapsto {}^t(|u|_a)_{a \in \mathcal{A}}. \tag{5.2}$$

The *language* $\mathcal{L}_\omega$ of a sequence ω is given by

$$\mathcal{L}_\omega := \{u \in \mathcal{A}^* \,:\, u \text{ is a factor of } \omega\}.$$

A map $\sigma : \mathcal{A} \to \mathcal{A}^* \setminus \{\varepsilon\}$ is a *substitution* over the alphabet $\mathcal{A}$. The domain of σ can be extended to $\mathcal{A}^*$ by concatenating the images of each letter, that is, σ is an endomorphism over the free monoid $\mathcal{A}^*$. This allows even to naturally extend the domain of σ to the set of sequences $\mathcal{A}^{\mathbb{N}}$.

Given a substitution σ, we define its *incidence matrix* as the square matrix $M_\sigma = (|\sigma(j)|_i)_{i,j\in\mathcal{A}}$. This definition immediately implies that $\mathbf{l}(\sigma(u)) = M_\sigma \mathbf{l}(u)$ for every $u \in \mathcal{A}^*$. We say that σ is unimodular if $\det M_\sigma = \pm 1$. The substitutions that we consider will be non-unimodular in the case $N \geq 2$.

Definition 5.1.1 (Directive sequence and S-adic sequence)**.** *Let $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 1}$ be a sequence of substitutions $\sigma_n : \mathcal{A}^* \to \mathcal{A}^*$ over the alphabet $\mathcal{A}$. We denote the set of substitutions as $\mathcal{S} = \{\sigma_n : n \geq 1\}$; this set may be finite or infinite. We say that $\boldsymbol{\sigma}$ is a directive sequence.*

A sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ is an S-adic sequence (or limit sequence) of the directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 1}$ if there exist $\omega^{(1)}, \omega^{(2)}, \ldots \in \mathcal{A}^{\mathbb{N}}$ such that

$$\omega^{(1)} = \omega, \qquad \omega^{(n)} = \sigma_n(\omega^{(n+1)}) \qquad \textit{for all } n \geq 1.$$

See [79] and [18] for results on S-adic sequences in the unimodular case.

Definition 5.1.2 (Generalized right eigenvector)**.** *Denote by $\mathbb{R}_+^d$ the set of vectors with positive entries in $\mathbb{R}^d$. Let $(M_n)_{n\geq 1}$ be a sequence of matrices in $\mathbb{N}^{d\times d}$. A vector $\mathbf{u} \in \mathbb{R}_+^d$ with $\|\mathbf{u}\|_1 = 1$ is said to be a generalized right eigenvector for $(M_n)_{n\geq 1}$ if*

$$\bigcap_{n\geq 1} M_1 \cdots M_n \mathbb{R}_+^d = \mathbb{R}_+ \mathbf{u}. \tag{5.3}$$

Given a directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 1}$ on an alphabet $\mathcal{A}$, we say that $\mathbf{u} \in \mathbb{R}_+^{|\mathcal{A}|}$ is a generalized right eigenvector of $\boldsymbol{\sigma}$ if $\mathbf{u}$ is a generalized right eigenvector of the corresponding sequence of incidence matrices $(M_{\sigma_n})_{n\geq 1}$.

On the topological space $\mathcal{A}^{\mathbb{N}}$ we consider the left shift Σ defined by $\Sigma(\omega_0\omega_1\cdots) = \omega_1\omega_2\cdots$, where $\omega_j \in \mathcal{A}$. Given a sequence ω on the alphabet $\mathcal{A}$, consider the closed set

$$X_\omega := \overline{\{\Sigma^n(\omega) : n \in \mathbb{N}\}}.$$

Then (X_ω, Σ) constitutes a topological dynamical system called a *subshift*. We are interested in ergodic properties of this type of dynamical systems for NCF sequences, which translate to the existence of word frequencies.

5.2 N-continued fraction sequences

In this section, we will define our main objects of study, which are two families of binary sequences called NCF sequences and dual NCF sequences. To do this, for each irrational $x \in [0, 1]$ we will first construct two S-adic sequences. The choice of these sequences is what is known as a *substitution selection* for the NCF algorithm and its natural extension in the sense of [17]. We start with relating directive sequences of substitutions to N-continued fraction expansions.

Definition 5.2.1 (Directive sequences for N-continued fraction expansions)**.** *Let $N \geq 1$ and let $x = [0; d_1, d_2, \ldots]_N \in [0,1] \setminus \mathbb{Q}$.*

1. *For each $n \geq 1$, consider the substitutions*

$$\sigma_n : \begin{cases} 0 \to 0^{d_n} 1^N, \\ 1 \to 0. \end{cases}$$

   We assign to x the directive sequence $\boldsymbol{\sigma}_x = (\sigma_n)_{n \geq 1}$. We denote $\mathcal{S} = \{\sigma_n : n \geq 1\}$.

2. *For each $n \geq 1$, consider the dual substitutions*

$$\widehat{\sigma}_n : \begin{cases} 0 \to 0^{d_n} 1, \\ 1 \to 0^N. \end{cases}$$

   We assign to x the directive sequence $\widehat{\boldsymbol{\sigma}}_x = (\widehat{\sigma}_n)_{n \geq 1}$. We denote $\widehat{\mathcal{S}} = \{\widehat{\sigma}_n : n \geq 1\}$.

For each $n \geq 1$, the corresponding incidence matrices of σ_n and $\widehat{\sigma}_n$ are given by

$$M_{\sigma_n} = \begin{pmatrix} d_n & 1 \\ N & 0 \end{pmatrix}, \qquad M_{\widehat{\sigma}_n} = \begin{pmatrix} d_n & N \\ 1 & 0 \end{pmatrix}.$$

Note that they are the transpose of each other. Moreover, easy calculation shows that when $d_n \geq N$, which is the case in the greedy algorithm that we have chosen, the matrices are *Pisot*. Recall that a matrix is said to be Pisot if one of its eigenvalues is a real number greater than 1, and the rest of its eigenvalues have modulus less than 1. Pisot matrices are very relevant in the study of substitutions (see for instance [2]). We would also like to remark that these are not the matrices of the Möbius transformations associated to the inverse branches of $T_N(x)$. The difference is that the numbers of on the diagonal have to be swapped as well as on the anti-diagonal. One can achieve this by relabelling 0 as 1 and vice versa. However, this will not affect our results. We chose the substitutions as it is so that the dual substitutions $\widehat{\sigma}_n$ are a particular instance of so called β-substitutions for simple Parry numbers (see [15, Section 3.2] and [32]).

Definition 5.2.2 (N-continued fraction sequence and its dual)**.** *Let $N \geq 1$ and $x = [0; d_1, d_2, \ldots]_N \in [0,1] \setminus \mathbb{Q}$.*

1. *We define the NCF sequence $\omega(x, N)$ as the S-adic sequence of the directive sequence $\boldsymbol{\sigma}_x = (\sigma_n)_{n \geq 1}$. The finite words $(\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(1))_{n \geq 1}$ form a nested sequence of prefixes of $\omega(x, N)$ and they satisfy*

$$\omega(x, N) = \lim_{n\to\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(1).$$

2. *We define the dual NCF sequence $\widehat{\omega}(x, N)$ as the S-adic sequence of the directive sequence $\widehat{\boldsymbol{\sigma}}_x = (\widehat{\sigma}_n)_{n\geq 1}$. The finite words $(\widehat{\sigma}_1 \circ \widehat{\sigma}_2 \circ \cdots \circ \widehat{\sigma}_n(0))_{n\geq 1}$ form a nested sequence of prefixes of $\widehat{\omega}(x, N)$ and they satisfy*

$$\widehat{\omega}(x, N) = \lim_{n\to\infty} \widehat{\sigma}_1 \circ \widehat{\sigma}_2 \circ \cdots \circ \widehat{\sigma}_n(0).$$

By Definition 5.1.1, if $\omega(x, N)$ is the S-adic sequence of the directive sequence $\boldsymbol{\sigma}_x = (\sigma_n)_{n\geq 1}$, then there exist $\omega^{(1)}, \omega^{(2)}, \ldots \in \mathcal{A}^{\mathbb{N}}$ such that $\omega^{(1)} = \omega(x, N)$ and $\omega^{(n)} = \sigma_n(\omega^{(n+1)})$ for all $n \geq 1$. It is not hard to see that $\omega^{(n)} = \omega(T_N^{n-1}(x), N)$ because $T_N(x) = [0; d_2, d_3, \ldots]_N$. The same holds for $\widehat{\omega}(x, N)$.

The notion of limit used in Definition 5.2.2 is rather informal, since we have finite words "converging" to an infinite sequence, but in this case it simply means that we have a nested sequence of prefixes of increasing length.

We introduce the following notation. Define the words

$$\Sigma_0 := 1, \qquad \Sigma_n := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(1) \qquad \text{for } n \geq 1.$$

By the definition of the substitutions σ_n in Definition 5.2.1 (1) the words Σ_n satisfy the recurrence

$$\Sigma_{n+1} = \Sigma_n^{d_n} \Sigma_{n-1}^N \qquad \text{for } n \geq 1. \tag{5.4}$$

The first iterations are

$$\Sigma_0 = 1, \qquad \Sigma_1 = 0, \qquad \Sigma_2 = 0^{d_1}1^N, \qquad \Sigma_3 = \underbrace{0^{d_1}1^N 0^{d_1}1^N \cdots 0^{d_1}1^N}_{d_2 \text{ times}} 0^N,$$

$$\Sigma_4 = \underbrace{\underbrace{0^{d_1}1^N 0^{d_1}1^N \cdots 0^{d_1}1^N}_{d_2 \text{ times}} 0^N \cdots \underbrace{0^{d_1}1^N 0^{d_1}1^N \cdots 0^{d_1}1^N}_{d_2 \text{ times}} 0^N}_{d_3 \text{ times}} \underbrace{0^{d_1}1^N \cdots 0^{d_1}1^N}_{N \text{ times}}.$$

Analogously, let

$$\widehat{\Sigma}_0 := 1, \qquad \widehat{\Sigma}_1 := 0, \qquad \widehat{\Sigma}_{n+1} := \widehat{\sigma}_1 \circ \widehat{\sigma}_2 \circ \cdots \circ \widehat{\sigma}_n(0) \qquad \text{for } n \geq 1.$$

Then from Definition 5.2.1 (2) we immediately see that

$$\widehat{\Sigma}_{n+1} = \widehat{\Sigma}_n^{d_n} \widehat{\Sigma}_{n-1}^N \qquad \text{for } n \geq 2. \tag{5.5}$$

The first iterations are

$$\widehat{\Sigma}_0 = 1, \qquad \widehat{\Sigma}_1 = 0, \qquad \widehat{\Sigma}_2 = 0^{d_1}1, \qquad \widehat{\Sigma}_3 = \underbrace{0^{d_1}10^{d_1}1\cdots 0^{d_1}1}_{d_2\text{ times}}0^N,$$

$$\widehat{\Sigma}_4 = \underbrace{\underbrace{0^{d_1}10^{d_1}1\cdots 0^{d_1}1}_{d_2\text{ times}}0^N \cdots \underbrace{0^{d_1}10^{d_1}1\cdots 0^{d_1}1}_{d_2\text{ times}}0^N}_{d_3\text{ times}}\underbrace{0^{d_1}1\cdots 0^{d_1}1}_{N\text{ times}}.$$

Let $\omega \in \{0,1\}^{\mathbb{N}}$ be a sequence over two letters and $a, b \in \{0,1\}$ with $a \neq b$. We say that a factor $v = a\cdots a$ of ω is a *maximal a block* of ω if either vb is a prefix of ω of bvb is a factor of ω. The recurrences (5.4) and (5.5) immediately allow to characterize maximal a blocks for $\omega(x,N)$ and $\widehat{\omega}(x,N)$ according to the following lemma.

Lemma 5.2.3. *Let $N \geq 1$ and $x = [0; d_1, d_2, \ldots]_N$ be given.*

(1a) 1^N is the only maximal 1 block of $\omega(x,N)$.

(1b) 0^{d_1} and 0^{d_1+N} are the only maximal 0 blocks of $\omega(x,N)$.

(2a) 1 is the only maximal 1 block of $\widehat{\omega}(x,N)$.

(2b) 0^{d_1} and 0^{d_1+N} are the only maximal 0 blocks of $\widehat{\omega}(x,N)$.

Note that the NCF sequence $\omega(x,N)$ can be obtained from the dual NCF sequence $\widehat{\omega}(x,N)$ by substituting each occurrence of 1 in $\widehat{\omega}(x,N)$ by 1^N. This is true because the sequences $(\Sigma_n)_{n\geq 1}$ and $(\widehat{\Sigma}_n)_{n\geq 1}$ satisfy the same recurrence formula, and the only difference is that $\Sigma_2 = 0^{d_1}1^N$ and $\widehat{\Sigma}_2 = 0^{d_1}1$.

Formally, consider the substitution

$$\tau : \begin{cases} 0 \mapsto 0, \\ 1 \mapsto 1^N. \end{cases} \tag{5.6}$$

Then

$$\omega(x,N) = \tau(\widehat{\omega}(x,N)) \tag{5.7}$$

holds. Hence, the dual NCF has essentially the “same shape” as the regular one but is a bit easier to work with. We will later use this correspondence to be able to transfer properties of one sequence to the other. The sequence $\omega(x,N)$ has the advantage that the ratio of the letter frequencies converges to x, which makes it a more natural choice.

5.2.1 Letter frequency and generalized right eigenvector

Let $N \geq 1$ and consider the expansion $x = [0; d_1, d_2, \ldots]_N \in [0,1] \setminus \mathbb{Q}$. Define the convergents $c_n = \frac{p_n}{q_n}$ for $n \geq 1$ as

$$\frac{p_n}{q_n} := [0; d_1, d_2, \ldots, d_n]_N,$$

and choose p_n and q_n so that they satisfy the following recurrence relations:

$$\begin{aligned} p_{-1} &= 1, & p_0 &= 0, & p_n &= d_n p_{n-1} + N p_{n-2}, \\ q_{-1} &= 0, & q_0 &= 1, & q_n &= d_n q_{n-1} + N q_{n-2}. \end{aligned} \tag{5.8}$$

Then we have $x = \lim_{n\to\infty} \frac{p_n}{q_n}$. Consider the directive sequence $\boldsymbol{\sigma}_x = (\sigma_n)_{n\geq 1}$ and the corresponding sequence of incidence matrices $(M_{\sigma_n})_{n\geq 1}$.

Set $M_{[1,n]} = M_{\sigma_1} M_{\sigma_2} \cdots M_{\sigma_n}$. Then we find that

$$M_{[1,n]} = \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix}.$$

Since $\mathbf{l}(1) = {}^t(0,1)$, we have

$$\mathbf{l}(\sigma_1 \circ \cdots \circ \sigma_n(1)) = M_{[1,n]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} q_{n-1} \\ p_{n-1} \end{pmatrix}$$

and, hence, we gain

$$\lim_{n\to\infty} \frac{|\sigma_1 \circ \cdots \circ \sigma_n(1)|_1}{|\sigma_1 \circ \cdots \circ \sigma_n(1)|_0} = \lim_{n\to\infty} \frac{p_{n-1}}{q_{n-1}} = x. \tag{5.9}$$

We define the *frequency* of a letter $a \in \mathcal{A}$ in the sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ as

$$f_a := \lim_{|p|\to\infty} \frac{|p|_a}{|p|}, \tag{5.10}$$

provided that the limit, which is taken over the prefixes p of ω, exists. If the limit does not exist, we say that a does not have a frequency in ω. Equation (5.9) implies that the N-continued fraction sequence $\omega(x,N)$ has letter frequencies and the *frequency vector* is given by $(f_0, f_1) = \left(\frac{1}{x+1}, \frac{x}{x+1}\right)$. We show next that this vector is in fact a generalized right eigenvector for $\boldsymbol{\sigma}_x$. We will use the following auxiliary lemma.

Lemma 5.2.4 (Birkhoff [20]). *Let $(B_n)_{n\geq 1}$ be a sequence of matrices with non-negative entries. If there exists a matrix B with strictly positive entries, an integer $h > 0$, and a strictly increasing sequence $(m_i)_{i\geq 1}$ of positive integers such that $B = B_{m_i} \cdots B_{m_i+h}$ for each $i \geq 1$, then $(B_n)_{n\geq 1}$ has a generalized right eigenvector.*

See also Furstenberg [33] and for instance the proof of [79, Proposition 3.5.5] where a slightly weaker statement given. Lemma 5.2.4 is used in the proof of the following result.

Lemma 5.2.5. *The directive sequence $\boldsymbol{\sigma}_x = (\sigma_n)_{n\geq 1}$ of NCF substitutions has a generalized right eigenvector given by $(f_0, f_1) = \left(\frac{1}{x+1}, \frac{x}{x+1}\right)$.*

Proof. Let

$$A = \begin{pmatrix} 1 & \frac{1}{N} \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} N & 1 \\ N & 0 \end{pmatrix}.$$

Note that, for every $n \geq 1$,

$$M_{\sigma_n} = \begin{pmatrix} d_n & 1 \\ N & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{d_n - N}{N} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} N & 1 \\ N & 0 \end{pmatrix} = A^{d_n - N} D.$$

Therefore the sequence of matrices

$$(M'_n)_{n\geq 1} = (\underbrace{A, \dots, A}_{d_1 - N \text{ times}}, D, \underbrace{A, \dots, A}_{d_2 - N \text{ times}}, D, \dots)$$

satisfies $\bigcap_{n\geq 0} M_{\sigma_n}\mathbb{R}^2_+ = \bigcap_{n\geq 0} M'_n \mathbb{R}^2_+$. Thus $\boldsymbol{\sigma}_x = (\sigma_n)_{n\geq 1}$ has a generalized right eigenvector if and only if $(M'_n)_{n\geq 1}$ has one.

If the sequence of NCF digits $(d_n)_{n\geq 1}$ is eventually equal to N, then $(M'_n)_{n\geq 1}$ is eventually equal to D. Because D^2 has only positive entries, the conditions of Lemma 5.2.4 are satisfied and $(M'_n)_{n\geq 1}$ has a generalized right eigenvector. Otherwise, the sequence $(M'_n)_{n\geq 1}$ changes infinitely many times between A and D, hence there exists a strictly increasing sequence of integers $(m_i)_{i\geq 1}$ such that $M'_{m_i} M'_{m_i+1} = DA$, which has strictly positive entries. Thus the existence of a generalized right eigenvector follows again from Lemma 5.2.4.

We conclude that $(M'_n)_{n\geq 1}$, and hence $\boldsymbol{\sigma}_x$, have a generalized right eigenvector. It follows from the proof of [79, Lemma 3.5.10] that, whenever $\boldsymbol{\sigma}_x$ has a generalized right eigenvector, its entries correspond to that of the letter frequency vector of the limit sequence. This finishes the proof. □

5.2.2 Substitution selection for the NCF algorithm

We now want to relate the sequences of substitutions $\boldsymbol{\sigma}_x$ and $\widehat{\boldsymbol{\sigma}}_x$ to the NCF algorithm. It turns out that these sequences of substitutions can be regarded as a combinatorial version of the NCF algorithm and its dual. For Sturmian sequences and their directive sequences, their multi-faceted interplay with the classical continued fraction algorithm is well-known (see *e.g.* Arnoux and Rauzy [11] or Arnoux and Fisher [10]). Berthé *et al.* [17] generalized this to unimodular multidimensional continued fraction algorithms by introducing the concept of *substitution selection*. The novelty of our setting is that we are working with non-unimodular matrices.

Let $N \geq 1$. We want to define a version of the NCF algorithm that works on the subset $\mathbb{P}_< = \{[1 : x] : 0 < x < 1\}$ of the projective space $\mathbb{P}$. Let $\mathbf{x} = [1 : x] \in \mathbb{P}_<$ and consider the matrix

$$C_N(\mathbf{x}) = \begin{pmatrix} \lfloor \frac{N}{x} \rfloor & N \\ 1 & 0 \end{pmatrix}.$$

Then the map

$$G_N : \mathbb{P}_< \to \mathbb{P}_<, \qquad \mathbf{x} \mapsto {}^t C_N(\mathbf{x})^{-1}\mathbf{x}$$

is called the *linear multiplicative N-continued fraction algorithm.* It has the form

$$G_N([1:x]) = \begin{pmatrix} 0 & \frac{1}{N} \\ 1 & -\frac{1}{N}\left\lfloor \frac{N}{x} \right\rfloor \end{pmatrix} \cdot [1:x] = \left[\frac{x}{N} : 1 - \frac{x}{N}\left\lfloor \frac{N}{x}\right\rfloor\right] = \left[1 : \frac{N}{x} - \left\lfloor \frac{N}{x}\right\rfloor\right],$$

which is a projectivization of T_N in the sense that the original mapping T_N can be seen in the second coordinate of G_N if the first coordinate is normalized to 1. If $x = [0; d_1, d_2, \ldots]_N \notin \mathbb{Q}$ then we have for each $n \geq 1$ that

$$^tC_N(G_N^{n-1}(\mathbf{x})) = {}^tC_N([1 : T_N^{n-1}(x)]) = \begin{pmatrix} \left\lfloor \frac{N}{T_N^{n-1}(x)} \right\rfloor & 1 \\ N & 0 \end{pmatrix} = \begin{pmatrix} d_n & 1 \\ N & 0 \end{pmatrix} = M_{\sigma_n}. \quad (5.11)$$

Iteration yields

$$\begin{aligned} \mathbf{x} &= {}^tC_N(\mathbf{x})G_N(\mathbf{x}) = {}^tC_N(\mathbf{x})\, {}^tC_N(G_N(\mathbf{x}))G_N^2(\mathbf{x}) = \ldots \\ &= {}^tC_N(\mathbf{x}) \cdots {}^tC_N(G_N^{n-1}(\mathbf{x}))G_N^n(\mathbf{x}) \end{aligned}$$

and therefore

$$\mathbf{x} = M_{\sigma_1}G_N(\mathbf{x}) = M_{\sigma_1}M_{\sigma_2}G_N^2(\mathbf{x}) = \cdots = M_{\sigma_1}\cdots M_{\sigma_n}G_N^n(\mathbf{x}).$$

Thus the NCF algorithm applied to $x = [0; d_1, d_2, \ldots]_N$ produces the incidence matrices of the substitutions $\sigma_1, \sigma_2, \ldots$ given in Definition 5.2.1 (1). Moreover, by Lemma 5.2.5, the ray $[1 : x]$ corresponds to the direction of the generalized right eigenvector of these substitutions. In this sense, the directive sequence $\boldsymbol{\sigma}_x$ can be regarded as a *substitution selection* of the NCF algorithm (see the definition of substitution selection in [17, Definition 2.2]). Indeed, given $\mathcal{S} = \{\sigma_n : n \geq 1\}$, we consider the map

$$\varphi : [0,1] \setminus \mathbb{Q} \to \mathcal{S}^{\mathbb{N}}, \qquad x \mapsto \boldsymbol{\sigma}_x = (\sigma_n)_{n\geq 1}$$

and endow the space $\mathcal{S}^{\mathbb{N}}$ with the left shift Σ, that is, $\Sigma((\sigma_n)_{n\geq1}) = (\sigma_{n+1})_{n\geq1}$. Then the following diagram commutes:

$$\begin{tikzcd} {[0,1] \setminus \mathbb{Q}} \arrow[r, "T_N"] \arrow[d, "\varphi"'] & {[0,1] \setminus \mathbb{Q}} \arrow[d, "\varphi"] \\ \mathcal{S}^{\mathbb{N}} \arrow[r, "\Sigma"] & \mathcal{S}^{\mathbb{N}} \end{tikzcd} \quad (5.12)$$

Thus we can associate the limit sequence $\omega(x, N)$ of $\boldsymbol{\sigma}_x$ to the NCF expansion of x in the same way as Sturmian sequences are associated to the classical continued fraction expansion of x, *e.g.* in [1, 10, 79].

As in the classical case (see [10]) we go one step further and associate the symbolic sequence $\widehat{\omega}(x, N)$ to the past of the natural extension of T_N. The associated directive sequence $\widehat{\boldsymbol{\sigma}}_x$ is in some sense a dual of $\boldsymbol{\sigma}_x$. Let $\mathbf{x} = [1 : x]$ and $\mathbf{y} = [1 : y]$ be elements of $\mathbb{P}_<$. Following [13], a natural extension of the map G_N is given by

$$\widetilde{G}_N : \mathbb{P}^2_< \to \mathbb{P}^2_<, \qquad \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \mapsto \begin{pmatrix} {}^tC_N(\mathbf{x})^{-1} & 0 \\ 0 & C_N(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \left[1 : \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor\right] \\ \left[1 : \frac{1}{N\cdot y + \left\lfloor \frac{N}{x} \right\rfloor}\right] \end{pmatrix}. \tag{5.13}$$

Taking second coordinates and inspecting the range of $\mathbf{y}$ this immediately yields the following result.

Proposition 5.2.6. *A natural extension of the map $T_N : [0, 1] \to [0, 1]$ is given by*

$$\begin{aligned} \widetilde{T}_N :&[0,1] \times \left[0, \frac{1}{N}\right] \to [0,1] \times \left[0, \frac{1}{N}\right], \\ &(x, y) \mapsto \begin{cases} \left(T_N(x), \frac{1}{N\cdot y + \left\lfloor \frac{N}{x} \right\rfloor}\right) & x \neq 0, \\ (0,0) & x = 0, \end{cases} \end{aligned} \tag{5.14}$$

with $\frac{dxdy}{(1+xy)^2}$ as invariant measure.

We mention that in [25], a different natural extension of T_N, which is isomorphic to ours, is given (the difference is that y is replaced by $\frac{y}{N}$). As we see from (5.13), the "past" of this natural extension (that is, the second coordinate of $\widetilde{T}_N$) is associated to $C_N(\mathbf{x})$. Moreover, $C_N(G_N^{n-1}(\mathbf{x})) = M_{\widehat{\sigma}_n}$ for every $n \geq 1$ (this is straight forward from (5.11)). Therefore, the natural extension of the NCF algorithm is related the incidence matrices of the dual substitutions $\widehat{\sigma}_n$, and we can associate it with directive sequences of the form $\widehat{\boldsymbol{\sigma}}_x = (\widehat{\sigma}_n)_{n\geq 1}$ and, hence, with the S-adic words $\widehat{\omega}(x, N)$.

These "substitution selections" superimpose a combinatorial structure on the NCF algorithm and its natural extension. As is detailed in [1, 10, 9, 79], this combinatorial structure gives information about the underlying continued fraction algorithm and vice versa. This motivates our study of the S-adic sequences $\omega(x, N)$ and $\widehat{\omega}(x, N)$.

5.3 Results on balance

In this section, we prove that the sequences $\omega(x, N)$ and $\widehat{\omega}(x, N)$ are finitely balanced. For each fixed $N \geq 1$ we provide an upper bound for the balance constant that is valid for each x. We refine this result by defining some sets in terms of the size of the NCF digits for which the balance constant can be improved. After that, we provide lower bounds for the balance constant. We refer the reader to [1] for some notions on balance of sequences.

We begin with the definition of balance.

Definition 5.3.1 (Balance)**.** *Given $C > 0$, we say that a pair (u, v) of words over the alphabet $\mathcal{A}$ is C-balanced if $|u| = |v|$ and*

$$-C \leq |u|_a - |v|_a \leq C \qquad \text{for every } a \in \mathcal{A}.$$

We say that a sequence $\nu \in \mathcal{A}^{\mathbb{N}}$ is C-balanced if every pair (u, v) of factors of ν with $|u| = |v|$ is C-balanced. We say that ν is finitely balanced if it is C-balanced for some $C > 0$.

Before we prove finite balancedness for NCF sequences and their duals, we introduce the following definition of a minimal pair.

Definition 5.3.2 (Minimal pair)**.** *Let $\nu \in \mathcal{A}^{\mathbb{N}}$, two factors $u, v \subset \nu$ and $C > 0$. We say that (u, v) is a minimal pair of not C-balanced factors if $|u| = |v|$, $||v|_a - |u|_a| > C$ for some $a \in \mathcal{A}$, and the length of u and v is minimal with respect to this property among all factors of ν.*

We have mentioned that it is easier to work with the dual sequence $\widehat{\omega}(x, N)$ than with the related sequence $\omega(x, N)$. Thus our strategy is to prove results for dual NCF sequences and translate them to NCF sequences afterwards. The following lemma allows us to transfer the property of balancedness in this way.

Lemma 5.3.3. *If $\widehat{\omega}(x, N)$ is C-balanced for some $C > 0$, then $\omega(x, N)$ is $N \cdot C$-balanced.*

Proof. Suppose this is not true, then given that $\widehat{\omega}(x, N)$ is C-balanced for some $C > 0$ there exists a minimal pair (u, v) of not $N \cdot C$-balanced factors of $\omega(x, N)$. It is clear that u and v cannot start with the same letter nor end with the same letter. Assume w.l.o.g. that $|v|_1 > |u|_1$, then v must start and end with 1 and u must start and end with 0. Thus according to Lemma 5.2.3 we can write $v = 1^j 0^{d_1} V 0^{d_1} 1^k$ for some word V and $j, k \in \{1, \ldots, N\}$. We distinguish two cases.

If $j + k \leq N$, then Lemma 5.2.3 implies that $\widetilde{v} = 0^{d_1+j+k-N} V 0^{d_1} 1^N$ is also in $\omega(x, N)$ and satisfies $|\widetilde{v}|_1 \geq |v|_1$ and $|\widetilde{v}| = |v|$. Then because $(u, \widetilde{v})$ is not $N \cdot C$-balanced and both words start with 0, (u, v) is not a minimal pair of not $N \cdot C$-balanced factors, a contradiction.

Suppose now that $N + 1 \leq j + k \leq 2N$. It is clear that, if (u, v) is a minimal pair of not $N \cdot C$-balanced factors, then $|v|_1 - |u|_1 = N \cdot C + 1$. Because u starts and ends with 0, $|u|_1$ is a multiple of N, so we must have $j + k = N + 1$. Hence, we can assume w.l.o.g. that $v = 1^N 0^{d_1} V 0^{d_1} 1$, because according to Lemma 5.2.3 we can adjust the values of j and k by "shifting" v. Consider the substitution τ defined in (5.6). Then by (5.7) we have that $\omega(x, N) = \tau(\widehat{\omega}(x, N))$. Find $\widehat{u}, \widehat{v} \subset \widehat{\omega}(x, N)$ such that $\tau(\widehat{u}) = u$ and $\tau(\widehat{v}) = v1^{N-1}$. Then $|\widehat{v}|_1 > |\widehat{u}|_1$ and $|\widehat{v}| < |\widehat{u}|$. Find a word $\widehat{V}$ such that $\widehat{v}\widehat{V}$ is a factor of $\widehat{\omega}(x, N)$ and $|\widehat{v}\widehat{V}| = |\widehat{u}|$. Then, because $\widehat{\omega}(x, N)$ is C-balanced by hypothesis, $|\widehat{v}\widehat{V}|_1 - |\widehat{u}|_1 \leq C$. Therefore, by the definition of τ, $|\tau(\widehat{v}\widehat{V})|_1 - |\tau(\widehat{u})|_1 \leq N \cdot C$. But

$|v|_1 \le |\tau(\widehat{v}\widehat{V})|_1$ and $\tau(\widehat{u}) = u$, which implies $0 < |v|_1 - |u|_1 \le N \cdot C$. This is again a contradiction to the assumption that (u, v) is a minimal pair of not $N \cdot C$-balanced factors of $\omega(x, N)$. □

The following theorem states the existence of balance constants for NCF sequences and their duals, which depend on N and on a lower bound for the NCF digits of x. We mention that this generalizes [80, Theorem 4.1], where the result is proven for the case where all the NCF digits are the same.

Theorem 5.3.4. *Let $N \ge 1$ be fixed and set $K \ge N$ and $C = \lfloor \frac{K-1}{K+1-N} \rfloor + 1$. If we set*

$$W_{K,N} := \{[0; d_1, d_2, \ldots]_N \in [0,1] \setminus \mathbb{Q} \,:\, d_n \ge K \text{ for all } n \ge 1\}$$

then the following assertions hold.

1. *For all $x \in W_{K,N}$ the dual NCF sequence $\widehat{\omega}(x, N)$ is C-balanced.*
2. *For all $x \in W_{K,N}$ the NCF sequence $\omega(x, N)$ is $N \cdot C$-balanced.*

Proof. We start with the proof of (1). Because the result is well-known for $N = 1$ we may assume that $N \ge 2$. Suppose this assertion is not true for some fixed K and N with $K \ge N \ge 2$. Then there exists $x \in W_{K,N}$, such that the sequence $\widehat{\omega} = \widehat{\omega}(x, N)$ admits a minimal pair (u, v) of not C-balanced factors. We assume that x is chosen in a way that (u, v) has minimal length among all not C-balanced pairs of factors of $\widehat{\omega}(y, N)$ with $y \in W_{K,N}$. We will reach a contradiction by finding a not C-balanced pair of shorter length.

Let $d = d_1$. By minimality of (u, v) we have $||v|_1 - |u|_1| = C + 1$. Also, u and v cannot start with the same letter nor end with the same letter. Assume w.l.o.g. that $|v|_1 > |u|_1$, then v must start and end with 1. Let 0^s1 be a prefix of u (it is easy to see that u has to contain an occurrence of 1 because otherwise $||v|_1 - |u|_1| \le 2 < C + 1$). Then $s \ge d+1$, otherwise we could always remove the prefix 0^s1 from u and the suffix 0^s1 from v (note that for $s \le d$ the word 0^s1 has to be a suffix of v by Lemma 5.2.3) and find a shorter pair of not C-balanced words. Analogously, if 10^t is a suffix of u then $t \ge d + 1$. Moreover, $s + t \ge 2d + N + 1$, otherwise we could “shift” u and find a word $\widetilde{u}$ with the prefix 0^s1 with $s \le d$ such that $(\widetilde{u}, v)$ is a minimal pair of not C-balanced words, which contradicts what we just proved.

Summing up, we may assume w.l.o.g. $u = 0^s1 \cdots 10^{d+N}$, $d+1 \le s \le d+N$, and $v = 10^d \cdots 0^d1$. For a factor w of ω denote by $|w|_{0^d*}$ the number of occurrences of a maximal 0 block of length 0^d in w, that is, $|w|_{0^d*} := |1w1|_{10^d1}$. Note that we need to use $|\cdot|_{10^{d+N}}$ because otherwise the prefix 0^s in u is counted twice if $s = d + N$. Lemma 5.2.3 implies that

$$|v|_0 = (d + N)\,|v|_{10^{d+N}} + d\,|v|_{0^d*}$$

and

$$|u|_0 = s + (d+N)\,|u|_{10^{d+N}} + d\,|u|_{0^{d}*}.$$

Then we have,

$$\begin{aligned} |u|_0 - |v|_0 &= s + (|u|_{10^{d+N}} - |v|_{10^{d+N}})(d+N) + (|u|_{0^{d}*} - |v|_{0^{d}*})\,d \\ = s + (|u|_{10^{d+N}} &- |v|_{10^{d+N}})\,N + ((|u|_{10^{d+N}} + |u|_{0^{d}*}) - (|v|_{10^{d+N}} + |v|_{0^{d}*}))\,d. \end{aligned} \tag{5.15}$$

Every maximal 0 block of $\widehat{\omega}$ lies between 1's. Since v starts and ends with 1, the number of maximal 0 blocks of v is $|v|_1 - 1$, and since u starts and ends with 0, the number of 0 blocks in u is $|u|_1 + 1$, of which exactly one of them (the prefix 0^s of u) is not counted by the terms $|u|_{10^{d+N}}$ and $|u|_{0^{d}*}$. Now, because $|v|_1 - |u|_1 = C + 1$, this yields

$$(|v|_{10^{d+N}} + |v|_{0^{d}*}) - (|u|_{10^{d+N}} + |u|_{0^{d}*}) = C. \tag{5.16}$$

Let

$$a := |u|_{10^{d+N}} - |v|_{10^{d+N}}.$$

Then from (5.15) and (5.16) we obtain

$$|u|_0 - |v|_0 = s + a \cdot N - d \cdot C. \tag{5.17}$$

Also, $|u| = |v|$ and hence

$$|u|_0 - |v|_0 = |v|_1 - |u|_1 = C + 1, \tag{5.18}$$

so combining (5.17) and (5.18) yields

$$a = \frac{(d+1)\,C + 1 - s}{N}. \tag{5.19}$$

Since $\widehat{\omega}$ is an S-adic sequence, by Definition 5.1.1 we have $\widehat{\omega} = \widehat{\sigma}_1(\widehat{\omega}^{(2)})$ where $\widehat{\omega}^{(2)} = \widehat{\omega}(T_N(x), N)$ with $T_N(x) = [0; d_2, d_3, \ldots]_N \in W_{K,N}$. As a consequence of the shape of $\widehat{\sigma}_1$ there exist words $u^{(2)}, v^{(2)} \subset \widehat{\omega}^{(2)}$ such that

$$\widehat{\sigma}_1(u^{(2)}) = 0^{d+N-s}u1, \qquad \widehat{\sigma}_1(v^{(2)}) = 0^{d}v.$$

We claim that

$$|v^{(2)}| \le |u^{(2)}|. \tag{5.20}$$

To prove this, suppose on the contrary that $|v^{(2)}| - |u^{(2)}| \ge 1$. Note that the letter 1 appears in $\widehat{\sigma}_1(u^{(2)})$ (resp. $\widehat{\sigma}_1(v^{(2)})$) whenever a 0 appears in $u^{(2)}$ (resp. $v^{(2)}$). Hence, (5.18) yields

$$|v^{(2)}|_0 - |u^{(2)}|_0 = |0^{d}v|_1 - |0^{d+N-s}u1|_1 = C. \tag{5.21}$$

Hence,

$$|v^{(2)}|_1 - |u^{(2)}|_1 = |v^{(2)}| - |u^{(2)}| + |u^{(2)}|_0 - |v^{(2)}|_0 \ge 1 - C.$$

By the definition of $\widehat{\sigma}_1$ and using that $C \geq 1$ this implies that

$$\begin{aligned}|\widehat{\sigma}_1(v^{(2)})| - |\widehat{\sigma}_1(u^{(2)})| &= (d+1)(|v^{(2)}|_0 - |u^{(2)}|_0) + N(|v^{(2)}|_1 - |u^{(2)}|_1)\\ &\geq (d+1)\,C + (1-C)\,N = C\,(d+1-N) + N \geq d+1.\end{aligned} \tag{5.22}$$

This is impossible because $s \leq d+N$ and so

$$|\widehat{\sigma}_1(v^{(2)})| - |\widehat{\sigma}_1(u^{(2)})| = |0^d v| - |0^{N+d-s}u1| = s - N - 1 \ \leq d-1.$$

This proves the claim.

Note that the block 0^{d+N} appears in $\widehat{\sigma}_1(u^{(2)})$ (resp. $\widehat{\sigma}_1(v^{(2)})$), whenever a 1 appears in $u^{(2)}$ (resp. $v^{(2)}$) because each 1 is followed by 0 in $u^{(2)}$ (resp. $v^{(2)}$). From the definition of a we obtain

$$|u^{(2)}|_1 - |v^{(2)}|_1 = |0^{d+N-s}u1|_{0^{d+N}} - |0^d v|_{0^{d+N}} = (|u|_{10^{d+N}}+1) - |v|_{10^{d+N}} = a+1 \tag{5.23}$$

(the "+1" comes from the fact that the prefix 0^{d+N} of $0^{d+N-s}u1$ is not counted in $|u|_{10^{d+N}}$).

Combining (5.20), (5.21), and (5.23) yields

$$0 \leq |u^{(2)}| - |v^{(2)}| = a + 1 - C. \tag{5.24}$$

Let $\widetilde{u}^{(2)}$ be the prefix of $u^{(2)}$ such that $|\widetilde{u}^{(2)}| = |v^{(2)}|$, *i.e.*, remove the last $a+1-C$ letters of $u^{(2)}$, of which at most $\left\lceil \frac{a+1-C}{d_2+1} \right\rceil$ are 1's, to obtain $\widetilde{u}^{(2)}$. Suppose that $a \geq C+1$. Then from (5.23) we get

$$|\widetilde{u}^{(2)}|_1 - |v^{(2)}|_1 \geq a + 1 - \left\lceil \frac{a+1-C}{d_2+1} \right\rceil \geq C+1.$$

This means that $(\widetilde{u}^{(2)}, v^{(2)})$ is a not C-balanced pair of factors of $\widehat{\omega}^{(2)} = \widehat{\omega}(T_N(x), N)$ with $T_N(x) \in W_{K,N}$. But $(\widetilde{u}^{(2)}, v^{(2)})$ has shorter length than (u, v), contradicting the minimality of (u, v). Together with (5.24) this yields $a \in \{C-1, C\}$. But if $a = C$, we need to remove only the last letter from $u^{(2)}$ (which is 0) to obtain $\widetilde{u}^{(2)}$. Thus in this case we get from (5.23) that

$$|\widetilde{u}^{(2)}|_1 - |v^{(2)}|_1 \geq a + 1 = C + 1$$

which is a contradiction again. Thus $a = C-1$ and we see from (5.19) that in this case we have $C - 1 = \frac{(d+1)\,C+1-s}{N}$. Because $s \leq d+N$ this implies that

$$C = \frac{s-N-1}{d+1-N} \leq \frac{d-1}{d+1-N} \leq \frac{K-1}{K+1-N}.$$

Thus a minimal pair (u,v) which is not C-balanced can only exist for $C \le \lfloor \frac{K-1}{K+1-N} \rfloor$. This contradicts our choice of C and the theorem is proved.

Statement (2) is straightforward from (1) and Lemma 5.3.3. □

The previous theorem has an immediate corollary that gives balance constants that depend only on N.

Corollary 5.3.5.

1. *For all $x \in [0,1] \setminus \mathbb{Q}$ we have that $\widehat{\omega}(x,N)$ is N-balanced. Furthermore, for every $N \ge 2$ there are uncountable many $x \in [0,1] \setminus \mathbb{Q}$ such that $\widehat{\omega}(x,N)$ is 2-balanced.*
2. *For all $x \in [0,1] \setminus \mathbb{Q}$ we have that $\omega(x,N)$ is N^2-balanced. Furthermore, for every $N \ge 2$ there are uncountable many $x \in [0,1] \setminus \mathbb{Q}$ such that $\omega(x,N)$ is $2N$-balanced.*

Proof. In (1), the first statement is immediate, since the greedy NCF expansion satisfies, for every $x \in [0,1] \setminus \mathbb{Q}$, that the digits are larger or equal to N, hence $x \in W_{N,N}$. The second statement follows from the fact that, for $x \in W_{2N,N}$, the dual S-adic sequence is 2-balanced. Furthermore, it is not hard to check that $W_{2N,N}$ is uncountable.

Assertion (2) immediately follows from (1) and Lemma 5.3.3. □

We have given upper bounds for C such that the S-adic sequences are C-balanced. Next we will give lower bounds. Note that in view of Corollary 5.3.5 this bound is optimal for $N = 2$.

Proposition 5.3.6. *Let $N \ge 2$ and $x \in [0,1] \setminus \mathbb{Q}$. Then the following assertions hold.*

1. *$\widehat{\omega}(x,N)$ is not 1-balanced.*
2. *$\omega(x,N)$ is not $(2N-1)$-balanced.*

Proof. Let $x = [0; d_1, d_2, \ldots]_N$.

(1) Recall that $\widehat{\omega}(x,N) = \widehat{\sigma}_1(\widehat{\omega}^{(2)})$, where $\widehat{\omega}^{(2)} = \widehat{\omega}(T_N(x),N)$. It is easy to check that 10 is a factor of $\widehat{\omega}^{(2)}$, hence $\widehat{\sigma}_1(10) = 0^N 0^{d_1} 1$ is a factor of $\widehat{\omega}(x,N)$. Since $N \ge 2$, this yields the factor $u = 0^{d_1+2}$ of $\widehat{\omega}(x,N)$.

Analogously, 00 is a factor of $\widehat{\omega}^{(2)}$, hence $\widehat{\sigma}_1(00) = 0^{d_1} 1 0^{d_1} 1$ is a factor of $\widehat{\omega}(x,N)$. This yields the factor $v = 10^{d_1}1$ of $\widehat{\omega}(x,N)$. Because u and v are of the same length and $|v|_1 - |u|_1 = 2$, assertion (1) is proved.

(2) Recall that $\omega(x,N) = \sigma_1(\omega^{(2)}) = \sigma_1 \circ \sigma_2(\omega^{(3)})$ where $\omega^{(2)} = \omega(T_N(x),N)$ and $\omega^{(3)} = \omega(T_N^2(x),N)$. It is not hard to check that 001 is a factor of $\omega^{(3)}$. We have

$$\begin{aligned}\sigma_1 \circ \sigma_2(001) &= \sigma_1(0^{d_2} 1^N 0^{d_2} 1^N 0) \\ &= (0^{d_1} 1^N)^{d_2} 0^N (0^{d_1} 1^N)^{d_2} 0^N 0^{d_1} 1^N,\end{aligned}$$

which gives us the factor $u = 0^N(0^{d_1}1^N)^{d_2}0^{N+d_1}$ of $\omega(x,N)$.

On the other hand, since $N \geq 2$ it is not hard to check that 110 is a factor of $\omega^{(3)}$. We have

$$\begin{aligned}\sigma_1 \circ \sigma_2(110) &= \sigma_1(0^{d_2+2}1^N) \\ &= 0^{d_1}1^N(0^{d_1}1^N)^{d_2}0^{d_1}1^N0^N,\end{aligned}$$

which gives us the factor $v = 1^N(0^{d_1}1^N)^{d_2}0^{d_1}1^N$ of $\omega(x,N)$. Since $|u| = |v|$ and $|v|_1 - |u|_1 = 2N$, assertion (2) follows. □

Remark 5.3.7. *Suppose that, given $x \in [0,1]$, we consider infinite NCF expansions that are different from the greedy expansion, that is, we allow $d_n < N$. Note that such sequences also exist for rational values of x. For each expansion (d_n) of x we can then define the S-adic sequence $\widetilde{\omega}(x,(d_n),N)$ of the substitutions (σ_n) associated to d_n for all $n \geq 1$. Then we have the following.*

1. *The property of N^2-balancedness does not hold in general for $\widetilde{\omega}(x,(d_n),N)$. For instance, let $N = 2$ and $x = [0;1,1,1,\ldots]_2$, i.e., the sequence of digits is given by $(d_n) = (1)$. Then $u = (0^31^2)^3(01^2)^20^31^20^3$ and $v = (1^20)^30^2(1^20)^30^2(1^20)^21$ are both factors of $\widetilde{\omega}(x,(d_n),N)$ of the same length and $|v|_1 - |u|_1 = 5 = N^2+1$.*

2. *Suppose $(d_n) = (d)$ for some $d < N$. Then the associated substitution σ is not Pisot, and hence the sequence $\widetilde{\omega}(x,N)$ is imbalanced, that is, it is not C-balanced for any $C > 0$. This is a consequence of [1, Theorem 13].*

3. *S-adic sequences corresponding to eventually greedy NCF expansions are finitely balanced. Suppose $d_n \geq N$ for all $n \geq K$ for some $K \geq 1$. Let $\sigma = \sigma_1 \circ \cdots \circ \sigma_{K-1}$ and let $\tilde{x} = [0; d_K, d_{K+1}, \ldots]_N$. Then $\widetilde{\omega}(x,(d_n),N) = \sigma(\omega(\tilde{x},N))$ and $\omega(\tilde{x},N)$ is N^2-balanced because it is an NCF sequence. It is not hard to check that, for any given substitution σ, if a sequence ν is C-balanced then $\sigma(\nu)$ is C'-balanced for some C'. Hence the statement follows.*

5.4 Factor complexity and unique ergodicity

Another property worth studying when working with sequences is their *factor complexity function.* This function counts how many different factors of a given length $n \in \mathbb{N}$ appear in a sequence.

Definition 5.4.1 (Factor complexity function)**.** *Given a sequence $\nu \in \mathcal{A}^{\mathbb{N}}$ over the finite alphabet $\mathcal{A}$ and $n \in \mathbb{N}$, set*

$$\mathcal{L}_n(\nu) := \{u \in \mathcal{A}^* \,:\, |u| = n,\ u \textit{ is a factor of } \nu\}.$$

Define the factor complexity function $p_\nu : \mathbb{N} \to \mathbb{N}$ as $p_\nu(n) = |\mathcal{L}_n(\nu)|$.

The factor complexity function has the trivial upper bound $p_\nu(n) \leq |\mathcal{A}|^n$, and ν is periodic if and only if there exists $n \in \mathbb{N}$ such that $p_\nu(n) \leq n$ (for the non-trivial direction see [22, Corollary 4.3.2]). As mentioned in the introduction, a sequence ν is said to be Sturmian if it is not eventually periodic and its factor complexity function is given by $p_\nu(n) = n + 1$. This means that Sturmian sequences are the aperiodic sequences with the smallest possible complexity. We will show that NCF sequences and their duals have low complexity as well and we will explicitly compute a formula for their factor complexity function. We follow [22] and [32].

A useful way to study the complexity function is to figure out how to obtain $\mathcal{L}_{n+1}(\nu)$ from $\mathcal{L}_n(\nu)$, and for that we make use of *left special factors*. Moreover, in what follows we will study three particular types of left special factors: infinite and maximal left special factors, and total bispecial factors. We introduce the corresponding definitions.

Definition 5.4.2. *Consider a sequence $\nu \in \mathcal{A}^{\mathbb{N}}$ over a finite alphabet $\mathcal{A}$.*

1. *Given a factor u of ν, we say that a letter $a \in \mathcal{A}$ is a left extension of u if au is a factor of ν.*
2. *We say that a factor u of ν is a left special factor of ν if there are two distinct letters $a, b \in \mathcal{A}$ that are left extensions of u. We set*
$$LS_n(\nu) := \{u \in \mathcal{A}^* : |u| = n, \ u \text{ is a left special factor of } \nu\}.$$
3. *An infinite word $u \in \mathcal{A}^{\mathbb{N}}$ is called an infinite left special factor of ν if every prefix of u is a left special factor of ν.*
4. *A left special factor u of ν is called a maximal left special factor if ua is not a left special factor for any $a \in \mathcal{A}$.*
5. *A left special factor u of ν is called a total bispecial factor if there exist $a, b \in \mathcal{A}$ with $a \neq b$ such that ua and ub are both left special factors of ν.*

In analogy with left extension, left special factor, infinite left special factor and maximal left special factor we can define right extension, right special factor, etc. If we describe the occurrences of special factors in $\mathcal{L}_n(\nu)$, we can determine $\mathcal{L}_{n+1}(\nu)$. The following result is a direct consequence of [32, Proposition 2.1].

Lemma 5.4.3. *Let ν be a sequence over a two letter alphabet and let p_ν be its factor complexity function. Then, for every $n \in \mathbb{N}$,*

$$p_\nu(n+1) - p_\nu(n) = |LS_n(\nu)|. \tag{5.25}$$

For example, a Sturmian sequence ν has exactly one left special factor of length n for every n, which implies $p_\nu(n+1) - p_\nu(n) = 1$, and because $p_\nu(1) = 2$ this implies that $p_\nu(n) = n + 1$.

5.4.1 Characterization of left special factors

In order to obtain the factor complexity function for dual NCF sequences, we will give a characterization of their left special factors. Fix $x \in [0,1] \setminus \mathbb{Q}$ and $N \geq 1$ and consider the dual NCF sequence $\widehat{\omega} = \widehat{\omega}(x, N)$. Recall that this is an S-adic sequence for the substitutions $\widehat{\boldsymbol{\sigma}}_x = (\widehat{\sigma}_n)_{n\geq 1}$, which means there exist infinite words $\widehat{\omega}^{(1)}, \widehat{\omega}^{(2)}, \ldots$ such that $\widehat{\omega} = \widehat{\omega}^{(1)}$ and $\widehat{\omega}^{(n)} = \widehat{\sigma}_n(\widehat{\omega}^{(n+1)})$ for every $n \geq 1$ (see Definition 5.1.1). This desubstitution process is crucial for this section because we will show that many properties of special factors are invariant under the substitutions $\widehat{\sigma}_n$.

Lemma 5.4.4. *The following assertions hold for every $n \geq 1$.*

1. *Let v be a left special factor of $\widehat{\omega}^{(n+1)}$. Then $\widehat{\sigma}_n(v)$ is a left special factor of $\widehat{\omega}^{(n)}$.*
2. *Let v be a left special factor of $\widehat{\omega}^{(n)}$ ending in the letter 1. Then there exists a left special factor u of $\widehat{\omega}^{(n+1)}$ such that $\widehat{\sigma}_n(u) = v$.*

Proof. (1) We have $0v, 1v \subset \widehat{\omega}^{(n+1)}$. Then $\widehat{\sigma}_n(0v) = 0^{d_n}1\widehat{\sigma}_n(v)$ and $\widehat{\sigma}_n(1v) = 0^N\widehat{\sigma}_n(v)$, which implies that $\widehat{\sigma}_n(v)$ is a left special factor of $\widehat{\omega}^{(n)}$.

(2) Consider a word $u \subset \widehat{\omega}^{(n+1)}$ of minimal length such that $v \subset \widehat{\sigma}_n(u)$. Since v ends with 1, by minimality of u it holds that v is a suffix of $\widehat{\sigma}_n(u)$. Since v is a left special factor, $1v$ is a factor of $\widehat{\omega}^{(n)}$, so it follows from the shape of the substitution $\widehat{\sigma}_n$ that $1v \subset \widehat{\sigma}_n(0u)$. The minimality of u implies $\widehat{\sigma}_n(u) = v$. Since $0v$ is also a factor of $\widehat{\omega}^{(n)}$, it turns out that $0v \subset \widehat{\sigma}_n(1u)$, and therefore u is a left special factor of $\widehat{\omega}^{(n+1)}$. □

The following characterization of infinite left special factors is based on [32, Section 3].

Lemma 5.4.5. *The only infinite left special factor of $\widehat{\omega}$ is $\widehat{\omega}$ itself.*

Proof. Recall that $(\widehat{\Sigma}_k)_{k\in\mathbb{N}}$ is a nested sequence of prefixes of $\widehat{\omega}$ satisfying (5.5). First, note that the last letter of $\widehat{\Sigma}_k$ is congruent to $k+1$ modulo 2. Also, for $k \geq 2$, both $\widehat{\Sigma}_k\widehat{\Sigma}_k \subset \widehat{\omega}$ and $\widehat{\Sigma}_{k-1}\widehat{\Sigma}_k \subset \widehat{\omega}$, giving us that $\widehat{\Sigma}_k$ is a left special factor for every $k \geq 1$. This implies that every prefix of $\widehat{\omega}$ is a left special factor of $\widehat{\omega}$ and hence $\widehat{\omega}$ is an infinite left special factor of itself.

It remains to show uniqueness. Let ν be an infinite left special factor of $\widehat{\omega}$. Recall that $\widehat{\omega} = \widehat{\sigma}_1(\widehat{\omega}^{(2)})$. We show first that there exists an infinite left special factor of $\widehat{\omega}^{(2)}$, namely $\nu^{(2)}$, such that $\nu = \widehat{\sigma}_1(\nu^{(2)})$. Consider a prefix v of ν ending in the letter 1. By hypothesis, this prefix is a left special factor. By part (2) of Lemma 5.4.4, there exists a left special factor $v^{(2)}$ of $\widehat{\omega}^{(2)}$ such that $\widehat{\sigma}_1(v^{(2)}) = v$. The prefix v can be chosen to be arbitrarily large, which means $v^{(2)}$ can be made to be arbitrarily large. This implies the existence of an infinite left special factor $\nu^{(2)}$ of $\widehat{\omega}^{(2)}$ such that $\nu = \widehat{\sigma}_1(\nu^{(2)})$. Iterating this reasoning, we can obtain a sequence $(\nu^{(n)})_{n\geq 1}$, $\nu^{(1)} = \nu$, such that for each n, $\nu^{(n)}$ is an infinite left special factor of $\widehat{\omega}^{(n)}$ and $\widehat{\sigma}_n(\nu^{(n+1)}) = \nu^{(n)}$. We use this to show that $\nu = \widehat{\omega}$.

Suppose $\nu \neq \widehat{\omega}$, then $\widehat{\omega}^{(n)} \neq \nu^{(n)}$ for each $n \geq 1$. This enables the definition of

$$d(\widehat{\omega}^{(n)}, \nu^{(n)}) := \min\{k : \widehat{\omega}_k^{(n)} \neq \nu_k^{(n)}\}.$$

Note that the substitutions $\widehat{\sigma}_n$ strictly increase the length of a word unless the word is 1 and $N = 1$. Since the image under $\widehat{\sigma}_n$ of both letters starts with 0, it follows that 1 is not a prefix of $\widehat{\omega}^{(n)}$ nor of $\nu^{(n)}$ for any $n \geq 1$. Therefore, it holds for all $n \geq 1$ that $d(\widehat{\omega}^{(n)}, \nu^{(n)}) > 1$ and $d(\widehat{\omega}^{(n+1)}, \nu^{(n+1)}) < d(\widehat{\omega}^{(n)}, \nu^{(n)})$, so the sequence of distances $(d(\widehat{\omega}^{(n)}, \nu^{(n)}))_{n\in\mathbb{N}}$ is strictly decreasing yet strictly positive, a contradiction. This shows that $\nu = \widehat{\omega}$. □

Next, we state some lemmas regarding maximal left special factors and total bispecial factors.

Lemma 5.4.6. *Let v be a maximal left special factor of $\widehat{\omega}^{(n)}$ containing the letter 1. Then there exists a maximal left special factor u of $\widehat{\omega}^{(n+1)}$ such that $v = \widehat{\sigma}_n(u)0^{d_n}$.*

Proof. First, note that if a left special factor has a unique right extension it cannot be maximal. Given a maximal left special factor v, since it has more than one right extension and contains a 1 by assumption, it must be of the form $v = v_0 v_1 \cdots v_s 10^{d_n}$ for some $s \in \mathbb{N}$ (see part (2b) of Lemma 5.2.3). By part (2) of Lemma 5.4.4, there exists a left special factor u of $\widehat{\omega}^{(n+1)}$ such that $v = \widehat{\sigma}_n(u)0^{d_n}$. It remains to show that u is maximal. Suppose it is not, then there exists a letter $a \in \mathcal{A}$ such that ua is also a left special factor. By part (1) of Lemma 5.4.4, $\widehat{\sigma}_n(ua)$ is a left special factor of $\widehat{\omega}^{(n)}$. If $a = 0$ then v is a proper prefix of the left special factor $\widehat{\sigma}_n(u0)$, contradicting the maximality of v. If $a = 1$ then $u10$ is a left special factor because 1 is always followed by 0, and in this case v is also a proper prefix of the left special factor $\widehat{\sigma}_n(u10)$, contradicting again the maximality of v. □

Lemma 5.4.7. *Let v be a total bispecial factor of $\widehat{\omega}^{(n)}$ containing the letter 1. Then there exists a total bispecial factor u of $\widehat{\omega}^{(n+1)}$ such that $v = \widehat{\sigma}_n(u)0^{d_n}$.*

Proof. Since v has more than one right extension and contains the letter 1 it must be of the form $v = v_0 v_1 \cdots v_s 10^{d_n}$ for some $s \in \mathbb{N}$ (see part (2b) of Lemma 5.2.3). By part (2) of Lemma 5.4.4, there exists a left special factor u of $\widehat{\omega}^{(n+1)}$ such that $v = \widehat{\sigma}_n(u)0^{d_n}$. It remains to show that u is a total bispecial factor. By hypothesis,

$0v0, 0v1, 1v0, 1v1 \subset \widehat{\omega}^{(n)}$. We have

$$\begin{aligned} 0\widehat{\sigma}_n(u)0^{d_n+1} &\subset \widehat{\sigma}_n(1u10), \\ 0\widehat{\sigma}_n(u)0^{d_n}1 &\subset \widehat{\sigma}_n(1u0), \\ 1\widehat{\sigma}_n(u)0^{d_n+1} &\subset \widehat{\sigma}_n(0u10), \\ 1\widehat{\sigma}_n(u)0^{d_n}1 &\subset \widehat{\sigma}_n(0u0). \end{aligned}$$

It follows from the shape of $\widehat{\sigma}_n$ that $0\widehat{\sigma}_n(u)$ can only occur as a factor of $\widehat{\sigma}_n(1u)$ and $1\widehat{\sigma}_n(u)$ can only occur as a factor of $\widehat{\sigma}_n(0u)$; moreover, $\widehat{\sigma}_n(u)0^{d_n+1}$ can only occur as a factor of $\widehat{\sigma}_n(u10)$ and $\widehat{\sigma}_n(u)0^{d_n}1$ can only occur as a factor of $\widehat{\sigma}_n(u0)$. This implies that u is a total bispecial factor of $\widehat{\omega}^{(n+1)}$. $\square$

5.4.2 Results on factor complexity

Fix $N \geq 2$ and $x \in [0,1] \setminus \mathbb{Q}$. Consider the NCF sequence $\omega(x, N)$ and its dual $\widehat{\omega}(x, N)$. Consider the corresponding nested sequences of prefixes $(\Sigma_n)_{n\geq 1}$ and $(\widehat{\Sigma}_n)_{n\geq 1}$ and the respective recurrence relations (5.4) and (5.5).

Define, for each $k \geq 1$, the words

$$\widehat{S}_k := \widehat{\Sigma}_k^{d_k}\widehat{\Sigma}_{k-1}^{d_{k-1}}\cdots\widehat{\Sigma}_1^{d_1}, \qquad \widehat{T}_k := \widehat{\Sigma}_k^{N-1}\widehat{S}_k$$

and

$$S_k := \Sigma_k^{d_k}\Sigma_{k-1}^{d_{k-1}}\cdots\Sigma_1^{d_1}, \qquad T_0 := 1^{N-1}, \qquad T_k := \Sigma_k^{N-1}S_k.$$

Define the numbers $\widehat{t}_0 < \widehat{s}_1 < \widehat{t}_1 < \widehat{s}_2 < \widehat{t}_2 < \cdots$ as

$$\widehat{s}_k := |\widehat{S}_k|, \qquad \widehat{t}_0 = 0, \qquad \widehat{t}_k := |\widehat{T}_k| \qquad (k \geq 1).$$

Define the numbers $t_0 < s_1 < t_1 < s_2 < t_2 < \cdots$ as

$$s_k := |S_k| \qquad (k \geq 1), \qquad t_k := |T_k| \qquad (k \geq 0).$$

Recall the definition of p_n and q_n with $n \geq -1$ given in (5.8). The main result of this section is the following.

Theorem 5.4.8. *The factor complexity functions of $\omega = \omega(x, N)$ and $\widehat{\omega} = \widehat{\omega}(x, N)$ satisfy*

$$p_\omega(n) \leq 2n \qquad \textit{and} \qquad p_{\widehat{\omega}}(n) \leq 2n \qquad\qquad (n \geq 1).$$

In particular, they are given by

$$p_\omega(n) = \begin{cases} 1, & n = 0, \\ 2n, & 1 \leq n \leq N-1, \\ n+1+\sum_{j=-1}^{k-1}(p_j+q_j)(N-1), & t_k < n \leq s_{k+1}, \\ 2n+1+\sum_{j=-1}^{k-1}(p_j+q_j)(N-1)-s_k, & s_k < n \leq t_k \end{cases} \tag{5.26}$$

and

$$p_{\widehat{\omega}}(n) = \begin{cases} 1, & n = 0, \\ n+1+\sum_{j=0}^{k-2}(\frac{p_j}{N}+q_j)(N-1), & \widehat{t}_{k-1} < n \leq \widehat{s}_k, \\ 2n+1+\sum_{j=0}^{k-2}(\frac{p_j}{N}+q_j)(N-1)-\widehat{s}_k, & \widehat{s}_k < n \leq \widehat{t}_k. \end{cases} \tag{5.27}$$

Before proceeding to the proof, we state and prove some lemmas.

Lemma 5.4.9. *If v is a maximal left special factor of $\widehat{\omega} = \widehat{\omega}(x, N)$, then it is of the form $\widehat{T}_k$ for some $k \geq 1$.*

Proof. Let $v = v^{(1)}$ be a maximal left special factor of $\widehat{\omega}$. Then either it contains a 1 or not. Suppose it does not contain a 1. It is not hard to check that the only maximal left special factor of $\widehat{\omega}^{(n)}$ that does not contain the letter 1 is $0^{d_n+N-1} = \widehat{T}_1$. Now suppose it does contain a 1. By Lemma 5.4.6, we can write $v^{(1)} = \widehat{\sigma}_1(v^{(2)})0^{d_1}$ for a maximal left special factor $v^{(2)} \subset \widehat{\omega}^{(2)}$. If $v^{(2)}$ also contains a 1, we can desubstitute again and obtain a maximal left special factor $v^{(3)} \subset \widehat{\omega}^{(3)}$ such that $v^{(2)} = \widehat{\sigma}_2(v^{(3)})0^{d_2}$. Iterating this process, because at each desubstitution the length of the words decreases, we eventually reach a maximal left special factor $v^{(n)}$ of $\widehat{\omega}^{(n)}$ that does not contain the letter 1 which is 0^{d_n+N-1}. We get

$$\begin{aligned} v &= \widehat{\sigma}_1(\widehat{\sigma}_2(\cdots\widehat{\sigma}_{n-1}(0^{d_n+N-1})0^{d_{n-1}}\cdots)0^{d_2})0^{d_1} \\ &= (\widehat{\sigma}_1 \circ \cdots \circ \widehat{\sigma}_{n-1}(0))^{d_n+N-1}(\widehat{\sigma}_1 \circ \cdots \circ \widehat{\sigma}_{n-2}(0))^{d_{n-1}} \cdots 0^{d_1} \\ &= \widehat{\Sigma}_n^{d_n+N-1}\widehat{\Sigma}_{n-1}^{d_{n-1}} \cdots \widehat{\Sigma}_1^{d_1} = \widehat{T}_n \end{aligned} \tag{5.28}$$

and the result follows. □

We now have a complete understanding of maximal left special factors of $\widehat{\omega}$. It is clear that any left special factor is either a prefix of $\widehat{\omega}$ (the only infinite left special factor) or a prefix of $\widehat{T}_k$ for some $k \geq 1$. The following result follows from the definition of $\widehat{S}_k$ and of $\widehat{\Sigma}_k$ and uses our previous results on total bispecial factors.

Lemma 5.4.10. *For each $k \geq 1$, the maximal common prefix of $\widehat{T}_k$ and $\widehat{\omega}$ is $\widehat{S}_k$.*

Proof. Fix $k \geq 1$ and suppose v is the maximal common prefix of $\widehat{T}_k$ and $\widehat{\omega}$. Then v is a strict prefix of $\widehat{T}_k$ because $\widehat{T}_k$ is not a prefix of $\widehat{\omega}$. By maximality, we have that there are $a, b \in \{0, 1\}$ with $a \neq b$ such that va is a prefix of $\widehat{T}_k$ and vb is a prefix of

$\widehat{\omega}$. Because all prefixes of $\widehat{T}_k$ and all prefixes of $\widehat{\omega}$ are left special factors, v is a total bispecial factor of $\widehat{\omega}$. If $k=1$ then $\widehat{T}_1 = 0^{d_1+N-1}$ and $v = \widehat{S}_1 = 0^{d_1}$. If $k \geq 2$ then v contains the letter 1. By Lemma 5.4.7, we can write $v = v^{(1)} = \widehat{\sigma}_1(v^{(2)})0^{d_1}$ for a total bispecial factor $v^{(2)} \subset \widehat{\omega}^{(2)}$. If $v^{(2)}$ also contains a 1, we can desubstitute again and obtain a total bispecial factor $v^{(3)} \subset \widehat{\omega}^{(3)}$ such that $v^{(2)} = \widehat{\sigma}_2(v^{(3)})0^{d_2}$. Iterating this process, because at each desubstitution the length of the words decreases, we eventually reach a total bispecial factor $v^{(n)}$ of $\widehat{\omega}^{(n)}$ that does not contain the letter 1. It is not hard to check that it must be of the form $v^{(n)} = 0^{d_n}$. Since v is a prefix of $\widehat{T}_k$, using the desubstitution of $\widehat{T}_k$ given in (5.28), we have $n \leq k$. By maximality of v, we obtain $n = k$. We get

$$\begin{aligned}
v &= \widehat{\sigma}_1(\widehat{\sigma}_2(\cdots\widehat{\sigma}_{k-1}(0^{d_k})0^{d_{k-1}}\cdots)0^{d_2})0^{d_1} \\
&= (\widehat{\sigma}_1 \circ \cdots \circ \widehat{\sigma}_{k-1}(0))^{d_k}(\widehat{\sigma}_1 \circ \cdots \circ \widehat{\sigma}_{k-2}(0))^{d_{k-1}} \cdots 0^{d_1} \\
&= \widehat{\Sigma}_k^{d_k}\widehat{\Sigma}_{k-1}^{d_{k-1}} \cdots \widehat{\Sigma}_1^{d_1} = \widehat{S}_k.
\end{aligned}$$ □

The following lemma computes the difference between consecutive terms of both complexity functions using left special factors.

Lemma 5.4.11. *Let $\widehat{\omega} = \widehat{\omega}(x, N)$ with factor complexity function $p_{\widehat{\omega}}$. Then, for every $n \geq 1$,*

$$p_{\widehat{\omega}}(n+1) - p_{\widehat{\omega}}(n) = \begin{cases} 1, & \widehat{t}_{k-1} < n \leq \widehat{s}_k, \\ 2, & \widehat{s}_k < n \leq \widehat{t}_k. \end{cases} \tag{5.29}$$

Proof. Let $n \geq 1$. By Lemma 5.4.5, the prefix of $\widehat{\omega}$ of length n is a left special factor. We have that

$$|LS_n(\widehat{\omega})| = 1 + |LS_n^*(\widehat{\omega})|,$$

where $LS_n^*(\widehat{\omega})$ is the set of left special factors that are not prefixes of $\widehat{\omega}$. Given a left special factor v of length n, Lemma 5.4.9 implies it is a prefix of $\widehat{T}_k$ for some k. Take k so that $\widehat{t}_{k-1} < n \leq \widehat{t}_k$. Suppose $\widehat{t}_{k-1} < n \leq \widehat{s}_k$, then v is a prefix of $\widehat{S}_k$. It follows from Lemma 5.4.10 that v is a prefix of $\widehat{\omega}$, therefore $LS_n^*(\widehat{\omega}) = \varnothing$ and so $|LS_n(\widehat{\omega})| = 1$.

Suppose now that $\widehat{s}_k < n \leq \widehat{t}_k$. In this case, $\widehat{T}_k$ and $\widehat{\omega}$ do not coincide in the first n letters, hence $|LS_n^*(\widehat{\omega})| = 1$ and therefore $|LS_n(\widehat{\omega})| = 2$.

Then

$$|LS_n(\widehat{\omega})| = \begin{cases} 1, & \widehat{t}_{k-1} < n \leq \widehat{s}_k, \\ 2, & \widehat{s}_k < n \leq \widehat{t}_k. \end{cases}$$

Then (5.29) follows from (5.25). □

We are in position to prove the main theorem of this section.

Proof of Theorem 5.4.8. We first compute the factor complexity function of $\widehat{\omega}$. Using that $p_{\widehat{\omega}}(0) = 1$ (then only word of length 0 is ε) and $p_{\widehat{\omega}}(1) = 2$ (the alphabet has two letters), (5.27) follows from (5.29) by direct computation. Indeed, let $n \geq 1$ and suppose first that $\widehat{t}_{k-1} < n \leq \widehat{s}_k$ for some k. We obtain from Lemma 5.4.11 that

$$\begin{aligned} p_{\widehat{\omega}}(n) &= 1 + (\widehat{s}_1 - \widehat{t}_0) + 2(\widehat{t}_1 - \widehat{s}_1) + (\widehat{s}_2 - \widehat{t}_1) + \cdots + 2(\widehat{t}_{k-1} - \widehat{s}_{k-1}) + (n - \widehat{t}_{k-1}) \\ &= n + 1 + (\widehat{t}_1 - \widehat{s}_1) + \cdots + (\widehat{t}_{k-1} - \widehat{s}_{k-1}). \end{aligned}$$

Note that $\widehat{t}_j = |\widehat{\Sigma}_j|(N-1) + \widehat{s}_j$ for all $j \geq 1$. Hence,

$$p_{\widehat{\omega}}(n) = n + 1 + \sum_{j=1}^{k-1} |\widehat{\Sigma}_j|(N-1).$$

Suppose now that $\widehat{s}_k < n \leq \widehat{t}_k$. Then

$$\begin{aligned} p_{\widehat{\omega}}(n) &= 1 + (\widehat{s}_1 - \widehat{t}_0) + 2(\widehat{t}_1 - \widehat{s}_1) + (\widehat{s}_2 - \widehat{t}_1) + \cdots + (\widehat{s}_k - \widehat{t}_{k-1}) + 2(n - \widehat{s}_k) \\ &= 2n + 1 + (\widehat{t}_1 - \widehat{s}_1) + \cdots + (\widehat{t}_{k-1} - \widehat{s}_{k-1}) - \widehat{s}_k \\ &= 2n + 1 + \sum_{j=1}^{k-1} |\widehat{\Sigma}_j|(N-1) - \widehat{s}_k. \end{aligned}$$

Recall from (5.9) that, given $j \geq 0$, $|\Sigma_j|_0 = q_{j-1}$ and $|\Sigma_j|_1 = p_{j-1}$, hence $|\Sigma_j| = p_{j-1} + q_{j-1}$. Recall also that, using the substitution τ defined in (5.6) that maps $\widehat{\omega}$ to ω, there is a factor 1^N in Σ_j whenever there is a 1 in $\widehat{\Sigma}_j$. Thus $|\widehat{\Sigma}_j|_0 = |\Sigma_j|_0$ and $|\widehat{\Sigma}_j|_1 = \frac{|\Sigma_j|_1}{N}$, and therefore $|\widehat{\Sigma}_j| = \frac{p_{j-1}}{N} + q_{j-1}$. This yields (5.27).

We can obtain the complexity of ω just like we did for $\widehat{\omega}$. We have $p_\omega(0) = 1$ and $p_\omega(1) = 2$. Note that $T_0 = 1^{N-1}$ is a maximal left special factor of ω. If $1 \leq n \leq N-1$, it holds that $LS_n(\omega) = \{0^n, 1^n\}$, that is, there are two left special factors; this implies $p_\omega(n) = 2n$. Note that, if a word $u \subset \widehat{\omega}$ starts with 0, then au is a left extension of u in $\widehat{\omega}$ if and only if $a\tau(u)$ is a left extension of $\tau(u)$ in ω. Hence, it is easy to check that u is a left special factor of $\widehat{\omega}$ if and only if $\tau(u)$ is a left special factor of ω. Moreover, u is a maximal left special factor of $\widehat{\omega}$ if and only if $\tau(u)$ is a maximal left special factor of ω. Also, ω is the unique infinite left special factor of itself, by the same arguments as in the proof of Lemma 5.4.5. Moreover, Lemmas 5.4.9 and 5.4.10 are analogous for ω: the maximal left special factors of ω are the words T_k for $k \geq 1$ (to which we add $T_0 = 1^{N-1}$), and the maximal common prefix of ω and T_k is S_k. Proceeding like in the proof of Lemma 5.4.11, this yields, for $n \geq N$,

$$|LS_n(\omega)| = \begin{cases} 1, & t_k < n \leq s_{k+1}, \\ 2, & s_k < n \leq t_k. \end{cases} \tag{5.30}$$

Finally, (5.26) follows from (5.30). □

5.4.3 Uniform word frequencies and unique ergodicity

Results on word combinatorics can contribute to the study of associated symbolic dynamical systems.

Definition 5.4.12 (Unique ergodicity)**.** *We say that a topological dynamical system is uniquely ergodic if it has only one invariant probability measure.*

Corollary 5.4.13. *The dynamical systems (X_ω, Σ) and $(X_{\widehat{\omega}}, \Sigma)$ are uniquely ergodic.*

Proof. A criterion of Boshernitzan states that unique ergodicity for a minimal subshift (X_ν, Σ) holds if the factor complexity function satisfies $\limsup \frac{p_\nu(n)}{n} < 3$. By [29, Proposition 7.1.5], the subshift (X_ν, Σ) is minimal if and only if ν is uniformly recurrent, that is, if every factor of ν occurs in an infinite number of places with bounded gaps.

Any factor u of $\omega(x, N)$ is a factor of Σ_k for some k, and each Σ_k appears infinitely often with bounded gaps, hence $\omega(x, N)$ is uniformly recurrent and hence the subshift (X_ω, Σ) is minimal. Since $p_\omega(n) \leq 2n$, the assertion follows. The reasoning for $\widehat{\omega}(x, N)$ is analogous. $\square$

Definition 5.4.14 (Uniform word and letter frequency)**.** *Consider a sequence $\nu = \nu_0\nu_1 \cdots \in \mathcal{A}^{\mathbb{N}}$ with $\nu_i \in \mathcal{A}$. Recall that for each $k, l \in \mathbb{N}$ and $u \in \mathcal{A}^*$, we denote by $|\nu_k \cdots \nu_{k+l-1}|_u$ the number of occurrences of the word u in the factor $\nu_k \cdots \nu_{k+l-1} \subset \nu$. We say that ν has uniform word frequency if for each $u \in \mathcal{A}^*$ there exists $f_\nu(u) \in \mathbb{R}$ which does not depend on k such that*

$$\lim_{l\to\infty} \frac{|\nu_k \cdots \nu_{k+l-1}|_u}{l} = f_\nu(u),$$

and this limit is uniform on k.

Corollary 5.4.15. *$\omega(x, N)$ and $\widehat{\omega}(x, N)$ have uniform word frequency. Moreover, this holds for all the elements of X_ω and $X_{\widehat{\omega}}$.*

Proof. It follows from [29, Theorem 7.2.10] since the respective dynamical systems are uniquely ergodic. $\square$

5.5 Entropy, growth rate, and a Farey map

5.5.1 Entropy and growth rate

We now calculate how fast the words $\Sigma_n = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(1)$ grow when n tends to ∞. To this end we need the entropy $h(T_N)$ of the N-continued fraction map T_N. This entropy, which is calculated in [25], is given by

$$h(T_N) = \frac{\frac{\pi^2}{3} + 2Li_2(N+1) + \log(N+1)\log(N)}{\log\left(\frac{N+1}{N}\right)}$$

where Li_2 denotes the dilogarithm function defined by

$$Li_2(x) = \int_0^x \frac{\log(t)}{1-t} dt.$$

Our result reads as follows.

Proposition 5.5.1. *For the growth rate of the lengths of the words $\Sigma_n = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(1)$ we obtain the formula*

$$\lim_{n\to\infty} \frac{1}{n} \log(|\Sigma_n|) = -\frac{1}{2} \left(h(T_N) + \log(N)\right). \tag{5.31}$$

Proof. Recall that $|\Sigma_n|_1 = p_{n-1}$ and $|\Sigma_n|_0 = q_{n-1}$. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log(|\Sigma_n|) &= \lim_{n\to\infty} \frac{1}{n} \log(p_{n-1} + q_{n-1}) \\ &= \lim_{n\to\infty} \frac{1}{n} \log\left(q_{n-1}\left(\frac{p_{n-1}}{q_{n-1}} + 1\right)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log(q_n). \end{aligned}$$

Thus it remains to show that

$$\lim_{n\to\infty} \frac{1}{n} \log(q_n) = -\frac{1}{2} \left(h(T_N) + \log(N)\right), \tag{5.32}$$

where $h(T_N)$ is the entropy of T_N. Let

$$\Delta(d_1, d_2, \ldots, d_n) = \{y \in [0,1] \,:\, y = [0; d_1, d_2, \ldots, d_n, \ldots]_N\}$$

be cylinders of order n and

$$\Delta_n(x) = \{y \in [0,1] \,:\, y = [0; d_1(x), d_2(x), \ldots, d_n(x), \ldots]_N\}.$$

Since the cylinders are a finite generator for T_N and give a finite countable partition, we have for almost every $x \in [0,1]$ that

$$\lim_{n\to\infty} -\frac{1}{n} \log\left(\mu_N(\Delta_n(x))\right) = h(T_N) \tag{5.33}$$

by using the Shannon-McMillan-Breiman-Chung Theorem and a Theorem of Kolmogorov and Sinai (see [24, Chapter 6]). Here μ_N is the absolutely continuous invariant measure. Since the measure μ_N and the Lebesgue measure λ are equivalent we have

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\mu_N(\Delta_n(x))\right) = \lim_{n\to\infty} \frac{1}{n} \log\left(\lambda(\Delta_n(x))\right).$$

Now, similar to the cylinders of the regular continued fraction, we have

$$\lambda(\Delta_n(x)) = \left|\frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n}{q_n}\right| = \frac{N^n}{q_n(q_n + q_{n-1})}.$$

We find

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log\left(\lambda(\Delta_n(x))\right) &= \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{N^n}{q_n(q_n+q_{n-1})}\right) \\
&= \log(N) - \lim_{n\to\infty} \frac{1}{n} \log\left(q_n(q_n+q_{n-1})\right) \\
&= \log(N) - \lim_{n\to\infty} \frac{1}{n} \log\left(q_n^2\left(1+\frac{q_{n-1}}{q_n}\right)\right) \\
&= \log(N) - \lim_{n\to\infty} \frac{2}{n} \log(q_n)
\end{aligned}$$

which gives us (5.32). □

5.5.2 A Farey-like map for greedy NCF expansions

Just as in the case of the Gauss map, there is also a slow version of the map T_N. Let $F_N : [0,1] \to [0,1]$ be defined as

$$F_N(x) = \begin{cases} \frac{N}{x} - N & \text{for } x \in \left(\frac{N}{N+1}, 1\right], \\ \frac{Nx}{N-x} & \text{for } x \in \left[0, \frac{N}{N+1}\right]; \end{cases}$$

see Figure 5.2.

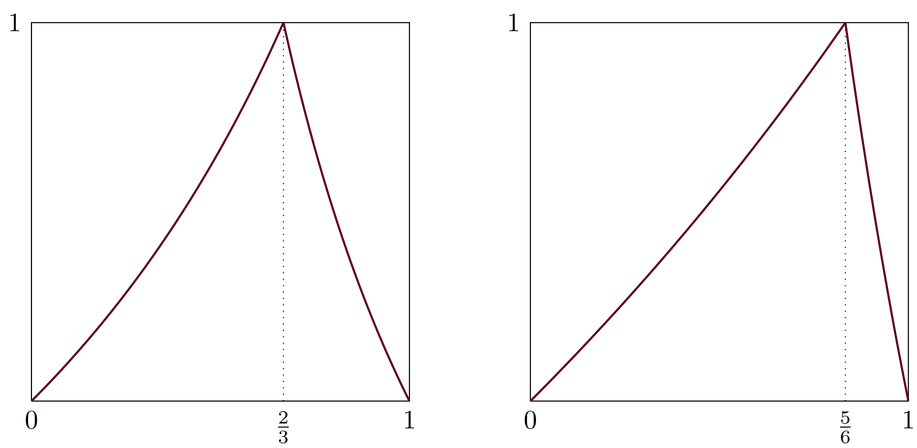


FIGURE 5.2: The map F_N for $N = 2$ on the left and $N = 5$ on the right.

Now let $x = [0; d_1, d_2, d_3, \ldots]_N$. It is clear that if $d_1 = N$ then $F_N(x) = [0; d_2, d_3, \ldots]_N$. If $d_1 > N$ we have

$$F_N(x) = \frac{Nx}{N-x} = \frac{N}{\frac{N}{x} - 1} = \cfrac{N}{d_1 - 1 + \cfrac{N}{d_2 + \ddots}}$$

so that $F_N(x) = [0; d_1 - 1, d_2, d_3 \ldots]_N$. This justifies the name Farey-like map. It is easy to check that the maps F_N are AFN-maps which are considered in [84]. Then by following the same arguments as in [41], every open interval contains a rational number and all rational numbers are eventually mapped to the indifferent fixed point, we can conclude that for every N there exists a unique absolutely continuous, infinite, σ-finite F_N-invariant measure μ_N that is ergodic and conservative for F_N. We now show the following result.

Proposition 5.5.2. *The infinite measure $\frac{dx}{x}$ is an invariant measure for the dynamical system $(F_N, [0,1], \mathcal{B})$. Here $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$.*

Proof. It suffices to show invariance for intervals $(a,b) \subset [0,1]$. Now

$$\mu\left((a,b)\right) = \int_a^b \frac{1}{x} dx = \log\left(\frac{b}{a}\right). \tag{5.34}$$

We have that $F_N^{-1}\left((a,b)\right) = \left(\frac{Na}{N+a}, \frac{Nb}{N+b}\right) \cup \left(\frac{N}{N+b}, \frac{N}{N+a}\right)$ which gives us

$$\begin{aligned}
\mu\left(F_N^{-1}(a,b)\right) &= \int_{\frac{Na}{N+a}}^{\frac{Nb}{N+b}} \frac{1}{x} dx + \int_{\frac{N}{N+b}}^{\frac{N}{N+a}} \frac{1}{x} dx \\
&= \log\left(\frac{\frac{Nb}{N+b}}{\frac{Na}{N+a}}\right) + \log\left(\frac{\frac{N}{N+a}}{\frac{N}{N+b}}\right) \\
&= \log\left(\frac{Nb(N+a)}{Na(N+b)} \frac{N(N+b)}{N(N+a)}\right) = \log\left(\frac{b}{a}\right)
\end{aligned}$$

which finishes the proof. □

To the dynamical system $(F_N, [0,1], \mathcal{B})$ we can also associate S-adic sequences in the same way as we did for the dynamical system defined by T_N. Note that if $x = [0; d_1, d_2, d_3, \ldots]_N$ then the sequence $1^{d_1 - N}, N, 1^{d_2-N}, N, 1^{d_3-N}, N, \ldots$ is the slow expansion of x corresponding to F_N. If we want a strong analogy with the NCF sequences that we studied, we should take the the Möbius transformations corresponding to the inverse branches of F_N and swap the numbers on the diagonal and anti-diagonal. This way we find

$$B = \begin{pmatrix} N & 1 \\ 0 & N \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} N & 1 \\ N & 0 \end{pmatrix}. \tag{5.35}$$

These matrices should be the incidence matrices of the substitutions. With this in mind we associate the slow NCF expansions with the following substitutions

$$\tau_B : \begin{cases} 0 \to 0^N, \\ 1 \to 01^N, \end{cases}$$

and

$$\tau_D : \begin{cases} 0 \to 0^N 1^N. \\ 1 \to 0, \end{cases}$$

For every irrational number x we can now find a directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 1}$, where $\sigma_n = \tau_B$ if the i^{th} digit in the slow expansion is 1 and $\sigma_n = \tau_D$ if it is N. It would be interesting to study the corresponding S-adic sequence.

Chapter 6

Rauzy fractals for non-unimodular S-adic substitutions

As a form of closure, we intend to show the strong connection between the different topics that have been presented in this thesis and establish a starting point for future lines of research. So far, Chapter 5 appears to be a bit detached from the rest: even though continued fractions also constitute a way to symbolically represent numbers, the geometrical and visual aspects were lacking. However, substitutions and self-affine tiles are intimately related: given a substitution that satisfies certain hypothesis (in particular, it is assumed that the incidence matrix is Pisot), one can define a self-affine set called the *central tile* or *Rauzy fractal*, introduced by Rauzy in [67]. We refer the reader to [8], [31] and [16] for results in the classical setting.

A first generalization of the classical Rauzy fractal is given by S-adic sequences, that is, instead of considering only one substitution, we take a sequence of them. This case was studied in [18] and a follow-up survey is given in [79]. They introduce Rauzy fractals in the S-adic case and study their dynamical and geometric properties. The classical case, as well as the S-adic setting from [18], is restrained to unimodular substitutions, that is, substitutions whose incidence matrix (which always has integer entries) has a determinant equal to ± 1.

If the classical definition of Rauzy fractal is applied to non-unimodular substitutions, overlaps occur and tiling results no longer hold. There is a very direct analogy with number systems: the non-unimodular case in substitutions is analogous to the rational case in number systems. A way to overcome the issue is, again, to introduce a representation space, which has an Euclidean factor and a factor defined in terms of a projective limit. Work has already been done in the field of non-unimodular Pisot substitutions in [61], where the authors introduced the so-called Dumont Thomas tiles associated with a (non-unimodular) substitution, which are very closely linked to Rauzy fractals. Related work was done for Pisot β numerations in [60].

Recall that NCF words and their duals are S-adic sequences. When $N \geq 2$, the determinant of the incidence matrix of each substitution equals N. This motivates the study of non-unimodular S-adic sequences and the corresponding Rauzy fractals.

We present an introduction, the main definitions, some lemmas, and some examples, yet the main results that one would expect of this theory (topological properties and tilings), are stated as conjectures.

6.1 The classical case

Rauzy fractals provide a bridge between geometry and substitutions. A brief and clear introduction can be found in [12]. Like self-affine tiles, Rauzy fractals are sets of non-empty interior whose boundary has, in general, non-integer Hausdorff dimension. They also satisfy a property of self-affinity, but it is slightly different than the one for number system tiles: in general, a Rauzy fractal is an almost disjoint union of smaller copies of itself, but these copies have all different scales. A thorough study of their geometric and dynamical properties is done in [16].

Consider a finite alphabet $\mathcal{A} = \{1, 2, \ldots, d\}$ and a substitution σ over this alphabet, i.e., a morphism on the free monoid $\mathcal{A}^*$ that gets extended to $\mathcal{A}^{\mathbb{N}}$ such that the image of each letter is non-empty. As before, we define its *incidence matrix* as the square matrix $M_\sigma = (|\sigma(j)|_i)_{i,j\in\mathcal{A}} \in \mathbb{N}^{d\times d}$, where $|\cdot|$ denoted the number of letters of a word. We say that σ is *unimodular* if $\det M_\sigma = \pm 1$, we say it is *irreducible* if the characteristic polynomial of M_σ is irreducible, and we say it is *Pisot* if M_σ has an eigenvalue that is a Pisot number (that is, it has a dominant real eigenvalue with modulus greater than one, and all the other eigenvalues have modulus less than one).

Assume that σ is unimodular, irreducible, and Pisot. This is the classical Rauzy fractal setting. We follow [16]. The matrix M_σ has a dominant eigenvalue $\beta > 1$ and $d-1$ (complex) eigenvalues with modulus less than one. Then the space $\mathbb{R}^d$ decomposes in the following way: there is a hyperplane $\mathbb{H}_c$ of $\mathbb{R}^d$ such that the restriction of M_σ to it is a contraction, and it is generated by the eigenvectors of the eigenvalues different from β. It is called the contracting hyperplane. There is one eigenspace associated with the eigenvalue β, and the restriction of M_σ to this one-dimensional eigenspace is expanding. We denote it by $\mathbb{H}_e$. Then there is a natural decomposition $\mathbb{R}^d = \mathbb{H}_c \bigoplus \mathbb{H}_e$. Let $\pi : \mathbb{R}^d \to \mathbb{H}_c$ be the projection onto $\mathbb{H}_c$ alongside $\mathbb{H}_e$.

Let $u = u_0u_1 \ldots \in \mathcal{A}^{\mathbb{N}}$ be an infinite word such that $\sigma(u) = u$, that is, u is a fixed point of the substitution. For the sake of simplicity, we assume that this word exists and is well-defined, and it causes no loss of generality (see [16, Remark 5.3.17]). Consider the abelianization map $\mathbf{l}$ from Definition 5.2. It turns out that the following definition is independent of the choice of u.

Definition 6.1.1. *We define the Rauzy fractal or central tile associated with the substitution σ as*

$$\mathcal{R}_\sigma := \overline{\{\pi(\mathbf{l}(u_0 \ldots u_{n-1})) \,:\, n \in \mathbb{N}\}}. \tag{6.1}$$

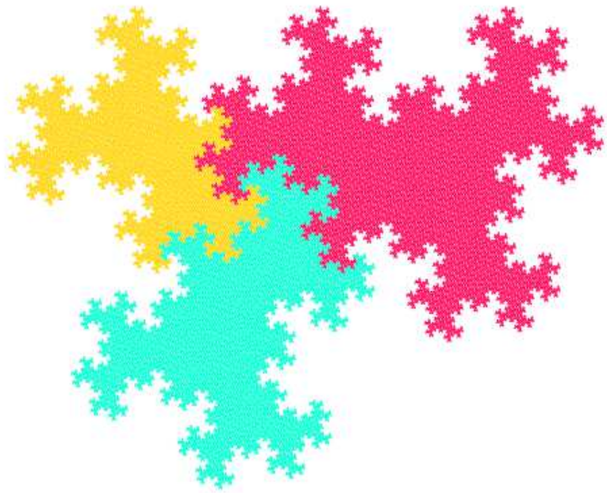


FIGURE 6.1: Rauzy fractal for the unimodular substitution $\sigma(1) = 13, \sigma(2) = 1, \sigma(3) = 2$.

The overline denotes the topological closure, with respect to the usual Euclidean metric of the hyperplane $\mathbb{H}_c$. We define the subtiles $\mathcal{R}_\sigma(i)$ of the Rauzy fractal for $i \in \mathcal{A}$ as

$$\mathcal{R}_\sigma(i) := \overline{\{\pi(\mathbf{l}(u_0 \ldots u_{n-1})) \,:\, n \in \mathbb{N},\, u_n = i\}}. \tag{6.2}$$

By definition, the central tile is the finite union of its subtiles:

$$\mathcal{R}_\sigma = \bigcup_{i \in \mathcal{A}} \mathcal{R}_\sigma(i).$$

Under certain hypotheses, that we will omit for now, this union is essentially disjoint. In particular, all of these sets are the closure of their interiors and their boundary has measure zero. It is shown in [16, Theorem 5.3.13] that the subtiles give a multiple tiling of the hyperplane $\mathbb{H}_c$.

An illustration of a Rauzy fractal is given in Figure 6.1. It corresponds to the substitution σ on the three-letter alphabet $\mathcal{A} = \{1, 2, 3\}$ given by $\sigma(1) = 13$, $\sigma(2) = 1$, $\sigma(3) = 2$. The three different colors correspond to the three different subtiles. The Rauzy fractal is defined in a hyperplane of $\mathbb{R}^3$, which is of course isomorphic to $\mathbb{R}^2$, so the figure is illustrated in the plane. The incidence matrix of this substitution is $M_\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and it satisfies $\det(M_\sigma) = 1$. We can see that all the subtiles have the same shape at different scales, which also coincides with the shape of the whole central tile. The reason behind this is that Rauzy fractals are solutions of IFS defined in terms of a graph. The essential disjointedness of the subtiles can be seen in this illustration, and a necessary condition for this is that the determinant of the matrix is a unit.

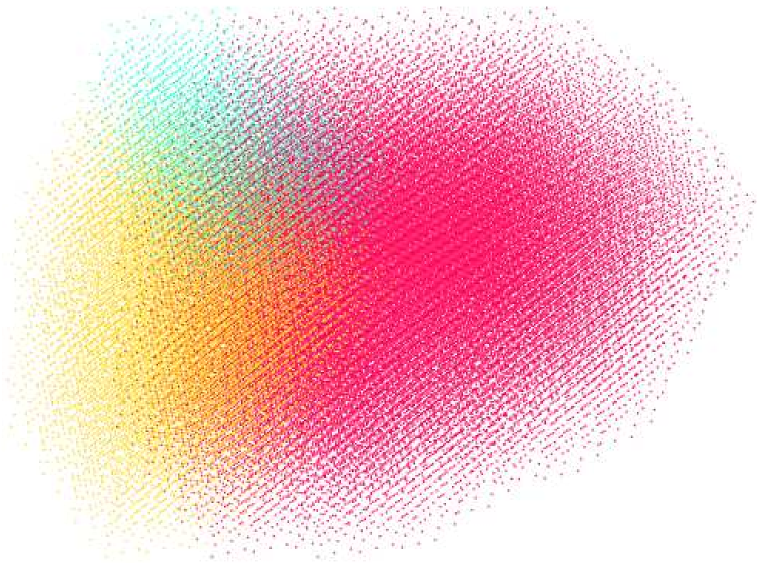


FIGURE 6.2: Set related to the non-unimodular substitution $\sigma(1) = 1112, \sigma(2) = 113, \sigma(3) = 11$.

In figure 6.2, we illustrate a set associated with a substitution σ, again on the three-letter alphabet $\mathcal{A} = \{1, 2, 3\}$, given by $\sigma(1) = 1112, \sigma(2) = 113, \sigma(3) = 11$. The incidence matrix for this example is $M_\sigma = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and it satisfies $\det(M_\sigma) = 2$. This set is defined in the same way as the classical Rauzy fractal, but we do not call it a Rauzy fractal or a central tile because it does not satisfy the condition that the substitution is unimodular. The picture resembles a lot Figure 1.5, where we showed a set associated with a number system with a non-integer base in $\mathbb{C}$. In the same way, this example shows points that are not evenly distributed, giving it this blurry appearance (even though it was computed with the same precision as Figure 6.2), and the different subtiles clearly overlap. The first part of the thesis was devoted to overcoming this issue by introducing a representation space with a p-adic component, or more generally, a projective limit. This is exactly what we wish to do in the Rauzy fractals setting.

If M_σ is not unimodular but has an eigenvalue β that is a Pisot number, some results have already been established, and we refer to [15] and [61] for β expansions (which is essentially an equivalent setting) and to [60] and [77] for substitutions.

6.2 The S-adic case

We will consider a larger family of Rauzy fractals, that are obtained not through a fixed point of a single substitution, but in terms of an S-adic sequence, as was done in [18]. A very complete survey of the dynamical, arithmetic, and geometrical properties of S-adic sequences can be found in [79].

Let $\mathcal{A} = \{1, 2, \ldots, d\}$ be a finite alphabet and let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a sequence of substitutions $\sigma_n : \mathcal{A}^* \to \mathcal{A}^*$. We denote the set of substitutions as $\mathcal{S} = \{\sigma_n : n \in \mathbb{N}\}$; this set may be finite or infinite. We call $\boldsymbol{\sigma}$ a directive sequence for $\mathcal{S}$. For simplicity, denote $M_n = M_{\sigma_n}$ for $n \in \mathbb{N}$ and we express consecutive products of substitutions and incidence matrices as

$$\sigma_{[k,l)} = \sigma_k \sigma_{k+1} \ldots \sigma_l \qquad \text{and} \qquad M_{[k,l)} = M_k M_{k+1} \ldots M_l$$

for $0 \le l \le k$.

To define Rauzy fractals for the S-adic case, we do not have a straightforward notion of fixed point, and we need some hypothesis on the directive sequence. For that reason, we introduce the following definitions.

Definition 6.2.1. *Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a sequence of substitutions. Then:*

1. *$\boldsymbol{\sigma}$ is said to be algebraically irreducible if, for each $k \in \mathbb{N}$, the characteristic polynomial of $M_{[k,l)}$ is irreducible for every sufficiently large $l > k$.*
2. *$\boldsymbol{\sigma}$ is said to be primitive if, for each $k \in \mathbb{N}$, $M_{[k,l)}$ has positive entries for some $l > k$.*
3. *$\boldsymbol{\sigma}$ is said to be recurrent if, for each $m \in \mathbb{N}$, there exists $n \ge 1$ such that $(\sigma_0, \ldots, \sigma_{m-1}) = (\sigma_n, \ldots, \sigma_{n+m-1})$.*
4. *$\boldsymbol{\sigma}$ is said to be C-balanced if there exists $C > 0$ such that, for $m \ge 1$ there exists n as in (3) and it holds that the language*

$$\mathcal{L}_{\boldsymbol{\sigma}}^{(m+n)} = \{u \in \mathcal{A}^* : u \text{ is a factor of } \sigma_{[m+n,\ell)}(i) \text{ for some } i \in \mathcal{A}, \ell \in \mathbb{N}\}$$

   is C-balanced.
5. *$\boldsymbol{\sigma}$ is said to be non-unit (equivalently non-monic or non-unimodular) if* $\det M_n \neq \pm 1$ *for some $n \in \mathbb{N}$.*

Recall that an infinite word or sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ is an *S-adic sequence*of $\boldsymbol{\sigma} = (\sigma_n)_{n\ge1}$ if there exist $\omega^{(1)}, \omega^{(2)}, \ldots \in \mathcal{A}^{\mathbb{N}}$ such that

$$\omega^{(1)} = \omega, \qquad \omega^{(n)} = \sigma_n(\omega^{(n+1)}) \qquad \text{for all } n \ge 1.$$

Let i_n denote the first letter of the word $\omega^{(n)}$ for each $n \in \mathbb{N}$. Assume

$$\lim_{n\to\infty} |\sigma_{[0,n)}(i_n)| = \infty,$$

which holds in particular when $\boldsymbol{\sigma}$ is primitive. Then we can write, with a slight abuse of notation (this is an infinite word that arises as a limit of finite words)

$$\omega = \lim_{n\to\infty} \sigma_{[0,n)}(i_n). \tag{6.3}$$

Proposition 6.2.2. *Let $\boldsymbol{\sigma}$ be a primitive and recurrent sequence of substitutions. Then there exists a generalized right eigenvector $\mathbf{u}$ as in Definition 5.1.2.*

Proof. See [79, Proposition 3.5.5]. □

Recall the notion of balance from Definition 5.3.1 and of letter frequency from Definition 5.10. Balancedness implies the existence of letter frequencies (see [79, Lemma 3.5.10]). The following Lemma was already mentioned before in the context of NCF sequences but we include the proof here in a more general context.

Lemma 6.2.3. *If $\boldsymbol{\sigma}$ is primitive and recurrent and ω is a limit word, then the letter frequency vector f_ω is a generalized right eigenvector for $\boldsymbol{\sigma}$.*

Proof. If $\boldsymbol{\sigma}$ is primitive and recurrent then there exists a generalized right eigenvector $\mathbf{u}$ and a limit word ω. By the definition of $\mathbf{u}$, the vectors $M_{[0,n)}e_i$ approach $\mathbb{R}_+\mathbf{u}$ when $n \to \infty$ (here e_i, $1 \le i \le d$, corresponds to the i-th canonical base vector). Note that $M_{[0,n)}e_i = \mathbf{l}(\sigma_{[0,n)}(i))$ for $i \in \mathcal{A}$, $n \in \mathbb{N}$. By primitivity, the prefixes of ω are the prefixes of $\sigma_{[0,n)}(i)$ for $i \in \mathcal{A}$, $n \in \mathbb{N}$, hence $\mathbf{l}(p)$ when p is a prefix of ω tends to $\mathbb{R}_+\mathbf{u}$ as $|p| \to \infty$, therefore $\frac{1}{|p|}\mathbf{l}(p)$ tends to $\mathbf{u}$ as $|p| \to \infty$ because $\|f_\omega\|_1 = 1$, i.e., $f_\omega = \mathbf{u}$. □

We define the language $\mathcal{L}_{\boldsymbol{\sigma}}$ of $\boldsymbol{\sigma}$ to be

$$\mathcal{L}_{\boldsymbol{\sigma}} := \{u \in \mathcal{A}^* \,:\, u \text{ is a factor of } \sigma_{[0,n)}(i) \text{ for some } i \in \mathcal{A},\, n \in \mathbb{N}\}.$$

Define

$$\mathcal{P}_{\boldsymbol{\sigma}} := \{p \in \mathcal{A}^* \,:\, p \text{ is a prefix of } \sigma_{[0,n)}(i) \text{ for some } i \in \mathcal{A},\, n \in \mathbb{N}\}.$$

Lemma 6.2.4. *Assume that $\boldsymbol{\sigma}$ is primitive and let $\omega \in \mathcal{A}^{\mathbb{N}}$ be an arbitrary limit word of $\boldsymbol{\sigma}$. Then*

1. $\mathcal{L}_{\boldsymbol{\sigma}} = \{u \in \mathcal{A}^* \,:\, u \textit{ is a factor of } \omega\}$.
2. $\mathcal{P}_{\boldsymbol{\sigma}} = \{p \in \mathcal{A}^* \,:\, p \textit{ is a prefix of } \omega\}$.

Proof. This is a consequence of (6.3). □

This enables the definition of the Rauzy fractal in the S-adic case, as follows. Consider the vector $\mathbf{1} = {}^t(1, \ldots, 1) \in \mathbb{R}^d$ and let $\mathbb{H} = \mathbf{1}^\perp$ be the hyperplane orthogonal to $\mathbf{1}$ containing the origin. Assume that there exists a generalized right eigenvector $\mathbf{u}$. Let $\pi : \mathbb{R}^d \to \mathbb{H}$ be the projection along the direction of $\mathbf{u}$ onto the hyperplane $\mathbb{H}$. Recall the abelianization map $\mathbf{l}$ defined in (5.2).

Definition 6.2.5. *We define the Rauzy fractal $\mathcal{R}_{\boldsymbol{\sigma}}$ on the space $\mathbb{H}$ as*

$$\mathcal{R}_{\boldsymbol{\sigma}} := \overline{\{\pi(\mathbf{l}(p)) \,:\, p \in \mathcal{P}_{\boldsymbol{\sigma}}\}}. \tag{6.4}$$

We define the subtiles $\mathcal{R}_{\boldsymbol{\sigma}}(i)$ of the Rauzy fractal for $i \in \mathcal{A}$ as

$$\mathcal{R}_{\boldsymbol{\sigma}}(i) := \overline{\{\pi(\mathbf{l}(p)) \,:\, pi \in \mathcal{P}_{\boldsymbol{\sigma}}\}}. \tag{6.5}$$

We state now some results that can be found in [18, Theorem 3.1].

Theorem 6.2.6. *Let $\boldsymbol{\sigma}$ be a sequence of unimodular substitutions. Assume that it is primitive, recurrent, algebraically irreducible, and C-balanced. Then:*

1. *Each subtile $\mathcal{R}(i)$ of the Rauzy fractal $\mathcal{R}$ is a compact set that is the closure of its interior and its boundary has Lebesgue measure zero.*
2. *The collection $\mathcal{C} = \{\boldsymbol{x} + \mathcal{R}(i) \,:\, \boldsymbol{x} \in \mathbb{Z}^d \cap \mathbb{H}, i \in \mathcal{A}\}$ forms a multiple tiling of $\mathbb{H}$.*

6.3 The non-unimodular case

We proceed to introduce a new setting for the representation of central tiles for non-unimodular substitutions, which contains also the S-adic case. The reader will see that the definitions are analogous to the representation space presented in Chapter 4.

Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a sequence of substitutions and $M_{[0,n)}$ be the product of the first n incidence matrices. We have the inclusions

$$\mathbb{Z}^d \supset M_{[0,1)}\mathbb{Z}^d \supset M_{[0,2)}\mathbb{Z}^d \supset \dots$$

We define the valuation $\nu_{\boldsymbol{\sigma}} : \mathbb{Z}^d \to \mathbb{R}_+ \cup \{\infty\}$ as

$$\nu_{\boldsymbol{\sigma}}(z) := \inf_{n\in\mathbb{N}}\{z \notin M_{[0,n+1)}\mathbb{Z}^d\}.$$

Let

$$\mathbf{d}_\nu(z,z') := 2^{-\nu_{\boldsymbol{\sigma}}(z-z')},$$

and we set $2^{-\infty} = 0$. Then $\mathbf{d}_\nu$ is a metric over $\mathbb{Z}^d$ that can be interpreted as two points being close if their difference is in $M_{[0,n)}\mathbb{Z}^d$ for a large $n \in \mathbb{N}$. The choice of the number 2 is arbitrary, since the choice of another real number greater than 1 would yield an equivalent metric.

Definition 6.3.1. *We define $\mathbb{Z}_{\boldsymbol{\sigma}}$ to be the completion of $\mathbb{Z}^d$ with respect to the metric $\mathbf{d}_{\boldsymbol{\sigma}}$. The valuation $\nu_{\boldsymbol{\sigma}}$ can be extended to every $z \in \mathbb{Z}_{\boldsymbol{\sigma}}$ by noting that $\mathbf{d}_\nu(0,z) = 2^{-\nu_{\boldsymbol{\sigma}}(z)}$. Moreover, $\mathbb{Z}_{\boldsymbol{\sigma}}$ is a locally compact abelian group and is therefore endowed with a normalized Haar measure μ.*

More generally, we can think of $\mathbb{Z}_{\boldsymbol{\sigma}}$ as a projective limit. We have, for $n \geq 1$, the canonical projections

$$\pi_n : \mathbb{Z}^d/M_{[0,n+1)}\mathbb{Z}^d \to \mathbb{Z}^d/M_{[0,n)}\mathbb{Z}^d.$$

Therefore, we have the projective system

$$\cdots \longrightarrow \mathbb{Z}^d/M_{[0,n+1)}\mathbb{Z}^d \xrightarrow{\pi_n} \mathbb{Z}^d/M_{[0,n)}\mathbb{Z}^d \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_1} \mathbb{Z}^d/M_0\mathbb{Z}^d.$$

This entitles the existence of the projective limit given by

$$\varprojlim_{n\geq 1} \mathbb{Z}^d/M_{[0,n)}\mathbb{Z}^d = \{(y_n)_{n\geq 1} \,:\, y_n \in \mathbb{Z}^d/M_{[0,n)}\mathbb{Z}^d \text{ and } \pi_n(y_{n+1}) = y_n \text{ for every } n \geq 1\}.$$

Then

$$\mathbb{Z}_{\boldsymbol{\sigma}} \simeq \varprojlim_{n\geq 1} \mathbb{Z}^d/M_{[0,n)}\mathbb{Z}^d.$$

Definition 6.3.2. *We definite our representation space to be*

$$\mathbb{K}_{\boldsymbol{\sigma}} := \mathbb{R}^d \times \mathbb{Z}_{\boldsymbol{\sigma}}$$

with component-wise addition.

We endow $\mathbb{K}_{\boldsymbol{\sigma}}$ with the metric

$$\mathbf{d}_{\boldsymbol{\sigma}}((x,y),(x',y')) := \max\{\|x-x'\|, 2^{-\nu_{\boldsymbol{\sigma}}(y-y')}\}$$

and product Haar measure $\mu_{\boldsymbol{\sigma}} = \lambda \times \mu$, where λ is the d-dimensional Lebesgue measure.

For every $n \geq 1$, the matrix $M_{[0,n)}$ acts on $\mathbb{K}_{\boldsymbol{\sigma}}$ by multiplication, by setting

$$M_{[0,n)}(x,y) = (M_{[0,n)}x, M_{[0,n)}y).$$

6.3.1 Generalized Rauzy fractals

Consider the vector $\mathbf{1} = {}^t(1,\ldots,1) \in \mathbb{R}^d$ and let $\mathbb{H} = \mathbf{1}^\perp$ be the hyperplane orthogonal to $\mathbf{1}$ containing the origin. Assume that there exists a generalized right eigenvector $\mathbf{u}$. Let $\pi : \mathbb{R}^d \to \mathbb{H}$ be the projection along the direction of $\mathbf{u}$ onto the hyperplane $\mathbb{H}$. Then we define the map

$$\widetilde{\pi} : \mathbb{Z}^d \to \mathbb{H} \times \mathbb{Z}_{\boldsymbol{\sigma}} \subset \mathbb{K}_{\boldsymbol{\sigma}}, \qquad z \mapsto (\pi(z), z). \tag{6.6}$$

Definition 6.3.3. *We define the generalized Rauzy fractal $\widetilde{\mathcal{R}}_{\boldsymbol{\sigma}}$ on the space $\mathbb{H} \times \mathbb{Z}_{\boldsymbol{\sigma}}$ as*

$$\widetilde{\mathcal{R}}_{\boldsymbol{\sigma}} := \overline{\{\widetilde{\pi}(\mathbf{l}(p)) \,:\, p \in \mathcal{P}_{\boldsymbol{\sigma}}\}}. \tag{6.7}$$

FIGURE 6.3: Generalized Rauzy fractal $\widetilde{\mathcal{R}}_{\widehat{\boldsymbol{\sigma}}}$ where $\widehat{\boldsymbol{\sigma}}$ is the dual NCF directive sequence for $N = 3$, $x = [0; 3, 4, 5, 3, 4, 3, \ldots]_3$.

We define the subtiles $\widetilde{\mathcal{R}}_{\boldsymbol{\sigma}}(i)$ of the generalized Rauzy fractal for $i \in \mathcal{A}$ as

$$\widetilde{\mathcal{R}}_{\boldsymbol{\sigma}}(i) := \overline{\{\widetilde{\pi}(\mathbf{l}(p))\ :\ pi \in \mathcal{P}_{\boldsymbol{\sigma}}\}}. \tag{6.8}$$

It is clear that

$$\widetilde{\mathcal{R}}_{\boldsymbol{\sigma}} = \bigcup_{i \in \mathcal{A}} \widetilde{\mathcal{R}}_{\boldsymbol{\sigma}}(i).$$

This new definition generalizes the classical Rauzy fractal to the non-unimodular case. We conjecture a generalization of Theorem 6.2.6.

Conjecture 6.3.4. *Let $\boldsymbol{\sigma}$ be a sequence of substitutions. Assume that it is primitive, recurrent, algebraically irreducible, and C-balanced. Then:*

1. *Each subtile $\widetilde{\mathcal{R}}_{\boldsymbol{\sigma}}(i)$ of the generalized Rauzy fractal $\widetilde{\mathcal{R}}_{\boldsymbol{\sigma}}$ is a compact set that is the closure of its interior and its boundary has $\mu_{\boldsymbol{\sigma}}$ measure zero.*

2. *There exists some lattice $\Lambda \subset \mathbb{H} \times \mathbb{Z}_{\boldsymbol{\sigma}}$ such that the collection $\mathcal{C} = \{\boldsymbol{x} + \widetilde{\mathcal{R}}_{\boldsymbol{\sigma}}(i)\ :\ \boldsymbol{x} \in \Lambda, i \in \mathcal{A}\}$ forms a multiple tiling of $\mathbb{H} \times \mathbb{Z}_{\boldsymbol{\sigma}}$.*

6.3.2 Rauzy fractals for NCF substitutions

As mentioned, NCF substitutions are S-adic and non-unimodular. We show an example of how the Rauzy fractal looks in this case. Let $N \geq 2$ be an integer, and let $x = [0; d_1, d_2, \ldots]_N \in [0, 1] \setminus \mathbb{Q}$. For each $n \geq 1$, consider the dual substitutions

$$\widehat{\sigma}_n : \begin{cases} 0 \to 0^{d_n}1, \\ 1 \to 0^N. \end{cases}$$

We have the directive sequence $\widehat{\boldsymbol{\sigma}} = (\widehat{\sigma}_n)_{n\geq 1}$. Essentially, we can think of the space $\mathbb{Z}_{\widehat{\boldsymbol{\sigma}}}$ as an N-adic space (even though it is not exactly $\mathbb{Q}_N$). Each matrix $M_{\widehat{\sigma}_n}$ has determinant N, therefore $\mathbb{Z}^2/M_{\widehat{\sigma}_n}\mathbb{Z}^2$ has N residues, which are given by $\mathcal{E} = \left\{\binom{0}{0}, \dots, \binom{N-1}{0}\right\}$. If we set $M_{[0,n)} = M_{\widehat{\sigma}_0}M_{\widehat{\sigma}_1}\cdots M_{\widehat{\sigma}_{n-1}}$, then points of $\mathbb{Z}_{\widehat{\boldsymbol{\sigma}}}$ can be expanded in the form

$$z = \sum_{n\geq \nu_{\widehat{\boldsymbol{\sigma}}}(z)} M_{[0,n+1)} \begin{pmatrix} e_n \\ 0 \end{pmatrix}, \tag{6.9}$$

with $e_n \in \{0, \dots, N-1\}$ for each $n \geq \nu_{\widehat{\boldsymbol{\sigma}}}(z)$ and $z_{\nu_{\widehat{\boldsymbol{\sigma}}}(z)} \neq 0$ for $z \neq 0$. These series converge with respect to the metric $\mathbf{d}_{\boldsymbol{\sigma}}$, and the expansion of any non-zero $z \in \mathbb{Z}_{\widehat{\boldsymbol{\sigma}}}$ in the form (4.2.1) is unique.

Consider the generalized Rauzy fractal $\widetilde{\mathcal{R}}_{\widehat{\boldsymbol{\sigma}}}$. In this case, $d = 2$ so the representation space is $\mathbb{R}^2 \times \mathbb{Z}_{\widehat{\boldsymbol{\sigma}}}$ and the Rauzy fractal lives in $\mathbb{H} \times \mathbb{Z}_{\widehat{\boldsymbol{\sigma}}}$ where $\mathbb{H} \simeq \mathbb{R}$. We consider the embedding

$$\psi : \mathbb{Z}_{\widehat{\boldsymbol{\sigma}}} \to \mathbb{R}, \qquad z \mapsto \sum_{n\geq \nu_{\widehat{\boldsymbol{\sigma}}}(z)} \frac{1}{N} e_n,$$

where e_n is as in (6.9). This allows us to represent $\widetilde{\mathcal{R}}_{\widehat{\boldsymbol{\sigma}}}$ in a plane. An example is given in Figure 6.3 for $N = 3$ and $x = [0; 3, 4, 5, 3, 4, 3, \dots]_3$. This is an illustration of the generalized Rauzy fractal $\widetilde{\mathcal{R}}_{\widehat{\boldsymbol{\sigma}}} \subset \mathbb{K}_{\widehat{\boldsymbol{\sigma}}}$ embedded in $\mathbb{R}^2$. The red corresponds to the subtile $\widetilde{\mathcal{R}}_{\widehat{\boldsymbol{\sigma}}}(0)$ and the yellow to $\widetilde{\mathcal{R}}_{\widehat{\boldsymbol{\sigma}}}(1)$. The reader may observe the resemblance with illustrations of rational self-affine tiles, in particular to the one corresponding to the negasemiternary number system of Chapter 2. This approximation seems to show that the subtiles only overlap in their boundaries. We conjecture that this is the case for all NCF directive sequences and their duals.

Chapter 7

Conclusions and open questions

7.1 Conclusions

We have made a contribution to the theory of self-affine tiles that can be summarized as follows: rational matrices behave the same as integer matrices when they are embedded in a space defined in terms of the appropriate completions. In this representation space, multiplication by this rational matrix has an integer scaling factor, allowing the definition of rational self-affine tiles. Whenever these tiles have positive measure (independently of the digit choice), they induce tilings, and this is a consequence of the self-affinity itself. Due to the non-Euclidean nature of the space, we incursion in the study of the tiles by "slicing them up", and discover interesting properties: such slices relate to each other by the property we called inter-affinity.

Our contributions to the area of combinatorics on words can be described in a few words as follows: knowing that Sturmian sequences are the non-eventually periodic one-balanced words, and knowing that they can be defined in terms of continued fraction expansions, we defined analogous words for N-continued fraction expansions and come up with balance constants depending on N that generalize the one-balance of Sturmian sequences. Moreover, knowing that Sturmian sequences are exactly those whose factor complexity function is $n \mapsto n + 1$, we compute a generalized complexity function formula for N-continued fraction sequences. We link this to dynamic results as well.

Finally, we form a bridge between seemingly distinct areas by means of generalized Rauzy fractals and show that the setting of non-unimodular substitutions (which contains N-continued fraction sequences) relates to that of rational self-affine tiles. We finish up by stating some conjectures.

Besides the aforementioned contributions to different areas of mathematics, this thesis has two other purposes that we hope the reader could appreciate: one of them was the illustration of the mathematical objects that we studied. First of all, we intended to come up with computer-generated images that were appealing, clear, and helpful to the understanding of the underlying concepts. This was taken further by creating a sculpture and a series of puzzles that hopefully display the beauty of these objects and in many cases make the observer curious about the underlying mathematics. This also relates to the other purpose of the thesis, which is that of exposition. We intended to develop our field of research by telling a story, starting from a simple case, moving to a more complex one, and landing in the full general setting. We believe that the content of our results is as valuable as the way it is presented and explained.

7.2 Open questions

To finalize, we state some open problems.

1. Consider an integer a with $|a| > 2$. Suppose that $\mathcal{D} \subset \mathbb{Z}$ is a digit set such that the set equation
$$aF = \bigcup_{d\in\mathcal{D}} (F + d)$$
has an attractor F of positive Lebesgue measure in $\mathbb{R}$. Clearly, $|\mathcal{D}| = a$. Let $\frac{a}{b}$ be a rational base with $|a| > b \geq 2$, a and b coprime integers. Following the definitions of Section 2.6, consider the representation space $\mathbb{K} = \mathbb{R} \times \mathbb{Q}_b$ associated with this rational number system and the diagonal embedding φ. Does the attractor $\mathcal{F}$ of the equation
$$\tfrac{a}{b}\mathcal{F} = \bigcup_{d\in\mathcal{D}} (\mathcal{F} + \varphi(d))$$
have positive Haar measure?

2. Let α be an expanding algebraic number of degree two. Consider a rational self-affine tile $\mathcal{F}(\alpha, \mathcal{D}) \subset \mathbb{K}_\alpha$ for some standard digit set $\mathcal{D} \subset \mathbb{Z}$. The intersective tiles $\mathcal{G}(z)$ of $\mathcal{F}(\alpha, \mathcal{D})$ form a tiling of $\mathbb{C}$ by means of Theorem 3.2.14. Can we relate this tiling to expansions in base α with digits in $\mathcal{D}$ that are unique almost everywhere? What properties do these digit expansions have?

3. Using the notion of generalized Rauzy fractal given in 6.3.3, can NCF sequences and their duals be expressed as codings of rotations on a torus, in an analogous way to Sturmian sequences? Does this induce pure discrete spectrum on the symbolic dynamical system induced by the shift?

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