On the Cardinal Product

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Danksagung

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Contents

Preamble

This work was motivated by investigations of approximate graph products. Such products arise in several entirely different contexts, for example theoretical biology and computational engineering. The latter is of considerable relevance at this University of Mining and Metallurgy.

In both contexts one of the problems is the design of fast factorization algorithms of graphs with respect to various products, which is the main topic of this dissertation.

To illuminate this connection, we include a short description of the envisioned applications in theoretical biology and computational engineering.

Theoretical biology

In theoretical biology graph products arise in two rather different contexts. The first context pertains to the **evolution of genetic sequences,** which is conveniently discussed in the framework of *sequence spaces*. Sequence spaces are Hamming graphs, that is Cartesian products of complete graphs, see Eigen [11], Dress and Rumschitzki [10]. It turns out to be of interest to understand the structure of localized subsets. Gavrilets [15], Grüner [16], and Reidys [30], for example, describe subgraphs in sequence space that correspond to the subset of viable genomes or to those sequences that give rise to the same phenotype. The structure of these subgraphs is intimately related to the dynamics of evolutionary processes [17, 29].

The second context pertains to a topological theory of the **relationships between genotypes and phenotypes** [13, 14, 37, 36, 35]. In this framework a so-called character (*Merkmal*) is identified with a factor of a generalized topological space that describes the variational properties of a phenotype. If recombination and sexual inheritance are disregarded, this framework reduces to strong products of graphs. Since characters are meaningfully defined only for subsets of phenotypes (for example, "only craniates have a noses") it is necessary to use a local definition [39]: A character corresponds to a factor in a factorizable induced subgraph with non-empty interior (where x is an interior vertex of $H \subset G$ if x and all its neighbors within G are contained in H.)

In both this and the previous application the graphs in question have to be either obtained from computer simulations (e.g. within the the RNA secondary structure model as in [13, 14, 8]) or they need to be estimated from biological data. In both cases they are known only approximately. In order to deal with such inaccuracies, a mathematical framework is needed that allows us to deal with graphs that are only approximately products and of which only subgraphs are (approximate) products.

Computational engineering

In the case of computational engineering the objects that one wishes to investigate are routinely modeled by grids. This has to be done with respect to the type of problem one wishes to solve and may result in rather complicated graphs. The structure of these graphs is then reflected in the systems of linear equations whose solutions have to be found repeatedly, fast, and accurately.

If the graphs are products or product-like one understands them sufficiently well in order to build efficient equation solvers. The reason is that data structures to store and algorithms to operate on sparse matrices are more efficient when the graph factors into a product or can be covered by a few product-like subgraphs. On a regular rectangular grid, for example, a matrix-vector multiplication will access data from memory with constant stride. On the other hand, a general sparse matrix algorithm would have to fetch the data by individual addressing in a more random way.

Frequently, when designing the computational grids, large parts of the underlying model could be covered by regular product-like grids, with some modifications or irregularities along the boundaries. To date there are no algorithms to optimize and exploit the approximate data structure, although this is certainly done manually when recognized by the programmers.

It will be highly significant to pursue this new direction and to develop good heuristics together with appropriate algorithms for the decomposition of large graphs into products or product-like subgraphs.

Chapter 1

Introduction

The subject of this dissertation are prime factorizations of directed graphs with respect to the cardinal product. This work is based on results of Sabidussi, McKenzie, Feigenbaum, Schäffer and Imrich. Sabidussi wrote several seminal papers on products of graphs, notably "The composition of graphs" [31], "Graph Multiplication" [32], "The lexicographic product of graphs" [33] and "Subdirect representations of graphs" [34]. Of special interest was the question whether the prime factor decomposition with respect to any of these products is unique. In case of the Cartesian product this problem was affirmatively answered for connected graphs by independent papers of Sabidussi [32] and Vizing [38].

Decompositions of graphs with respect to the cardinal product were first studied in the context of finite and infinite relational structures by McKenzie in 1971 [27]. For finite directed and undirected graphs McKenzie's results imply unique prime factorization under certain connectedness conditions. Since the development of complexity theory just goes back to the late 70's, it is not surprising that McKenzie does not address factorization algorithms.

For the strong product, which can be considered as a special case of the cardinal product, this problem was first settled by Feigenbaum and Schäffer. In [12] they presented a polynomial algorithm for the prime factorization of connected graphs with respect to the strong product. Their procedure consists of three parts: First the problem of factorizing a graph G is reduced to the factorization of a thin graph G/R. This follows the ideas of McKenzie [27]. Then G/R is factored. This is the main and most difficult part. It is effected by construction of the so-called Cartesian skeleton H and subsequent prime factor decomposition of H with respect to the Cartesian product. Finally the factorization of G/R is extended to the original graph G.

A variant of this algorithm was proposed by Imrich [19] for the prime factorization of undirected nonbipartite connected graphs with respect to the cardinal product.

The aim of this thesis is the generalization of Imrich's algorithm to directed graphs. Hence, in the first chapters we begin with a short description of fundamental properties of directed graphs and prove several lemmas concerning thinness and connectedness.

Then we list important results about prime factorizations with respect to products that are relevant in this thesis. Following the ideas of Feigenbaum and Schäffer we define the Cartesian skeleton and present an algorithm to compute it. It is the most important tool for the proof of Theorem 6.1.2, the first main result, which gives us a polynomial algorithm to compute the prime factor decomposition (PFD) with respect to the cardinal product for finite, N^+-connected and R^+-thin graphs.

As in the case of the cardinal product of undirected graphs, the proof of the correctness of the algorithm also shows that the prime factorization is unique. This is important, because the class of N^+-connected R^+-thin graphs is not identical with the class of N^+- and N^--connected thin graphs, for which McKenzie showed unique prime factorization. (McKenzie's connectivity condition is stronger, but his thinness condition weaker than ours.)

Thus, Theorem 6.1.2 extends the class of directed graphs that are known to have unique prime factorizations with respect to the cardinal product. To our knowledge this is the only such extension since 1971. Furthermore Theorem 6.1.4 describes the structure of automorphisms of finite, directed graphs that are N^+-connected and R^+-thin.

Chapter 7 is devoted to generalizations of Theorem 6.1.2 to graphs that are finite, N^+-connected, but not R^+-thin. To do this a new class of graphs, so-called $R^+_{s,r}$-graphs, is introduced. In the second section we characterize prime graphs and divisors of graphs in this class. Furthermore problems and examples concerning these graphs are considered in the third section.

In the next chapter we prove Theorem 8.2.4, the second main result. It tells us that the PFD with respect to the cardinal product of graphs, for which McKenzie showed uniqueness of the PFD and which fulfill a weak additional assumption, can be found in polynomial time.

Chapter 9 is concerned with the distinguishing number of products of graphs. The distinguishing number $D(G)$ of a graph G is defined as the least integer d such that G has a d-distinguishing labelling that has the property that the identity is the only label preserving automorphism. It was introduced 1996 by Albertson and Collins [2].

This number varies between 1 and $|V(G)|$. (It is 1 if G is asymmetric and $|V(G)|$ if G is complete.) One can, loosely speaking, consider d as the minimum number of colors needed to brake the symmetries of G. Thus, it should be easy to compute for graphs whose automorphism groups have a well understood structure. Clearly this is the case for products, as Theorems 6.1.3 and 6.1.4 demonstrate. Therefore it was a very natural task to apply the results on products of prime graphs and their automorphism groups to the computation of distinguishing numbers.

Our results are summarized in Corollary 9.2.2 and Theorem 9.2.4, in which we determine the distinguishing numbers of arbitrary finite or countable Cartesian products of K_2's and K_3's with at least four factors. Note that $Aut(\Pi^{\square}_{i\in I}K_3) = Aut(\Pi^{\times}_{i\in I}K_3)$, whence $D(\Pi^{\times}_{i\in I}K_3) = D(\Pi^{\square}_{i\in I}K_3)$.

In the last sections we consider variations of the distinguishing number: We prove that the distinguishing chromatic number of the 4-cube is four, which extends a result by Choi, Hartke and Kaul [6], and that the 1-local distinguishing number of the 4-cube is three.

In the final chapter we develop a local algorithm to compute prime factorizations with respect to the strong product. The purpose of this approach is to speed up the strong product PFD algorithm by Feigenbaum and Schäffer [12]. Furthermore we note that this algorithm can also be adapted for cardinal product decompositions if all subgraphs induced by closed neighborhoods are cardinal products of subgraphs of all factors.

Chapter 2

Definitions and Notations

2.1 Directed Graphs

A *directed graph*, or shortly *digraph*, $G = (V, E)$ is a set V together with a set E of ordered pairs $[x, y]$ of vertices of G. We allow that both $[x, y]$ and $[y, x]$ are in E and do not require x, y to be distinct. Thus, E is a subset of the Cartesian product $V \times V$.

V is the *vertex set* and E the *edge set* of G. The vertex x is the *origin* and y the *terminus* of $[x, y]$. In the case when $x = y$ we speak of a *loop*. In analogy to the undirected case we call a graph G with $E(G) = V(G) \times V(G)$ *complete*. If it has n vertices it will be denoted by K_n^d to distinguish it from the ordinary complete graph K_n (where any two distinct vertices are connected by an undirected edge.)

A graph is called *totally disconnected* if it has no edges (and thus also no loops). Clearly a cardinal product is totally disconnected if and only if at least one factor is totally disconnected. We call a graph *connected* if for all vertices x, y of G there is a finite sequence of vertices $(x_i)_{0 \leq i \leq n}$ so that $x_0 = x$, $x_n = y$ and $[x_i, x_{i+1}] \in E(G)$ for $(0 \leq i < n)$. A graph G is *bipartite* if there exists a partition $V_1 \cup V_2 = V(G)$ so that all edges of G can be written as $[x, y]$ or $[y, x]$, where $x \in V_1$ and $y \in V_2$.

We say $E(G)$ is *reflexive* if $E(G)$ contains all loops $[x, x]$, where $x \in V(G)$. It is *symmetric* if $[x, y] \in E(G)$ if and only if $[y, x] \in E(G)$. By abuse of language one also says that G is reflexive, respectively symmetric. Symmetric directed graphs correspond to undirected graphs by identification of pairs of edges with opposite directions.

The *out-neighborhood* $N^+(x)$ of a vertex x, compare Figure 2.1, is defined

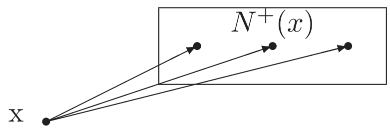

Figure 2.1: $N^+(x)$

as the set

$$\{y \in V \mid [x, y] \in E\}.$$

Analogously one defines the *in-neighborhood* $N^-(x) = \{y \in V \mid [y, x] \in E\}$. Sometimes we call $N^+(x)$ the N^+-neighborhood of x and $N^-(x)$ the N^--neighborhood. The cardinality of the out-neighborhood or in-neighborhood of a vertex x is called *out-degree* or *in-degree* of this vertex, respectively. The out-degree of x is denoted by $d^+(x)$, the in-degree by $d^-(x)$. Clearly a directed graph is uniquely defined by its vertex set and the out-neighborhoods of the vertices.

For symmetric graphs, i.e. undirected graphs, we have $N^+(x) = N^-(x)$ for all vertices x. Hence, we shortly speak of the *neighborhood* $N(x)$ of some vertex x. The cardinality of $N(x)$ defines the *degree* of the vertex x. The set $N[x]$ defined as $N(x) \cup \{x\}$ is called *closed neighborhood* of x. If all vertices have the same degree $n \in \mathbb{N}$, the graph is *n-regular*. Every n-regular graph is *regular*.

P_n $(n \in \mathbb{N})$ denotes the path on length n that is defined by $V(P_n) = \{0, 1, 2, ..., n\}$ and $E(P_n) = \{[a, b] \mid a, b \in V(P_n),\ |a - b| = 1\}$. C_n $(n > 3)$ is the circle of size n. It is defined by $V(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $E(C_n) = \{[a, b] \mid a, b \in V(P_n),\ a - b \equiv \pm 1\}$.

G is N^+-*connected* if for all $x, y \in V(G)$ an $n \in \mathbb{N}$ and a sequence $(x_i)_{0 \leq i \leq n}$ can be found such that $x_0 = x$, $x_n = y$ and

$$N^+(x_i) \cap N^+(x_{i+1}) \neq \emptyset \quad \text{for} \quad 0 \leq i < n. \tag{2.1}$$

If one replaces the out-neighborhoods in 2.1 by in-neighborhoods one gets the definition of N^--*connectedness*.

We continue with the definition of three binary relations: Two vertices a, b of G are in the relation R ($\approx$ in McKenzie's terminology) if both their out-neighborhoods and their in-neighborhoods are the same. We write aRb. A graph G is called *thin* if no two different vertices of G are in the relation R.

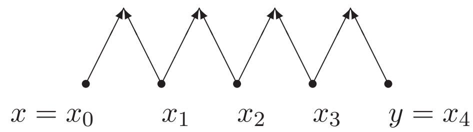

Figure 2.2: An x-y sequence in an N^+-connected graph

R is an equivalence relation on the set of vertices of G, which means that it is symmetric, reflexive and transitive. As usual the equivalence class $\overline{a}$ is defined as $\{b \in V(G) \mid aRb\}$, thus we can define the *quotient graph* G/R as follows: the vertex set of G/R is the set of all equivalence classes $\{\overline{x} \,|\, x \in V(G)\}$ of $V(G)$ with respect to R, and $[\overline{x}, \overline{y}] \in E(G/R)$ if there are vertices $a \in \overline{x}$, $b \in \overline{y}$ with $[a, b] \in A(G)$.

Two vertices of G are in the relation R^+ if their N^+-neighborhoods are the same. Clearly R^+ is an equivalence relation, too, R^- is defined analogously. A graph is then called R^+-*thin*, respectively R^--*thin*, if all equivalence classes of the relation R^+, respectively R^-, consist of just one element. The quotient graphs G/R^+ and G/R^- are defined in analogy to G/R.

Clearly a graph is thin if it is R^+- or R^--thin. However, a graph can be thin even if it is neither R^+- nor R^--thin, as the graph G in Figure 2.3 shows. $N^+(2) = N^+(5)$ and $N^-(3) = N^-(4)$. Thus G is neither R^+- nor R^--thin, but it is thin, because no two vertices have both equal out-neighborhoods and equal in-neighborhoods.

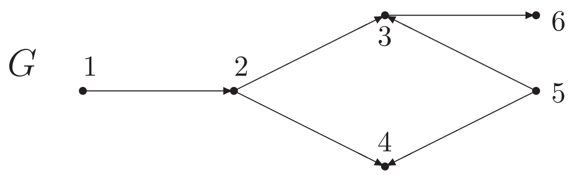

Figure 2.3: Thin, but neither R^+-thin nor R^--thin.

2.2 The Cartesian, the cardinal and the strong product

The *cardinal product* $G_1 \times G_2$ of two directed graphs G_1, G_2 is defined on the Cartesian product $V(G_1) \times V(G_2)$ of the vertex sets of the factors. The out-neighborhood of a vertex $(x_1, x_2) \in V(G_1) \times V(G_2)$ is the Cartesian product of the out-neighborhoods of x_1 in G_1 and x_2 in G_2:

$$N^+_{G_1 \times G_2}((x_1, x_2)) = N^+_{G_1}(x_1) \times N^+_{G_2}(x_2).$$

More general, we can define the cardinal product (consistent with the above definition) for arbitrarily many, also infinitely many factors:

Let I be a possibly infinite index set and $\{G_\iota\}_{\iota \in I}$ a set of digraphs. Then the *cardinal product* $G = \prod_{\iota \in I} G_\iota$ is defined as follows:

(i) $V(G)$ is the Cartesian product of the vertex sets of the factors. In other words, $V(G)$ is the set of functions $x : \iota \mapsto x_\iota \in V(G_\iota)$ of I into $\bigcup_{\iota \in I} V(G_\iota)$.

(ii) $E(G)$ consists of all unordered pairs $[x, y]$ of vertices of G such that $[x_\iota, y_\iota] \in E(G_\iota)$ for all $\iota \in I$.

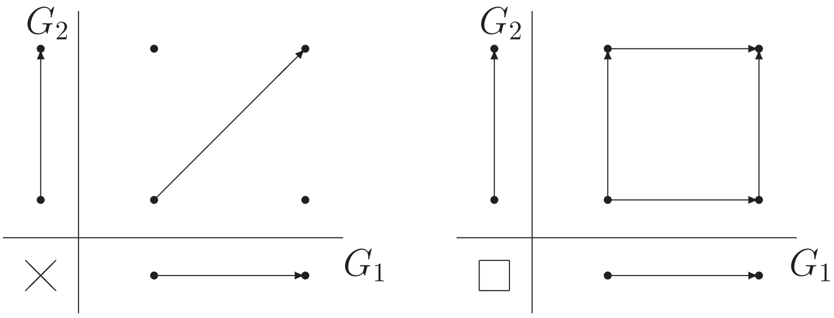

Figure 2.4: The cardinal and the Cartesian product

The *Cartesian product* $C = \prod^{\Box}_{\iota \in I} G_\iota$ and the *strong product* $H = \prod^{\boxtimes}_{\iota \in I} G_\iota$ are defined on the same vertex set as the cardinal product. The edge set $E(C)$ of the Cartesian product consists of all unordered pairs $[x, y]$ of vertices of G which have the property that there exists one $\iota \in I$ so that $[x_\iota, y_\iota] \in E(G_\iota)$ and $x_\mu = y_\mu$ for all $\mu \in I \setminus \{\iota\}$. For two factors G_1 and G_2 we denote the Cartesian product by $G_1 \Box G_2$. The edge set $E(H)$ of the strong product is the set of all Cartesian and all direct product edges $E(C) \cup E(\prod_{\iota \in I} G_\iota)$. An example can be viewed in Figure 2.5.

The three products are commutative and associative. The loop on one vertex is a unit for the cardinal product and the graph consisting of one vertex and no edge is a unit for the Cartesian and the strong product.

If $x \in V(\prod_{\iota \in I} G_\iota)$ we call the x_ι the *coordinates* of x or projections of x onto the factor G_ι. Note that the endpoints of every edge in a cardinal product of k graphs without loops differ in all k coordinates.

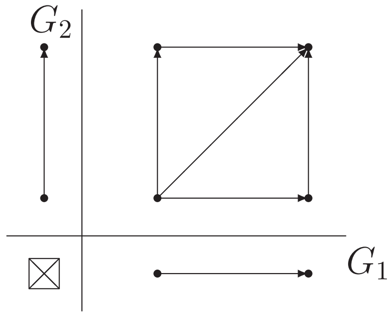

Figure 2.5: The strong product

Let be $G = \prod_{\iota \in I} G_\iota$ and $a \in V(G)$. Then the G_i-*layer* G_i^a is the subgraph of G induced by the vertex set $\{x \mid x_j = a_j \text{ for all } j \neq i\}$. If G has no loop these layers are totally disconnected.

For the Cartesian product layers are defined analogously and for this product it is easy to see that every G_i-layer is isomorphic to G_i and that every edge of G is in some G_i-layer ($i \in I$). Thus, layers of Cartesian products are also called copies (of the respective factors).

Chapter 3

Basic lemmas

Here we investigate under which conditions products inherit thinness and connectedness properties from their factors.

3.1 Thinness

The first lemma is due to McKenzie [27](Lemma 2.3).

Lemma 3.1.1 *Let G be a directed graph. Then:*

(i) *G/R is thin.*
(ii) *If $G = G_1 \times G_2$ is N^+- and N^--connected, then $G/R = G_1/R \times G_2/R$.*

Since we wish to study R^+-thin graphs, we need an analogous lemma for the relation R^+.

Lemma 3.1.2 *Let G be the cardinal product of two nontrivial directed graphs G_1 and G_2. If all out-neighborhoods of the vertices of G are nonempty, then*

$$G/R^+ = G_1/R^+ \times G_2/R^+.$$

Proof. Two vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in the relation R^+ if and only if $N^+(x) = N^+(y)$. This is equivalent to $N^+(x_1) \times N^+(x_2) = N^+(y_1) \times N^+(y_2)$. Since $N^+(x)$ and $N^+(y)$ are both nonempty this is possible if and only if $N^+(x_1) = N^+(y_1)$ and $N^+(x_2) = N^+(y_2)$, that is, if $x_1 R^+ y_1$ and $x_2 R^+ y_2$. □

Remark: N^+-connectivity implies that the N^+-neighborhoods are nonempty. Even in this case G/R^+ need not be R^+-thin as Figure 7.2 shows.

Corollary 3.1.3 *Let G be the cardinal product of two nontrivial directed graphs G_1 and G_2. If all N^+-neighborhoods of the vertices of G are non-empty, then the following statements are equivalent:*

(i) *G is R^+-thin.*
(ii) *G_1 and G_2 are R^+-thin.*

Clearly Lemma 3.1.2 and Corollary 3.1.3 remain valid if N^+ is replaced by N^-, and R^+ by R^-.

3.2 Connectedness

Lemma 3.2.1 *Let $G = G_1 \Box G_2 \Box \cdots \Box G_k$ be the Cartesian product of undirected graphs. Then the following conditions are equivalent:*

(i) *G is connected.*
(ii) *All factors G_i $(i \in \{1, 2, ..., k\})$ are connected.*

Lemma 3.2.2 *Let $G = G_1 \times G_2 \times \cdots \times G_k$ be the direct product of undirected graphs. Then the following conditions are equivalent:*

(i) *G is connected.*
(ii) *G_i $(i \in \{1, 2, ..., k\})$ is connected and at most one factor G_i $(i \in \{1, 2, ..., k\})$ is bipartite.*

Proof. Using induction this lemma follows immediately from the Theorem of Weichsel [40], whose content is the statement of this lemma for $k = 2$. $\Box$

Lemma 3.2.3 *Let $G = G_1 \times G_2 \times \cdots \times G_k$ be the cardinal product of directed graphs. Then the following conditions are equivalent:*

(i) *G is N^+-connected.*
(ii) *G_i $(i \in \{1, 2, ..., k\})$ is N^+-connected.*

Remark: The statement holds also if one substitutes N^+ by N^- in (i) and (ii).

Proof. By induction it is sufficient to prove the lemma for $k = 2$.

(i) $\Longrightarrow$ (ii): Given two vertices v_1, $w_1 \in G_1$. Let's take an arbitrary $v_2 \in G_2$, then there is by condition i) an $n \in \mathbb{N}$ and a sequence $(x_i)_{0 \leq i \leq n}$ with $x_0 = (v_1, v_2)$, $x_n = (w_1, v_2)$ and for all i $(0 \leq i < n)$ $N^+(x_i) \cap N^+(x_{i+1}) \neq \emptyset$. If $x_i = (x_{i,1}, x_{i,2})$ for $0 \leq i \leq n$, the sequence $(x_{i,1})_{0 \leq i \leq n}$ will have the

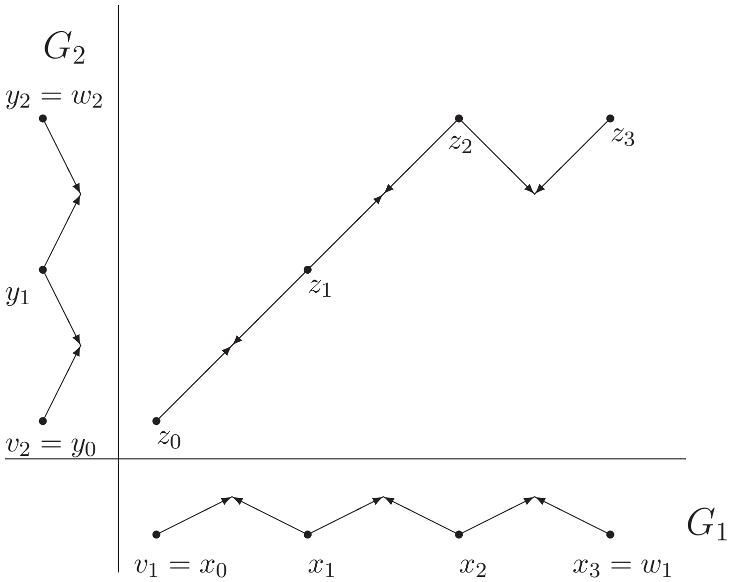

Figure 3.1: Connectedness

property $N_1^+(x_{i,1}) \cap N_1^+(x_{i+1,1}) \neq \emptyset$ for $0 \leq i < n$. Therefore G_1 is N^+-connected and by analogous projection of a sequence onto G_2 one can prove N^+-connectedness of G_2.

(ii) $\Longrightarrow$ (i): Given two vertices $v = (v_1, v_2)$, $w = (w_1, w_2) \in G$. Then there are $n, m \in \mathbb{N}$ and sequences $(x_i \in G_1)_{0 \leq i \leq n}$ and $(y_i \in G_2)_{0 \leq i \leq m}$ with $N_1^+(x_i) \cap N_1^+(x_{i+1}) \neq \emptyset$ for $0 \leq i < n$, $x_0 = v_1$, $x_n = w_1$, $N_2^+(y_i) \cap N_2^+(y_{i+1}) \neq \emptyset$ for $0 \leq i < m$, $y_0 = v_2$ and $y_n = w_2$. W.l.o.g. $n \leq m$. Let z_i denote (x_i, y_i) for $0 \leq i \leq n$ and (x_n, y_i) for $n < i \leq m$. Then $(x_i)_{0 \leq i \leq m}$ is a sequence from v to w in G which fulfills: $N^+(x_i) \cap N^+(x_{i+1}) \neq \emptyset$ for $0 \leq i < n$ as you can see in Figure 3.1. This means G is N^+-connected, too. $\square$

Chapter 4

Prime factorizations

Prime graphs will be defined, examples of graphs with non-unique prime factor decomposition will be given and results that guarantee uniqueness of the prime factorization under certain conditions will be listed.

4.1 The Cartesian product

Definition 4.1.1 *A graph G is* prime *with respect to the Cartesian, respectively the cardinal product, if it cannot be written as a Cartesian, respectively a cardinal product, of two nontrivial graphs, i.e. of two graphs with at least two vertices each.*

Any finite graph can clearly be represented as a product of prime graphs. If any two representations of a graph G as a product of prime graphs are the same up to isomorphisms and the order of the factors, we say that G has a unique prime factor decomposition (UPFD). For any of the two products considered there are graphs without UPFD.

Turning to the Cartesian product, denote the disjoint union of graphs by $+$ and, for the time being, the n-th power of a graph with respect to the Cartesian product by G^n. Then it is not hard to see that the identity

$$(K_1 + K_2 + K_2^2) \,\square\, (K_1 + K_2^3) = (K_1 + K_2^2 + K_2^4) \,\square\, (K_1 + K_2)$$

holds and that both sides of the identity are products of prime graphs, no two of which are isomorphic. Even though you can surely imagine that there are many graphs with a non-unique PFD, we know an old result about uniqueness:

Theorem 4.1.2 (Sabidussi [32] and Vizing [38]) *Let G be an undirected, connected and finite graph without loops. Then G has a unique representation as a Cartesian product of prime graphs, up to isomorphisms and the order of the factors.*

4.2 The cardinal product

When we are looking for a counterexample to the UPFD with respect to the cardinal product, we can take the C_6, the undirected cycle on six vertices. It is the product of the K_3 and the K_2, but also the product of the K_2 and P_2^*, where P_n^* ($n \in \mathbb{N}$) denotes the path of length n with two loops added to the end vertices.

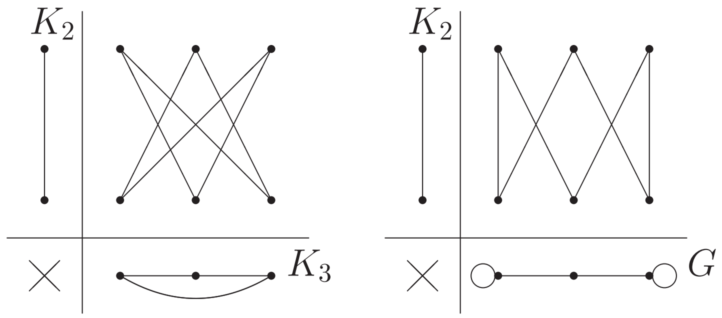

Figure 4.1: Two different C_6-decompositions

Using neighborhood considerations we can prove Lemma 4.2.1, which yields many examples of prime graphs.

Lemma 4.2.1 *Every undirected, connected graph G with an odd number of vertices and maximal degree less than or equal to 3 is prime.*

Proof. Let $|V(G)| = n$. If n is prime, we are done. Otherwise the prime factor decomposition of n consists only of odd numbers. Thus every nontrivial divisor of G has at least three vertices. We can conclude from connectedness that at least one vertex of this divisor has to have a degree greater than or equal to 2. Every vertex of G has a degree less than or equal to 3, hence G cannot have a nontrivial decomposition. □

Generally we can determine, distinguishing the size, all possible PFDs of circles by the next theorem.

Theorem 4.2.2 *Let $G = C_n$.*

(i) If n is odd, G is prime.
(ii) If $4|n$, G has a unique PFD with respect to the cardinal product, namely $P_1 \times P^*_{(n/2)-1}$.
(iii) If n is even and 4 no divisor of n, G has exactly two different PFDs, namely $P_1 \times P^*_{(n/2)-1}$ and $P_1 \times C_{n/2}$.

Proof. (i) All vertices of G clearly have degree 2, thus this statement is an immediate consequence of Lemma 4.2.1.

(ii) From the fact that every vertex of G has degree 2 and that neighborhoods in G are products of neighborhoods of factors of G we know that every G-decomposition consists of only one 2-regular factor and that every other factor must be 1-regular. But P_1 is the only nontrivial, 1-regular, connected graph. The only 2-regular, connected graphs are circles and P^*_k's. By multiplying one can see that the decomposition given in the statement is the only nontrivial one of G.

(iii) Analogous to (ii). □

In the same way we can determine the PFDs of all paths.

Theorem 4.2.3 *All P_n ($n \in N$) have a unique PFD with respect to the direct product. More precisely we have:*

(i) If n is even or $n = 1$, then P_n is prime.
(ii) If n is odd and greater than one, then the unique PFD of P_n is $K_2 \times P^l_{(n-1)/2}$, where P^l_m ($m \in \mathbb{N}$) denotes the path of length n with a loop added to one end vertex.

Proof. (i) This follows immediately from Lemma 4.2.1.

(ii) P_n contains $(n-1)$ vertices of degree 2 and two vertices of degree 1. Suppose $A \times B$ is a nontrivial decomposition of P_n. Then one factor, say A, has exactly one vertex of degree 1 the other, B, two such vertices. Only one factor can contain vertices of degree 2. This factor must be A, because otherwise A is trivial.

Thus we know B consists of two vertices of degree 1. From connectedness we conclude $B = K_2$.

Since all vertices of A have a degree less than 3, a spanning tree of A must be a path. Adding a loop to an end vertex is the only way of adding an edge that increases only the degree of one end vertex. □

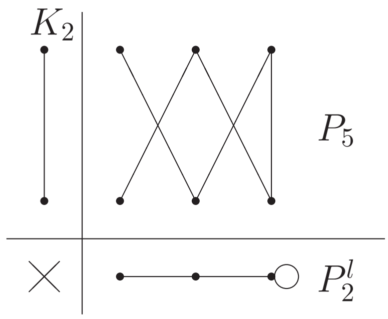

Figure 4.2: The P_5-decomposition

In the following lemma we consider hypercubes that are Cartesian powers of K_2.

Lemma 4.2.4 *Every nontrivial decomposition $A \times B$ of a hypercube Q contains a K_2.*

Proof. For all $x = (x_A, x_B) \in Q$ we know that $N(x) = N_A(x_A) \times N_B(x_B)$. It is not hard to see that for arbitrary $u,\ v \in N(x)$ there is a unique $y \in Q$ with $N(x) \cap N(y) = \{u, v\}$. Since neighborhoods but also intersections of neighborhoods are products of vertex sets in the factors, $p_A(u) = p_A(v)$ or $p_B(u) = p_B(v)$. Assume the first equation holds. Then for all $w \in N(x)$ $p_A(u) = p_A(w)$, because otherwise neither $p_A(v) = p_A(w)$ nor $p_B(v) = p_B(w)$ could hold. Thus, $|p_A(N(x))| = 1$.

We have shown that for every vertex in Q there is a projection (p_A or p_B) such that all vertices of the neighborhood are projected to one vertex. This is only possible if A or B equals K_2. □

After this short visit at undirected graphs we return to oriented ones, since we want to investigate oriented cycles $\overrightarrow{C_n}$, defined by $V(\overrightarrow{C_n}) = \{0, 1, ..., n-1\}$ and $E(\overrightarrow{C_n}) = \{ab \mid a, b \in V(\overrightarrow{C_n}) \ \wedge \ (a-b) \in \{-1, n-1\}\}$ for $n \geq 2$. Note that for each $x \in V(\overrightarrow{C_n})$ $d^+(x) = d^-(x) = 1$. From this we conclude that the same equations hold for every vertex of a divisor of $\overrightarrow{C_n}$, too. But this implies that all nontrivial divisors are oriented cycles.

In the next lemma we will show, how cardinal multiplication of oriented cycles works in general. Using it we can simply prove that $\overrightarrow{C_n}$ is prime if and only if n is a prime power and that there is a unique PFD of oriented cycles (Theorem 4.2.7). Before we start with the lemma we explain that for $n \in \mathbb{N}$, G a graph, $n * G$ is the graph defined as the disjoint union of n graphs, each one isomorphic to G.

Lemma 4.2.5 *Given $a, b \in \mathbb{N}$. Then $G = \overrightarrow{C_a} \times \overrightarrow{C_b} = gcd(a,b) * \overrightarrow{C}_{lcm(a,b)}$.*

Proof. The vertex set V of the given product is $\{(x,y) \mid 0 \leq x < a,\ 0 \leq y < b\}$ and we note that every $u \in V$ has in- and out-degree 1. This implies that the components of $\overrightarrow{C_a} \times \overrightarrow{C_b}$ are oriented cycles. Consider the vertex $(0,0)$, its only out-neighbor is $(1,1)$, but this vertex has also exactly one out-neighbor. Going from a vertex to its unique out-neighbor will be called step in the following.

Starting at $(0,0)$ it is clear that we reach the vertex $(n\ mod(a), n\ mod(b))$ after n steps. Now the question arises after which minimal natural number n of such steps do we come back to $(0,0)$, with other words, when does $(n\ mod(a), n\ mod(b)) = (0,0)$ hold? Since the considered product has $a * b$ vertices it is obvious that $n \leq a * b$. The stated question has a rather easy answer:

$$(n\ mod(a), n\ mod(b)) = (0,0) \iff a|n,\ b|n$$

$$\iff lcm(a,b)|n$$

So the minimal possible n is $lcm(a,b)$. Thus we know G contains $\overrightarrow{C}^0_{lcm(a,b)}$, which denotes the cycle of length $lcm(a,b)$ containing $(0,0)$, as one component. If $lcm(a,b) = a * b$ we are done, because in this case all vertices of G are in this cycle and $gcd(a,b) = 1$.

If $lcm(a,b) < a*b$, the vertex sets $V_i = \{(x+i, x) \mid 0 \leq x < lcm(a,b)\}$ $(1 \leq i < gcd(a,b))$induce cycles $\overrightarrow{C}^i_{lcm(a,b)}$ in G, which are exactly the remaining components of G. □

Corollary 4.2.6 *A graph $\overrightarrow{C}_n$ is prime with respect to the cardinal product if and only if n is a prime power.*

Proof. If n is a prime power, then every nontrivial decomposition $n = p * q$ has the property $gcd(p,q) > 1$, thus Lemma 4.2.5 implies $\overrightarrow{C}_p \times \overrightarrow{C}_q \neq \overrightarrow{C}_n$.

If n (> 1) is not a prime power, there exists a decomposition $n = p * q$ with $gcd(p, q) = 1$, hence Lemma 4.2.5 implies $\overrightarrow{C}_p \times \overrightarrow{C}_q = \overrightarrow{C}_n$. □

This corollary immediately implies the following theorem that represents a class of directed graphs, which have a unique PFD, although they are neither N^+- nor N^-- connected.

Theorem 4.2.7 *The PFD with respect to the cardinal product is unique for oriented cycles* $\overrightarrow{C}_n$.

Proof. If $n = p_1^{r_1} * p_2^{r_2} * ... * p_l^{r_l}$, where the prime numbers p_i are pairwise different, then Lemma 4.2.5 gives us the unique PFD $\overrightarrow{C}_n = \overrightarrow{C}_{p_1^{r_1}} \times \overrightarrow{C}_{p_2^{r_2}} \times ... \times \overrightarrow{C}_{p_l^{r_l}}$. □

The next theorem, due to McKenzie, proves uniqueness of the PFD for graphs fulfilling more general conditions. Interestingly oriented cycles do not fulfill the conditions of this theorem.

Theorem 4.2.8 (McKenzie [27]) *Let G be an N^+- and N^--connected finite graph. Then G has a unique representation as a cardinal product of prime graphs, up to isomorphisms and the order of the factors.*

Feigenbaum and Schäffer [12] showed that this factorization of a graph G can be found in polynomial time if $E(G)$ is reflexive and symmetric. Imrich [19] extended this result to graphs that are not reflexive. Of course the connectivity conditions still have to be met. We formulate this as a theorem.

Theorem 4.2.9 (Feigenbaum and Schäffer [12], Imrich [19]) *Let $G = (V, E)$ be an N^+- and N^--connected finite graph, where E is symmetric, that is, where $[x, y] \in E$ if and only if $[y, x] \in E$. Then the prime factor decomposition of G with respect to the cardinal product can be found in polynomial time.*

In the two following chapters a proof is presented that enlarges the class of graphs which have a unique prime factorization. It will be shown that all R^+-thin, N^+-connected finite graphs have a unique PFD and that it can be found in polynomial time for those graphs.

Chapter 5

The Cartesian skeleton

5.1 The idea

The idea of the proof is to reduce the problem (a) of finding the PFD of G with respect to the cardinal product to the problem (b) of finding the PFD of an undirected, connected finite graph H with respect to the Cartesian product. Problem (b) that is solved first by Sabidussi and Vizing, see Theorem 4.1.2. An in the number of edges linear algorithm to compute the PFD is due to Imrich and Peterin [23].

Definition 5.1.1 *We call an undirected graph H, defined on the set of vertices of G,* Cartesian skeleton *of G, if every decomposition $G_1 \times G_2$ of G with respect to the direct product induces a decomposition $H_1 \Box H_2$ of H such that $V(H_i) = V(G_i)$ $(i \in \{1, 2\})$.*

To find such a graph H we need two additional definitions:

Definition 5.1.2 *Let G be the cardinal product of two graphs G_1 and G_2. A pair $\{(x_1, x_2), (y_1, y_2)\}$ of distinct vertices in a product $G_1 \times G_2$ is* Cartesian *with respect to the decomposition $G_1 \times G_2$ if either $x_1 = y_1$ or $x_2 = y_2$. If G is a product of several factors $G_1 \times G_2 \times \cdots \times G_k$, then a pair of distinct vertices $\{(x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k)\}$ is* Cartesian *if there is an index j so that $x_i = y_i$ for $i \neq j$.*

The concept of Cartesian pairs is due to Feigenbaum and Schäffer [12] and was motivated by the fact that the edge set of strong products $G = G_1 \boxtimes G_2$ contains the edge set of the Cartesian product $G_1 \Box G_2$. (The strong product of G_1 and G_2 has the same vertex set as the Cartesian product, but its edge set is $E(G_1 \Box G_2) \cup E(G_1 \times G_2))$. The problem of factoring a graph with respect to the strong product can then be reduced to that of factoring a graph with respect to the Cartesian product if one can remove the non-Cartesian edges.

We try to proceed analogously. The difference will be that the Cartesian edges are in general not in $E(G_1 \times G_2)$. (They can be in the product if the factors contain loops). If H is a Cartesian product $H_1 \Box H_2$, all so-called copies $[(u_1, x_2)(u_1, y_2)]$ of edges $[(x_1, x_2)(x_1, y_2)] \in E(G)$ are in $E(G)$, too. This motivates the following definition, which describes a basic property of the Cartesian skeleton:

Definition 5.1.3 *We call a set F of pairs of distinct vertices of G* copy consistent *with respect to the decomposition $G_1 \times G_2$ of G if F consists of Cartesian pairs, and if for every pair $\{(x_1, x_2), (y_1, y_2)\}$ in F with $x_1 = y_1$ all pairs $\{(u_1, x_2), (u_1, y_2)\}$ for $u_1 \in V(G_1)$ are in F and, if $x_2 = y_2$, then $\{(x_1, u_2), (y_1, u_2)\} \in F$ for $u_2 \in V(G_2)$.*

5.2 Key lemmas

In this section we present two lemmas that can be used to find Cartesian pairs and sets of pairs that are copy consistent. Both of them are related to Lemma 2 and Lemma 3 of [20] and they will help us to compute the Cartesian skeleton.

The idea of the first key lemma is that the out-neighborhood $N^+(y_1, y_2)$ is a maximal subset of the set $N^+(x_1, x_2)$ among all proper subsets $N^+(z)$ of $N^+(x_1, x_2)$ if and only if $\{(x_1, x_2), (y_1, y_2)\}$ is a Cartesian pair:

Lemma 5.2.1 *Let G be a finite, R^+-thin, nontrivial cardinal product $G = G_1 \times G_2$ of directed graphs with the property that all out-neighborhoods are nonempty, F a set of Cartesian pairs of vertices of G, that is copy consistent with respect to the decomposition $G_1 \times G_2$ and H the undirected graph with the same set of vertices as G, and edge set F. Let*

$$Q(x) = \{y \mid N^+(y) \subset N^+(x)\}$$

and $P(x)$ denote the set of vertices in the connected component of H containing x. Furthermore define $\mathcal{J}(x) = \{N^+(y) \mid y \in Q(x) \setminus P(x)\}$.

Then the set F' of all pairs $\{x, y\}$, for which $N^+(y)$ is maximal in $\mathcal{J}(x)$ with respect to inclusion, is copy consistent. Hence $F \cup F'$ is copy consistent, too.

Proof. We show first that all pairs $\{x, y\} \in F'$ are Cartesian. Let $N^+(y)$ be maximal (with respect to inclusion) in $\mathcal{J}(x)$. Set $x = (x_1, x_2)$, $y = (y_1, y_2)$ and suppose that $\{x, y\}$ is not Cartesian. Then $x_1 \neq y_1$ and $x_2 \neq y_2$ by definition. Consider $y' = (y_1, x_2)$ and $y'' = (x_1, y_2)$.

If y' and y'' are in $P(x)$, then there exists a (simple) path from x to y' in H. By the copy consistency of F one can also find a path from x to y', that contains just vertices whose second component is x_2. If we substitute the second components of all vertices of this path by y_2, we obtain, by the copy consistency of F, a path from y'' to y in H. Together with a path from x to y'' this yields a path from x to y in H. Thus $y \in P(x)$, contrary to $N^+(y) \in \mathcal{J}(x)$. For this reason we can assume without loss of generality, that $y' \notin P(x)$.

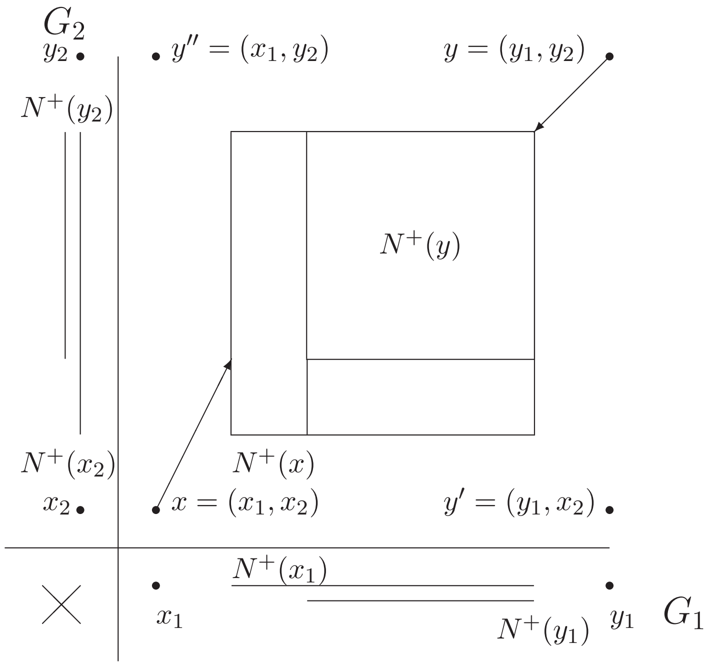

Figure 5.1: Situation of Lemma 5.2.1

Since $N_1^+(y_1) \times N_2^+(y_2) \subset N_1^+(x_1) \times N_2^+(x_2)$ we have $N_i^+(y_i) \subseteq N_i^+(x_i)$, see Figure 5.2. This implies $N_i^+(y_i) \subset N_i^+(x_i)$, since G_i is R^+-thin ($i \in \{1, 2\}$). But then

$$N^+(y') = N_1^+(y_1) \times N_2^+(x_2) \subset N_1^+(x_1) \times N_2^+(x_2) = N^+(x),$$

$y' \in Q(x) \setminus P(x)$ and $N^+(y) \subset N^+(y')$ contrary to the maximality of $N^+(y)$ in $\mathcal{J}(x)$.

For the proof of the copy consistency assume now that $\{u, v\} \in F'$, where u plays the role of x. We may assume w.l.o.g. that $\{u, v\}$ lies in a copy of G_1. Then $u = (u_1, u_2)$ and $v = (v_1, u_2)$ for some fixed u_1, v_1 and u_2. $N_1^+(v_1)$ must be maximal in $\{N_1^+(w_1) \mid (w_1, u_2) \in Q(u) \setminus P(u)\}$.

Consider another copy $\{u', v'\}$ of this pair. Then $u' = (u_1, u_2')$ and $v' = (v_1, u_2')$. Can any vertex $v'' = (v_1'', v_2'')$ in $Q(u') \setminus P(u')$ prevent the pair $\{u', v'\}$ from being in F'? If this were the case, this vertex v'' would have to satisfy $N^+(v') \subset N^+(v'') \subset N^+(u')$, or equivalently,

$$N_1^+(v_1) \times N_2^+(u_2') \subset N_1^+(v_1'') \times N_2^+(v_2'') \subset N_1^+(u_1) \times N_2^+(u_2').$$

But this is only possible if $N_2^+(v_2'') = N_2^+(u_2')$, whence $v_2'' = u_2'$ since G_2 is R^+-thin and $N_1^+(v_1) \subset N_1^+(v_1'') \subset N_1^+(u_1)$. This implies $(v_1'', u_2) \in Q(u)$. From the maximality of $N_1^+(v_1)$ we infer $(v_1'', u_2) \in P(u)$. By the copy consistency we thus have $v'' = (v_1'', u_2') \in P(u')$, which is not possible. □

The second lemma is a generalized version of the first. Simplified it tells us that $\{(x_1, x_2), (y_1, y_2)\}$ is a Cartesian pair if and only if the out-neighborhood $N^+(y_1, y_2)$ has under additional conditions maximal intersection with $N^+(x_1, x_2)$, see Figure 5.2.

Lemma 5.2.2 *Let G be a finite, N^+-connected, nontrivial cardinal product $G = G_1 \times G_2$ of R^+-thin, directed graphs, F a copy consistent set of Cartesian pairs of vertices of G that is closed under applications of Lemma 5.2.1, and H the undirected graph with the same set of vertices as G and edge set F. For every $x \in V(G)$ let, as in Lemma 5.2.1, $P(x)$ denote the set of vertices in the connected component of H that contains x. Furthermore, set $I(x, y) = N^+(x) \cap N^+(y)$ and*

$$\mathcal{I}(x) = \{I(x, y) \mid y \notin P(x), I(x, y) \neq \emptyset\}.$$

Let F' be the set of all pairs $\{x, y\}$ that satisfies one of the following conditions:

- *(i) $I(x, y)$ is strictly maximal in $\mathcal{I}(x)$ or*
- *(ii) $I(x, y)$ is nonstrictly maximal in $\mathcal{I}(x)$ and $N^+(z) \not\subset N^+(y)$ for all $z \notin P(x)$ with $I(x, z) = I(x, y)$.*

Then F' is copy consistent with respect to the decomposition $G_1 \times G_2$, and $F \cup F'$ is copy consistent. too.

Proof. We show first that all pairs $\{x, y\} \in F'$ are Cartesian. W.l.o.g. $I(x, y)$ is maximal in $\mathcal{I}(x)$ (and not necessarily in $\mathcal{I}(y)$). Set $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and suppose that $\{x, y\}$ is not Cartesian. Then $x_1 \neq y_1$ and $x_2 \neq y_2$ by definition. Consider $y' = (y_1, x_2)$ and $y'' = (x_1, y_2)$.

If y' and y'' are in $P(x)$, then $y \in P(x)$ by copy consistency (as in Lemma 5.2.1). Therefore we can assume without loss of generality that $y' \notin P(x)$. From

$$((N_1^+(x_1) \cap N_1^+(y_1)) \times (N_2^+(x_2) \cap N_2^+(y_2))$$
$$\subseteq ((N_1^+(x_1) \cap N_1^+(y_1)) \times N_2^+(x_2).$$

we know $I(x, y) \subseteq I(x, y')$, and therefore $I(x, y') \neq \emptyset$. The maximality of $I(x, y)$ implies

$$I(x, y) = I(x, y'),$$

and thus $N_2^+(x_2) \subseteq N_2^+(y_2)$. Since G_2 is R^+-thin, we infer $N_2(x_2) \subset N_2(y_2)$ and

$$N^+(y') = N_1^+(y_1) \times N_2^+(x_2) \subset N_1^+(y_1) \times N_2^+(y_2) = N^+(y),$$

contrary to $N^+(z) \not\subset N^+(y)$ for all $z \notin P(x)$ with $I(x, z) = I(x, y)$.

We continue with the copy consistency property of F' and first remark that $\mathcal{J}x)$, as defined in Lemma 5.2.1, must be empty for every $x \in G$ when H is closed under applications of Lemma 5.2.1.

Let $\{u, v\} \in F'$. Thus $I(u, v)$ satisfies condition (ii), where w.l.o.g. $I(u, v)$ is maximal in $\mathcal{I}(u)$. Furthermore, let $\{u', v'\}$ be a copy of $\{u, v\}$. Clearly $v' \notin P(u')$ by the copy consistency property of H and $I(u', v') \neq \emptyset$. Without loss of generality we can assume $u = (u_1, u_2)$, $v = (v_1, u_2)$, $u' = (u_1, u_2')$ and $v' = (v_1, u_2')$.

We wish to show that $(u', v') \in F'$. Suppose $(u', v') \notin F'$. Then there is a $z = (z_1, z_2) \notin P(u')$ with i) $I(u', z) \supset I(u', v')$ or ii) $I(u', z) = I(u', v')$ and $N^+(z) \subset N^+(v')$.

Assume $z_2 = u_2'$. Then $(u, (z_1, u_2)) \in F'$ instead of (u, v), so we know $z_2 \neq u_2'$. But $I(u', v') \subseteq I(u', z)$ means $(N_1^+(u_1) \cap N_1^+(v_1)) \times N_2^+(u_2') \subseteq (N_1^+(u_1) \cap N_1^+(z_1)) \times (N_2^+(u_2') \cap N_2^+(z_2))$. $\Rightarrow N_2^+(u_2') \subseteq N_2^+(z_2)$. G_2 R^+-thin $\Rightarrow N_2^+(u_2') \subset N_2^+(z_2)$. $\Rightarrow N^+(u') \subset N^+((u_1, z_2))$. F closed under applications of Lemma 5.2.1 $\Rightarrow$ a)$(u_1, z_2) \in P(u')$. This shows $z_1 \neq u_1$.

From $N_2^+(u_2') \subset N_2^+(z_2)$ we have $I(u', z) = I(u', (z_1, u_2'))$. $\Rightarrow I(u', (z_1, u_2')) \supseteq I(u', v')$ and in the case of "=" $N^+((z_1, u_2')) \subset N^+(z) \subset N^+(v') \Rightarrow$

$I(u, (z_1, u_2)) \supseteq I(u, v)$ and in the case of "=" $N^+((z_1, u_2)) \subset N^+(v)$. This is only possible if $(z_1, u_2) \in P(u)$.

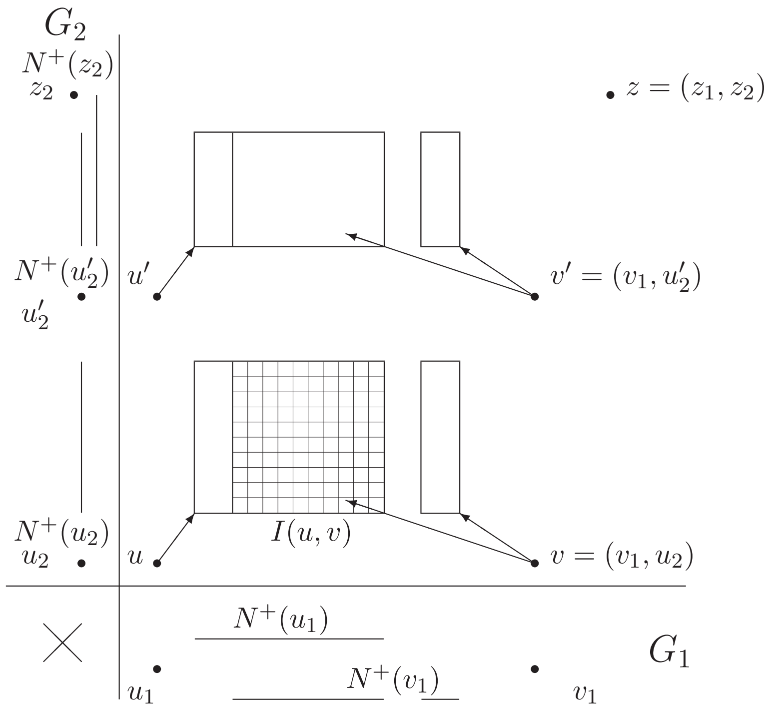

Figure 5.2: Situation of Lemma 5.2.2

But copy consistency of F implies b)$(z_1, u_2') \in P(u')$. From a), b) and the copy consistency again we know $z \in P(u')$, contradiction. □

5.3 The Cartesian skeleton algorithm

In this section we use the key lemmas to calculate the Cartesian skeleton. The following algorithm, which applies those lemmas is modelled after the marking algorithm of Imrich [19]: It gives us the edges of the Cartesian skeleton in form of found pairs of vertices. Further it will be proved in Lemma 5.3.2 that the Cartesian skeleton of a finite, R^+-thin and N^+-connected graph G is connected.

Algorithm 1

Input: A finite, R^+-thin and N^+-connected graph G.
Output: A set of marked pairs of vertices of G that is copy consistent with

respect to any decomposition of G.

Begin
  For Each $x \in V(G)$
    $P(x) := \{x\}$; Insert $P(x)$ into P; $Q(x) := \{y \mid N^+(y) \subset N^+(x)\}$
  Next x
  M1: While $\exists x \in V(G)$ for which $Q(x) \setminus P(x) \neq \emptyset$
    For Each such x
      $G(x) := \{N^+(y) \mid y \in Q(x) \setminus P(x)\}$
      If $N^+(y)$ is maximal in $G(x)$, then mark $\{x, y\}$
    Next x
    If $\{x, y\}$ has been marked, then join $P(x)$ and $P(y)$ in P
  End **M1:**
  Set all $I(x, y) = N^+(x) \cap N^+(y)$
  M2: While $\#P > 1$
    M3: For Each $x \in V(G)$
      $G(x) := \{I(x, y) \mid y \notin P(x),\ I(x, y) \neq \emptyset\}$
      If $G(x) \neq \emptyset$, Then
        For Each $y \notin P(x)$ with $I(x, y) \neq \emptyset$
          If ($I(x, y)$ is maximal in $G(x)$) And
            ($N^+(z) \not\subset N^+(y)$ For Each $z \notin P(x)$ with
            $I(x, y) = I(x, z)$) Then mark $\{x, y\}$
          End If
        Next y
      End If
    End **M3**
    If $\{x, y\}$ has been marked, Then join $P(x)$ and $P(y)$ in P
  End **M2**
End

Remark: All computations in the algorithm are polynomial in the number of vertices n.

Lemma 5.3.1 *Let G be finite, R^+-thin, N^+-connected and H an undirected graph with the same vertex-set as G. If the edges of H are the pairs of vertices marked by the Cartesian skeleton algorithm, then H is a Cartesian skeleton of G and connected.*

Proof. At the begin the Cartesian skeleton algorithm applies Lemma 5.2.1

for $F = \emptyset$, which is copy consistent with respect to any decomposition of G. But then we get copy consistency with respect to any decomposition of the set of found pairs step by step. $\Rightarrow$

The edge set of H is copy consistent with respect to any decomposition of G. By that reason we know that every cardinal decomposition of G induces a decomposition of H with respect to the Cartesian product, where the vertex sets of the factors are in both products the same. Thus H is a Cartesian skeleton of G.

To prove the connectedness of H it suffices to show that M2 of our Cartesian skeleton algorithm is no endless loop, which can be understood easily: If $\#P > 1$, we have two vertices x and y with $P(x) \neq P(y)$, but then we take a sequence $x = x_0, x_1, ..., x_{n-1}, x_n = y$ with $N^+(x_i) \cap N^+(x_{i+1}) \neq \emptyset$. There exists a minimal index $j \in \{1, 2, ..., n\}$ so that $P(x_{j-1}) \neq P(x_j)$. This implies $G(x_{j-1}) \neq \emptyset$. Hence the cardinality of P will be reduced when the loop M2 is finished. $\Box$

From now on we can use the unique PFD of H. By the uniqueness of the layers for the Cartesian PFD of H we have then also unique layers in the cardinal PFD of G. The problem of finding factors with respect to the cardinal product is reduced then to project the edges onto vertex sets of possible factors.

In the class of undirected graphs there exists an analogous result for non-bipartite graphs, but for bipartite ones (which implies that they are not N^+-connected) we can prove only an a little bit weaker result:

Lemma 5.3.2 *Let G be undirected, finite, connected, thin and H an undirected graph with the same vertex-set as G. If the edges of H are the pairs of vertices marked by Algorithm 1 of [19], then H is a Cartesian skeleton of G. It consists of two connected components if and only if G is bipartite .*

Proof. H is a Cartesian skeleton of G by the same reasons as in the last proof.

a) Suppose G is bipartite:

While $|P| > 2$, the cardinality of P will be reduced with every repetition of the loop M2, which means that the number of connected components of H is less or equal 2:

G connected $\Rightarrow$ There exists an edge $[x, y] \in E$ with $P(x) \neq P(y)$. $|P| > 2$ implies that there exists a vertex z with $P(x) \neq P(z) \neq P(y)$. Since G is connected, we can find a path C from y to z. $C = c_0 K_1 c_1 K_2 .... K_N c_N$, $c_0 = y$ and $c_N = z$. Let c_j ($j \in \{1, 2, ..., N\}$) be the first vertex of C, which is not in $P(x) \cup P(y)$.

$j > 1$: $c_j \notin P(c_{j-2})$, $c_{j-1} \in (N(c_j) \cap N(c_{j-2})) \neq \emptyset \Rightarrow |P|$ reduced.

$j = 1$: $c_1 \notin P(x)$, $y \in (N(c_1) \cap N(x)) \neq \emptyset \Rightarrow |P|$ reduced.

Next we show that H has at least two connected components:

Suppose $V(G) = A_1 \cup A_2$ is a partition of the vertex set such that every edge in G joins one vertex in A_1 with one vertex of A_2. In H then there can be no edge joining A_1 and A_2, because a vertex in A_1 and a vertex in A_2 have always empty neighborhood intersection.

b) Suppose H consists of two connected components, let us call them A and B.

Be aware that for $x \in A$ and $y \in B$ the neighborhood intersection $I(x, y)$ must be the empty set, otherwise y would be in $P(x)$ after applying Algorithm 1. Using this it is not hard to show that there can be no edge in $E(G)$ joining x with another vertex $a \in A$: By connectedness of G there is a path from a to a vertex $b \in B$. Without loss of generality we can assume that $[a, b]$ is an edge in G, but then $I(x, b) \neq \emptyset$, contradiction. □

Chapter 6

Factoring N^+-connected R^+-thin graphs

6.1 The first main result

In this section we show that every N^+-connected, R^+-thin finite graph has a unique PFD with respect to the cardinal product and that it can be found in polynomial time. For finite graphs the UPFD is an immediate consequence of the common refinement property, which will be explained in the next lemma.

Lemma 6.1.1 *Let G be a finite, N^+-connected and R^+-thin graph and let $A \times B$ and $C \times D$ be two decompositions of G with respect to the cardinal product. Then there exists a decomposition*

$$G = A_C \times A_D \times B_C \times B_D$$

so that $A = A_C \times A_D$, $B = B_C \times B_D$, $C = A_C \times B_C$ and $D = A_D \times B_D$.

We call the decomposition $A_C \times A_D \times B_C \times B_D$ a common refinement of the decompositions $A \times B$ and $C \times D$ of G.

We omit the proof of this lemma, because it is tedious and the same as the proof of Lemma 6 in [19]. Further it is used that the PFD with respect to Cartesian product can be found in polynomial time, or even, as a new result [23] shows, in linear time.

Theorem 6.1.2 *Every finite, N^+-connected and R^+-thin graph G has a unique PFD with respect to the cardinal product. It can be computed in polynomial time.*

Proof: We prove only uniqueness. Let us assume we have two PFDs

$$G_1 \times G_2 \times ... \times G_r = Q_1 \times Q_2 \times .... \times Q_s$$

of G $(r, s > 1)$. We proceed by induction with respect to the number of vertices, assuming that the statement is true for all graphs with fewer vertices than G. Let be $B = G_2 \times ... \times G_r$ and $D = Q_2 \times Q_3 \times .... \times Q_s$, so we clearly have $G = G_1 \times B = Q_1 \times D$. Now we use the existence of a common refinement: $G = A_C \times A_D \times B_C \times B_D$, where 1) $G_1 \cong A_C \times A_D$, 2) $B \cong B_C \times B_D$, 3) $Q_1 \cong A_C \times B_C$ and 4) $D \cong A_D \times B_D$. G_1 prime implies $G_1 \cong A_C$ or $G_1 \cong A_D$. If the first relation is true, we know from Q_1 prime and 3): $Q_1 \cong A_C \cong G_1$ and $B_C \cong A_D \cong K_1^s$. But then 2) and 4) show $B \cong B_D \cong D$ and the induction hypothesis (IH) proves the uniqueness of the PFD.

In the second case $G_1 \cong A_D$ implies $D \cong G_1 \times B_D \cong Q_2 \times ... \times Q_s$. IH $\Rightarrow$ w.l.o.g. $G_1 \cong Q_2$ and $B_D \cong Q_3 \times ... \times Q_s$. From $A_C \cong K_1^s$ we have $Q_1 \cong B_C$. By the definition of B and 2) we have then:

$$G_2 \times ... \times G_r \cong B \cong B_C \times B_D \cong Q_1 \times Q_3 \times Q_4 \times ... \times Q_s.$$

Again the induction hypothesis shows the uniqueness of the PFD. □

The proof of the second assertion is the same as the proof of Lemma 8 in [19]. It can also be found in [20]. Furthermore Imrich proved in [19] (Theorem 3) that every automorphism of a undirected cardinal product graph G which is nonbipartite, connected and R-thin is a permutation, possibly trivial, of isomorphic factors combined with automorphisms of the G-factors.

We want to generalize this theorem to directed graphs. Since it is an important tool we cite the following theorem which describes automorphisms of Cartesian product graphs.

Theorem 6.1.3 (Imrich [18], Miller [28]) *Let ϕ be an automorphism of a connected graph G with PFD $G_1 \Box G_2 \Box ... \Box G_k$. Then there exists a permutation π of $\{1, 2, ..., k\}$ together with isomorphisms $\psi_i : G_i \to G_{\pi i}$ such that*

$$\phi(v_1, v_2, ..., v_k) = (\psi_{\pi^{-1}1} v_{\pi^{-1}1}, \psi_{\pi^{-1}2} v_{\pi^{-1}2}, ..., \psi_{\pi^{-1}k} v_{\pi^{-1}k},).$$

Now we can prove the final theorem of this chapter:

Theorem 6.1.4 *Let $G_1 \times G_2 \times ... \times G_s$ be the PFD of the finite, directed, N^+-connected and R^+-thin graph G and ϕ an element of $Aut(G)$. Then* *there exists a permutation π of $\{1, 2, ..., s\}$ together with automorphisms $\psi_i \in Aut(G_i)$ such that*

$$\phi(v1_{r_1}, v2_{r_2}, ..., vs_{r_s}) = (\psi_1 v1_{r_{\pi(1)}}, \psi_2 v2_{r_{\pi(2)}}, ..., \psi_s vs_{r_{\pi(s)}}),$$

where the v_i are vertices of G_i.

Proof. Neighborhoods are mapped on neighborhoods by automorphisms. Therefore a pair of vertices with maximal neighborhood intersection will be mapped to a pair with maximal neighborhood intersection. This means that Cartesian pairs and also the Cartesian skeleton H of G are invariant under automorphisms.

With other words: Every automorphism of G induces an automorphism of H. But from the last theorem we know that the automorphisms of H are given by permutations of isomorphic factors of the Cartesian PFD of H together with automorphisms on the factors itself. Hence, automorphisms of G have the structure that we claimed in the theorem. □

Chapter 7

Graphs that are not R^+-thin

7.1 General considerations

It is really natural to ask if it is possible to generalize the result of Theorem 6.1.2 to graphs that are only finite and N^+-connected, because for undirected graphs we know this generalization: Suppose a graph G is not thin. Then we can apply the relation R and get a thin quotient graph G/R. The PFD of G/R can be found by Lemma 8 of [19] and there is a well described (blowing-up) procedure (McKenzie [27] and Imrich [19] did it in detail and we used it in the first part) that can be used to reconstruct the G-factorization from the (G/R)-factorization.

What about directed graphs? Of course there are thin graphs that are neither R^+- nor R^--thin, as Figure 2.3 and the following simpler example show, so Theorem 6.1.2 and its $-$-version cannot always be applied to obtain a (G/R)-factorization of a directed graph.

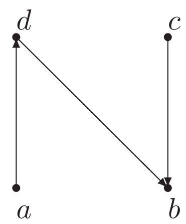

Figure 7.1: A thin graph G that is neither R^+- nor R^--thin

For this reason R does not seem to be the appropriate relation for directed graphs. Another idea is to use the relation R^+ instead of R, because the R^+-quotient graph of the graph in Figure 7.1 is R^+-thin. But unfortunately this does not always work: If you take a look at Figure 7.2, you see that R^+-quotient graphs need not be R^+-thin.

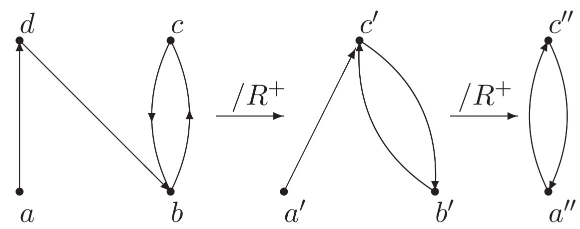

Figure 7.2: G/R^+ need not be R^+-thin

But be aware: We investigate finite graphs, thus we surely get some R^+-thin graph after a finite number of R^+-applications. Since we know induction the main problem is still to reconstruct the G-decomposition from the (G/R^+)-decomposition. If we are able to do this one time, we will be able to repeat this procedure for a finite number of times.
In the blowing up procedure of [19] (Section 8) complete factors were firstly extracted and vertices of the quotient graph then blown up to R-classes (Lemma 14) of a larger graph. We would like to do something analogous for R^+-classes, but we cannot do it if we do not know the structure of the graphs that are induced by vertices in such classes.

Definition 7.1.1 *Let r and s be integers, $r > 0,\ s \geq 0$. $R^+_{s,r}$ is a graph consisting of a complete subgraph K^d_s and $r-s$ other vertices that have empty in-neighborhoods and whose out-neighborhoods consist exactly of the vertices in the complete subgraph K^d_s.*

Remark: $R^+_{s,r}$ has totally r vertices and s loops (s is also the size of the largest complete subgraph), compare Figure 7.3.

Lemma 7.1.2 *Graphs induced by vertices in R^+-classes are $R^+_{s,r}$-graphs.*

Proof. For vertices in those graphs there exist two possibilities: Their in-neighborhoods can be empty or nonempty. In the second case the vertex must be in the out-neighborhood of all vertices in the R^+-class. Thus those vertices induce a complete subgraph K_s^d, the out-neighborhoods of the other vertices consist exactly of the vertices in the complete subgraph K_s^d and their in-neighborhoods are empty. □

7.2 $R^+_{s,r}$-graphs

Obviously the structure of those graphs is much richer than the structure of graphs induced by R-classes, which can only be totally disconnected or complete. A big difference can be seen if one considers the PFD. For R-classes the PFD corresponds to the PFD of natural numbers, but for graphs induced by R^+-classes PFD is difficult, although multiplication is easy:

$$R^+_{s_1,r_1} \times R^+_{s_2,r_2} = R^+_{s_1 * s_2, r_1 * r_2}.$$

Figure 7.3 shows that the PFD need not be unique.

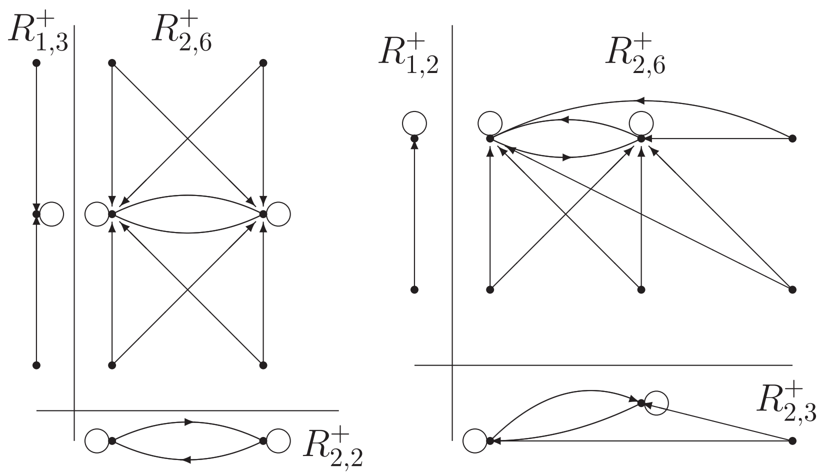

Figure 7.3: Non-unique PFD

If one looks carefully at this figure, one may recognize that the drawn graphs are N^+-connected. Thus it is in general not possible to prove uniqueness of the PFD for graphs that are only finite and N^+-connected. For this reason we do not plan to enlarge the class of graphs that are known to have a unique PFD one more time, but we will use the considerations of this chapter to find a polynomial algorithm to compute the PFD of graphs in a large subclass of the class for which McKenzie showed UPFD.

As we have seen in the last section, the subgraphs of a graph G induced by the equivalence classes of the relation R^+ are $R^+_{s,r}$-graphs. Of course computing with these graphs is more difficult than working with complete graphs. One reason is, that the PFD of $R^+_{s,r}$-graphs is not unique as Figure 7.3 shows. However, if we want to work with such graphs and their PFDs we have to characterize prime $R^+_{s,r}$:

Lemma 7.2.1 *Given the graph $R^+_{s,r}$. Then the following statements are equivalent:*

(i) $R^+_{s,r}$ *is not prime.*

(ii) *There is a nontrivial decomposition $r = r_1 * r_2$ and a (possibly trivial) decomposition $s = s_1 * s_2$ so that $s_1 \leq r_1$ and $s_2 \leq r_2$.*

Proof. (i)$\Longrightarrow$(ii): Assume $R^+_{s,r} = A \times B$ is a nontrivial factorization, then $r = |V(A)| * |V(B)|$ $(|V(A)| > 1, |V(B)| > 1)$. Let us call the number of loops of A s_A and the number of loops of B s_B. Obviously we have then: $s_A * s_B = s$, $s(A) \leq |V(A)|$ and $s(B) \leq |V(B)|$

(ii)$\Longrightarrow$(i): $R^+_{s_1,r_1} \times R^+_{s_2,r_2} = R^+_{s,r}$ □

In analogy to the undirected case we want to compute gcd's of $R^+_{s,r}$-graphs, but to do this we need a characterization of divisors of $R^+_{s,r}$, which will be given in the next lemma:

Lemma 7.2.2 *The following statements are equivalent:*

(i) $A|R^+_{s,r}$.

(ii) *There exist integers s_1, r_1 with $s_1 \geq 0$ and $r_1 > 0$, so that $A = R^+_{s_1,r_1}$ and the relations $s_1|s$, $r_1|r$ and $s/s_1 \leq r/r_1$ hold.*

The inequality means that the number of loops in the cofactor $R^+_{s,r}/A$ has to be smaller then the number of vertices.

Proof. (i)$\Longrightarrow$(ii): Let be $r_1 = |V(A)|$ and s_1 the number of loops of A. Then the first two relations of statement (ii) hold. The last must be true, because the cofactor can not have more loops than vertices. Projection of $R^+_{s,r}$ onto A shows that A contains a $K^d_{s_1}$ and that the out-neighborhoods of the other $r_1 - s_1$ vertices of A consist exactly of the vertices of the $K^d_{s_1}$ (Their in-neighborhoods are empty). Hence $A = R^+_{s_1,r_1}$.

(ii)$\Longrightarrow$(i): $R^+_{s_1,r_1} \times R^+_{s/s_1,r/r_1} = R^+_{s,r}$ □

Let us assume now that B is the set of all subgraphs $R^+_{s,r}$ of G induced by the classes of the equivalence relation R^+. How can we define a greatest common divisor for B and how can we compute it? The answer to the first part of the question is:

Definition 7.2.3 *Let B be a finite set of $R^+_{s,r}$-graphs. A* greatest common divisor*, shortly* gcd*, of B is a common divisor with a maximal number of vertices.*

For the second part we use the following simple but important consideration: The gcd z of $\{|b| \mid b \in B\}$ must be divided by the size of every common divisor. Hence we just have to check, if an $R^+_{s,z}$ with $0 \leq s \leq z$ divides all $b \in B$. In the case that no $R^+_{s,z}$ divides all $b \in B$ we check the $R^+_{s,y}$, where y is the second largest divisor of z and so on.
Remark: Unfortunately a gcd is not uniquely defined if we just want it to have a maximal number of vertices as the following example illustrates:

$$R^+_{1,3},\ R^+_{3,3} | R^+_{3,15},\ R^+_{3,21}.$$

7.3 Counterexamples and problems

The first example shows that the quotient graph G/R^+ of a thin graph G is not necessarily thin, which really leads to problems (additional assumptions) in the next chapter.

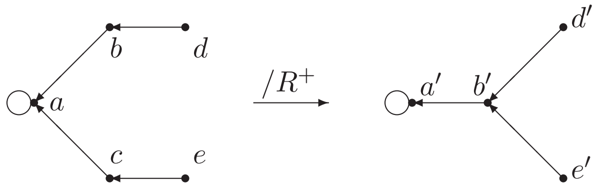

Figure 7.4: G/R^+ need not be R-thin

Even if we demand N^+- and N^--connectedness, we can find some R-thin graph G, that has a quotient graph G/R^+ which is not R-thin. We just have to draw some additional edges in the graphs of the last example as we see in Figure 7.5. G is thin, but the vertices d' and e' in the quotient graph have both equal in- and equal out-neighborhoods.

Interestingly this graph G of Figure 7.5 has a quotient graph G/R^- which is not R-thin, too. We can see this in Figure 7.6.

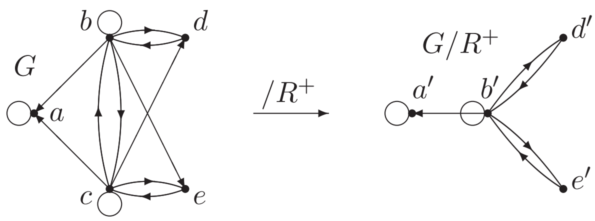

Figure 7.5: $N^+(b) = N^+(c)$.

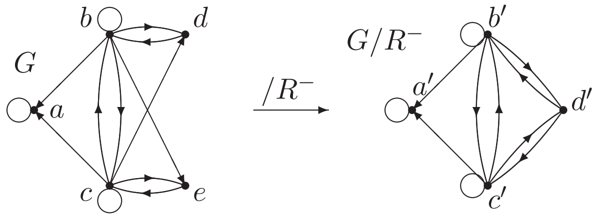

Figure 7.6: $N^-(d) = N^-(e)$.

When I studied gcd's of $R^+_{s,r}$-graphs, once I wanted to prove the following: If $b = (b_1, ..., b_m)'$ is a column- and $a = (a_1, ..., a_l)$ a row-vector consisting of $R^+_{s,r}$-graphs, $M = b \times a$ some $m \times l$-matrix and g_a, g_b gcd's of the graphs in a, respectively b. Then the product $g_a \times g_b$ is a gcd of the graphs in the matrix M.

After some successless tries I found this counterexample: $b = (R^+_{1,7}, R^+_{2,7})^t$, $a = (R^+_{1,2})$ and

$$M = b \times a = \begin{pmatrix} R^+_{1,14} \\ R^+_{2,14} \end{pmatrix}.$$

Here is $R^+_{1,1}$ a gcd of the set of all entries in b and $R^+_{1,2}$ a gcd of a, but $R^+_{1,7}$ is a gcd of the entries in M.
From Figure 7.3 we can learn that $R^+_{2,6}$ has two different PFDs. This leads to the following question: Is the number of different (non-isomorphic) PFDs of one $R^+_{s,n}$-graph bounded polynomially or even stronger linearly in n?

Yet we could find no counterexample to the conjecture in the linear and no proof in the polynomial case.

Chapter 8

Factoring graphs that are not R^+-thin

8.1 Blowing up

The only thing we are not so happy with in Theorem 6.1.2 is the strong thinness condition. As described in the first section the proof of a more general theorem is based on successive transformations of a given graph until we can apply the last theorem. Graphs are transformed in this paper by computing their R^+-quotient graphs, but G/R^+ need neither be R^+-thin as Figure 4 in [24], nor thin as Figure 7.4 in this paper shows. However, after a finite number of R^+-applications we obtain an R^+-thin graph. Therefore it is clear that the main task of this section is the reconstruction of the G- from the G/R^+-factorization. By the next lemmas we solve the problem in the case where the PFD of G/R^+ consists only of two factors.

This main task is realized by a blowing up procedure. To describe it we have introduced in the last chapter a new class of graphs, those $R^+_{s,r}$-graphs. It is not too hard to prove that graphs induced by R^+-classes are exactly $R^+_{s,r}$-graphs (for details see Lemma 7.1.2).

Let $A_1 \times B_1$ be the PFD of G/R^+ and the vertex sets labelled as follows: $V(A_1) = \{a_1, a_2, ..., a_l\}$, $V(B_1) = \{b_1, b_2, ..., b_m\}$ and $V(G/R^+) = \{v_1, v_2, ..., v_n\}$. In the proofs of the next lemmas we will also use a matrix M that consists of all subgraphs of G that are induced by the vertices in R^+-classes in G. The element in row j and column i be the graph induced by the vertices in the R^+-class (b_j, a_i). The idea is that every decomposition $G = B \times A$ induces a decomposition $M = b \times a$, where b is a column-vector containing the R^+-classes of B and a is a row-vector with the R^+-classes of

A. If G/R^+ is thin, the matrix M is unique up to the order of the rows and the columns.

In the following we try to obtain the G-decomposition of the not R^+-thin graph G by "blowing up" vertices of A_1 and B_1 to R^+-classes of G-factors. The question for an estimation of the number of different ways to blow up a given PFD $A_1 \times B_1$ will be answered by Lemma 8.1.1: It bounds this number polynomially.

Lemma 8.1.1 *Assumptions: Let G be an N^+- and N^--connected, R-thin finite graph. Further let $B_1 \times A_1$ be a PFD of the thin graph G/R^+, where $V(A_1) = \{a_1, a_2, ..., a_l\}$ and $V(B_1) = \{b_1, b_2, ..., b_m\}$.*

By $D(b_j, a_i)$ we denote the subgraph of G induced by the vertices in the R^+-equivalence class (b_j, a_i) in G/R^+. Next we define an $m \times l$ matrix M, where the element in row j and column i is the $R^+_{s,r}$-graph $D(b_j, a_i)$.

In the statements (ii) and (iii) we assume G not to be prime.

Then the following statements hold:

- *(i) G has at most two nontrivial prime divisors. Hence G is prime itself or it has a PFD $G = A \times B$, where $A/R^+ = A_1$ and $B/R^+ = B_1$.*
- *(ii) M is the product of a column-vector b and a row-vector a, therefore $rk(M) = 1$ holds.*
- *(iii) In general the number of M-decompositions $b \times a$ is bounded by n^2.*

Proof. (i) Since G is N^--connected, it does not allow an $R^+_{s,r}$-graph as divisor. For that reason every decomposition into three nontrivial factors of G induces by Lemma 3.1.2 a decomposition of G/R^+ into three nontrivial factors. Hence G has at most two prime divisors.

Now we assume that $G = A \times B$. Of course it is no restriction if we assume additionally $A/R^+ = A_1$ and $B/R^+ = B_1$.

(ii) We define the row-vector a as $(D(a_1), D(a_2), ..., D(a_l))$, where $D(a_i)$ is the subgraph of A that is induced by an R^+-class (a_i) of A, the column-vector b analogously. By Lemma 3.1.2 the following equations hold, which proves *(ii)*.

$$b \times a = \begin{pmatrix} D(b_1) \\ D(b_2) \\ \vdots \\ D(b_m) \end{pmatrix} \times \begin{pmatrix} D(a_1) & D(a_2) & \cdots & D(a_l) \end{pmatrix} =$$

$$\begin{pmatrix} D(b_1,a_1) & D(b_1,a_2) & \cdots & D(b_1,a_l) \\ D(b_2,a_1) & D(b_2,a_2) & \cdots & D(b_2,a_l) \\ \vdots & \vdots & \cdots & \vdots \\ D(b_m,a_1) & D(b_m,a_2) & \cdots & D(b_m,a_l) \end{pmatrix} = M$$

(iii) Let be a and b as in (ii). At first one must recognize that an entry $D(b_1)$ in b determines all other entries of b and a uniquely. $D(b_1)$ must be a divisor of $D(b_1, a_1)$, but however, the size of $D(b_1)$ is bounded by n and the number of loops, too. Thus the number of ways to define $D(b_1)$ is bounded by n^2, which bounds also the number of M-decompositions. □

Why did we assume G/R^+ to be thin in the last lemma? Well, if it is not thin, it may happen that one R-class r consists of the six vertices $v_1, v_2, ..., v_6$. By a lemma of McKenzie [27] we know that this class is a product of R-classes in A_1 resp. B_1. Let us assume they consist of two, resp. three vertices. Of course the order of the vertices can be chosen, so that $V = \{v_1, v_2, ..., v_6\} = \{b_1, b_2, b_3\} \times \{a_1, a_2\}$. For v_1 there are six different pairs of coordinates possible. Of course there are totally 6! ways to define the coordinates of the vertices in r and the problem is that this number is in general not polynomial bounded. This is the reason for which we assume G/R^+ to be thin. However, by McKenzie the PFD of G is unique as its coordinatization, so maybe one can use this once to prove the lemma without those annoying assumption.

Of course at this state the new graphs are not completely defined, because yet we know nothing about the edges between the R^+-classes.

Lemma 8.2.1 will tell us, how to find the edges between the R^+-classes of the blown up graphs and Lemma 8.2.2 shows that we do not get into troubles, if G/R^+ has more than 2 prime divisors.

8.2 The second main result

Lemma 8.2.1 *Let G be a finite, thin, N^+- and N^--connected graph and $A_1 \times B_1$ the PFD of G/R^+. We use the notations of Lemma 8.1.1 and assume M (uniquely defined only if G/R^+ is thin) to be the product $(D(a_1), ..., D(a_l))^t * (D(b_1), D(b_2), ..., D(b_k))$.*

Suppose there is a decomposition $G = B \times A$, where $V(A)$ consists of the vertices in the $D(a_i)$ $(i \in 1, 2, ..., l)$, $V(B)$ of the vertices in the $D(b_j)$. (The edge sets of A and B are unknown in the beginning.) Then the A- and

B-layers of G as well as A and B can be computed in polynomial time.

Proof. By the thinness of G we know that the A- and B-layers are unique and of course $x \in V(G)$ can be written as (x_A, x_B) and $y \in V(G)$ as (y_A, y_B), where $x_A,\ y_A \in A$ and $x_B,\ y_B \in B$. We denote the subgraph of G induced by all vertices in $D(b_j, a_i)$ with $j \in \{1, 2, ..., k\}$ by $l(a_i)$. In some sense it is a *generalized layer*: $l(a_1) = B \times D(a_1)$. The $l(b_j)$ be defined analogously. Before layers are determined we list three important basic facts:

(i) Layers are subgraphs of generalized layers.
(ii) From the definition of R^+-classes we also know that the in-neighborhoods are unions of R^+-classes.
(iii) If G is N^--connected $\Rightarrow$ All in-neighborhoods are nonempty. This holds also for all factors of G.

Let

$$I_A(x) = \{j \mid \exists i \in \{1, 2, ..., l\}\ s.t.\ (b_j, a_i) \subset N^-(x)\}.$$

From (i) - (iii) we can conclude, as proved below in detail, that:
Two vertices $x, y \in l(b_j)$ are in the same A-layer $\iff$ $I_A(x) = I_A(y)$.

"$\Rightarrow$": We suppose at first that $x, y \in l(b_j)$ are in the same A-layer, hence $x_B = y_B$. From (ii) we infer that

$$N^-(x_B) = N^-(y_B) = \bigcup_{j \in I} b_j$$

for some subset I of $\{1, 2, ..., k\}$. But one can obtain this information about the in-neighborhoods also from the known generalized layers, which allows to prove: $I_A(x) = I = I_A(y)$. We will show the first equation in detail, the second can be proved analogously.

Given some $j_0 \in I$. From (iii) we know that $N^-(x_A) \neq \emptyset$, hence we have from (ii) that there is some $i_0 \in \{1, 2, ..., l\}$, so that $a_{i_0} \subset N^-(x_A)$. This implies $(b_{j_0}, a_{i_0}) \subset N^-(x)$. Definition of $I_A(x)$ $\Rightarrow$ $j_0 \in I_A(x)$.

For any $j_0 \in I_A(x)$ the definition of $I_A(x)$ implies that there exists an i_0 such that $(b_{j_0}, a_{i_0}) \subset N^-(x)$. Thus the Cartesian product of R^+-classes $b_{j_0} \times a_{i_0}$ is a subset of $N^-(x_B) \times N^-(x_A)$. $a_{i_0} \neq \emptyset \Rightarrow$ $b_{j_0} \subset N^-(x_B)$. From the definition of I we know now that $j_0 \in I$, which completes the proof of the first equation.

"$\Leftarrow$": Here we suppose $I_A(x) = I_A(y)$. This implies $N^-(x_B) = N^-(y_B)$. [Otherwise there would exist without loss of generality some $b_{j_0} \subset N^-(x_B) \setminus$

$N^-(y_B)$. Then we also have some $(b_{j_0}, a_{i_0}) \subset N^-(x) \setminus N^-(y)$, which implies $j_0 \in I_A(x) \setminus I_A(y)$, contradiction.] But x and y are in the same generalized layer $l(b_j)$, which means $N^+(x_B) = N^+(y_B)$. G thin implies directly that all divisors of G are thin, too, which would follow under our assumptions also from Lemma 3.1.1. Thus x_B equals y_B. The vertices x and y are in the same A-layer.

By statement (i) layers are completely described now. Sets of vertices with equal sets $I_A(.)$ can be found in polynomial time. Let be $|V(G)| = n$, $|V(G/R^+)| = n'$. The algorithm for finding these layers consists of three parts:

1.) Compute for all vertices $x \in V(G)$ the index set $I_A(x)$.

For every x (factor: n)

    For every R^+-class (b_j, a_i) of G (factor: n')

        Take a $y \in (b_j, a_i)$ and check

        If $y \in N^-(x)$ (factor: n)

            $I_A(x) = I_A(x) \cup \{j\}$

2.) Sort all $I_A(x)$. The cardinality of those index sets is of course not bigger than $|B_1|$. Hence the effort for doing this part is $n * |B_1| * log(|B_1|)$, which is roughly bounded by $n^2 * log(n)$.

3.) Find maximal sets $L \subset V(G)$ which fulfill: $\forall x,\ y \in L:\ I_A(x) = I_A(y)$. Those sets induce layers. To find the A-layer through $x \in V(G)$, we have to check for all $y \in V(G)$ or more efficient for all y in the same generalized A-layer as x, if

$$I(x) = I(y).$$

The effort for checking the last equation is bounded by n, since the index sets are sorted. Totally we have a bound n^3 for the last part of the algorithm.

If all coordinates of G with respect to A and B are known, then it is easy to compute $E(A)$ and $E(B)$: Project $E(G)$ to the vertex sets of A and B. □

The following algorithm applies this lemma. It can be used to compute the PFD of a graph G that fulfills the conditions of Lemma 8.2.1.

Algorithm 2

Input: A graph G that fulfills the conditions of Lemma 8.2.1.

Output: The PFD of G.

If (There is no decompositions $b \times a$ of M.)

Return (G is prime).
Determine the A- and B-layers using the algorithm of Lemma 8.2.1.
If (All A-layers have the same cardinality and the same holds for all B-layers.)
$E(A) = p_A(E(G))$
$E(B) = p_B(E(G))$
If ($G = A \times B$)
Return (A and B are the prime factors of G.)
G is prime.

Remarks: If there is no M-decomposition, then G is prime, because a nontrivial G-decomposition would induce an M-decomposition. The function p_A, respectively p_B, is the projection of G onto A, resp. B. Since we know that $V(G) = V(A \times B)$ and $E(G) \subseteq E(A \times B)$, checking the equation $G = A \times B$ is easy:

$$G = A \times B \iff E(G) = E(A \times B) \iff |E(G)| = |E(A)| \cdot |E(B)|.$$

If the projection does not lead to a G-decomposition, then Lemma 8.2.1 implies that G is prime. Lemma 8.2.1 bounds the complexity of the layers computation polynomially in $|V(G)|$. Therefore Algorithm 2 is polynomial in $|V(G)|$.

Lemma 8.2.2 *If G is a finite, thin, N^+- and N^--connected graph and $A_1 \times .... \times A_l$ a PFD of the thin graph G/R^+, then the PFD of G can be found in polynomial time $O(n^4)$, where n is the number of vertices of G.*

Proof. For $l = 2$ the statement of the corollary follows immediately from Lemma 8.1.1, Lemma 8.2.1 and the last algorithm. In general we take all PF A_i ($i \in I = \{1, 2, ..., l\}$) times the cofactor G/A_i and try to blow up this decomposition of G/R^+ to a decomposition of G.

Let be $|V(G/R^+)| = n$. We have to find all layers at first. From Lemma 8.2.1 we know that the complexity of this computation is bounded by $O(n^3)$ and by Algorithm 2 we bound the effort for proving if blowing up the vertices of A_i really leads to a factor of G by $|E(G)|$.

If the blown up graph A'_{i_0} over A_{i_0} (this means $A'_{i_0}/R^+ = A_{i_0}$) is really a divisor of G it must be prime by Lemma 3.1.2. We extract it from G and cancel i_0 from I.

Then we try to blow up decompositions $A_i \times A_j$ ($i, j \in I$) times the cofactor with respect to G/R^+ to decompositions of G. Note that the factor over $A_i \times A_j$, if existing, must be prime, because otherwise we had found factors over A_i and A_j. Analogously we proceed with factors consisting of three PF's and so on.

Since G/R^+ has at most $log_2(n)$ PF's, it has at most n different divisors. For this reason Lemma 8.2.1 must be applied at most n times. Hence the complexity for finding the PFD of G is bounded by $O(n^6)$. □

Lemma 8.2.3 *If G is a finite, N^+- and N^--connected graph and $G/R = \prod_{i \in I} A_i$ a PFD, then the PFD of G can be found in $O(n^2)$ time, where n is the number of vertices of G.*

Proof. We can proceed as in the case of undirected graphs, compare [19]: Determine all minimal subsets S of $I = \{1, 2, ..., r\}$ so that there are graphs A and B with $G = A \times B$, $A/R = \prod_{j \in S} A_j$ and $B/R = \prod_{j \in J \setminus S} A_j$. From minimality of S we can conclude that A is prime. Now we can extract A and compute the other divisors of G analogously.

Those graphs A and B can be found by the blowing up procedure described in Lemma 13 of [19]. Since $r \leq log_2(n)$, the lemma must be applied at most n times.

To get one R-class of A respectively B, we just have to do one gcd-computation of less than $\sqrt{n}$ natural numbers, because it is not possible that both A and B consist of more than or of exactly $\sqrt{n}$ R-classes if G is not thin. We can get every other R-class of A and B by one division. Clearly the number of divisions is bounded by n.

The effort of the gcd-computation of less than $\sqrt{n}$ natural numbers smaller than n is bounded by $\sqrt{n} * log(n)$. Hence, the total complexity of this procedure is bounded by $O(n^2)$. □

Theorem 8.2.4 *Let G be a finite, thin, N^+- and N^--connected graph. Furthermore we assume that all quotient graphs $T_{i+1} = T_i/R^+$ $(k \geq i \geq 0$ and $T_0 = G)$ that we compute until we arrive at an R^+-thin graph T_k are thin. Then the PFD of G can be computed in $O(n^5)$ time.*

Proof. We apply the relation R^+ until we obtain an R^+-thin graph. The graph we got after the i-th quotient graph computation will be denoted by T_i. Thus $G = T_0$. After a finite number k of R^+-applications we obtain an R^+-thin graph T_k.

Now we have to compute the PFDs of the graphs T_i $(i \in \{0, 1, 2, ..., k\})$. To determine the PFD of T_k in polynomial time (complexity n^5) we use Theorem 6.1.2.

The idea at this point is of course to win the T_i-factorization from the T_{i+1}-factorization ($i \in \{0, 1, 2, ..., k-1\}$). Since

$$T_{i+1} = T_i/R^+ \tag{8.1}$$

we can apply Lemma 8.2.2. It tells us that the complexity for finding the PFD of T_i is bounded by $O(n^4)$.

After at most k ($k \leq n$) steps we know the PFD of G. Thus the complexity of our algorithm is bounded by $O(n^5)$. □

Corollary 8.2.5 *If the first thinness condition of Theorem 8.2.4 is not satisfied, then the PFD of G can still be computed in $O(n^5)$ time.*

Proof. If G is not thin we simply apply the relation R before we start, if necessary, with the R^+-applications. To get the G-decomposition from the G/R-decomposition we just have to apply Lemma 8.2.3. □

Corollary 8.2.5 is our most general result concerning PFDs with respect to directed cardinal products. It gives us a polynomial algorithm to compute the PFD of graphs that are N^+-, N^--connected - those are shown to have a unique PFD by McKenzie (Theorem 4.2.8) - and additionally fulfill the assumption that all quotient graphs in Theorem 8.2.4 are thin.

It is quite clear that all results in the last chapters, especially Theorem 6.1.2, also hold if we replace R^+ by R^-, N^+ by N^- and vice versa. In view of this further theorem we assume that it is possible to prove Theorem 8.2.4 without using the thinness conditions concerning the quotient graphs of G.

Chapter 9

Distinguishing product graphs

9.1 Definitions

Definition 9.1.1 *A labeling* $\ell : V(G) \rightarrow \{1, 2, ..., d\}$ *of a graph* G *is* $d-$ *distinguishing if no nontrivial automorphism of* G *preserves the labeling.*

Definition 9.1.2 *The distinguishing number* $D(G)$ *of a graph* G *is the least integer* d *so that* G *has a* d*-distinguishing labeling.*

This concept was introduced by Albertson and Collins in [2] and has received considerable attention, cf. [4] and [5].

All graphs in this chapter are assumed to be connected. Assuming connectivity is possible without loss of generality, because a graph and its complement have the same automorphism group (and hence equal distinguishing numbers) and because the complement of a disconnected graph is connected.

If a graph has no nontrivial automorphism its distinguishing number is 1. In other words, $D(G) = 1$ for asymmetric graphs. The other extreme, $D(G) = |G|$, occurs if and only if $G = K_n$. This follows from the fact that $D(G) \leq \Delta(G)$ for all graphs $G \neq K_n,\ K_{n,n}$ and C_5 (see [25]).

The Cartesian and the cardinal product of graphs have under certain conditions automorphism groups that are well understood. Hence it is not surprising that there exist many results about distinguishing numbers of product graphs.

It all started with the paper [3] of Bogstad and Cowen in which the distinguishing number of finite hypercubes Q_d was determined, where $Q_d = K_2^d$

and G^r means the rth power of G with respect to the Cartesian product. The result was: $D(Q_2) = D(Q_3) = 3$ and $D(Q_d) = 2$ for $d \geq 4$. In Section 9.2 we show the proof for their result, which will be generalized for arbitrary finite or countable products of K_2 and K_3 with at least four factors.

Then Albertson [1] proved that for a connected prime graph G, $D(G^r) = 2$ for $r \geq 4$ and, if $|V(G)| \geq 5$, then $D(G^r) = 2$ for all $r \geq 3$.

The state of the art for finite Cartesian powers is the following result by Imrich and Klavžar in [22]:

Theorem 9.1.3 *Let $G \neq K_2,\ K_3$ be a connected graph and $k \geq 2$. Then $D(G^k) = 2$.*

9.2 Finite and countable Cartesian products of K_2 and K_3

We start with a repetition of the Cartesian product definition of possibly infinitely many factors. To this end let I be an index set and $G_i,\ i \in I$, be a family of graphs. Then the *Cartesian product*

$$G = \prod_{i \in I}^{\Box} G_i$$

is defined on the set x of all functions $x : i \mapsto x_i,\ x_i \in V(G_i)$, where two vertices $x,\ y$ are adjacent if there exists a $k \in I$ such that $x_k y_k \in E(G_k)$ and $x_i = y_i$ for $i \in I \setminus \{k\}$.

For products of infinitely many nontrivial graphs G_i, we note the first fundamental difference to the finite case. If we have only finitely many factors, then the product is connected if and only if the factors are. If we have infinitely many nontrivial factors, there are vertices that differ in infinitely many coordinates x_i. One cannot connect them by paths of finite length, since the endpoints of every edge differ in just one coordinate. Therefore such products are disconnected and we call the components of G *weak Cartesian products*. To identify a component, it suffices to know a single vertex of it. Thus the weak Cartesian product

$$G = \prod_{i \in I}^{\Box\ a} G_i$$

is the connected component of $G = \prod_{i \in I}^{\Box} G_i$ containing the vertex a. Since we consider (only) countably infinite products, we can identify vertices with sequences, for example: The vertex $x : \mathbb{N} \to \bigcup_{i\in\mathbb{N}} V(G_i),\ i \mapsto x_i \in V(G_i)$ can be identified with the sequence $(x_1, x_2, ...)$.

The goal of this section is to prove $D(H) = 2$, where H is the weak Cartesian product $\prod_{i\in\mathbb{N}}^{\Box\ v_0} P_i$ with $P_i \in \{K_2, K_3\}$, $V(K_i) = \{0, 1, ..., i-1\}$, and $v_0 = (0, 0, ...)$. We begin with the labeling that was used by Bogstad and Cowen to show that the distinguishing number of the n-dimensional hypercube K_2^n is 2 for $n > 3$, since variants of this labeling will also be used to prove new results.

Theorem 9.2.1 (Bogstad and Cowen [3]) *$D(K_2^n) = 2$ for $n > 3$.*

Proof. Given $n \in \mathbb{N},\ n > 3$. We represent the vertices of K_2^n by all $0-1$ vectors of length n, denote the vertex all of whose coordinates are zero by v_0 and the vertices whose first i coordinates are 1 and all the others zero by v_i $(i = 1, 2, ..., n)$. Clearly $v_0v_1v_2...$ is a path P of length n that is isometrically embedded in K_2^n.

(a) We color all vertices of P and $v = (1, 0, 0, ..., 0, 1)$ white, the others black, and claim that this is a distinguishing coloring. The only white vertex with three white neighbors is v_1, thus it is fixed by any color preserving automorphism α. The vertices v, v_0 and v_n are the only white ones, which have exactly one white neighbor. From $n > 3$ we conclude that v_n has the largest distance to v_1 among them. Hence the the vertices v_1, v_2, ..., v_n are fixed by α. But then v_0 is fixed as the antipode of v_n and also v as the only remaining white vertex.

(b) Consider two different vertices x, y of the hypercube that are not on the path P. Suppose they differ in coordinate i: $x(i) = 1 \neq 0 = y(i)$. If they have different distance to v_i, x cannot be mapped on y by α. If they have equal distance to v_i, we know that $d(x, v_{i-1}) = d(x, v_i) + 1 = d(y, v_i) + 1 = d(y, v_{i-1}) + 2$, which means that x and y have different distance to v_{i-1}. Therefore we know again that x cannot be mapped on y by α. Since x and y were arbitrarily chosen, all vertices of K_2^n are fixed by α. $\Box$

The main additional idea of the following corollary is that two fixed vertices in a triangle also fix the third vertex in the triangle. Using this fact we can generalize the result of Bogstad and Cowen to arbitrary finite Cartesian products of K_2-s and K_3-s with more than three factors.

Corollary 9.2.2 *$D(\prod_{i\in S}^{\Box} P_i) = 2$ for $P_i \in \{K_2, K_3\}$ if S is a finite set with $|S| > 3$.*

Proof. $H = \prod_{i \in S}^{\square} P_i$, $|S| = n$. The vertex set of H be the set of all vectors of length n with entries 0, 1 or 2 in the coordinates i with $P_i = K_3$ and entries 0 or 1 in the coordinates j with $P_j = K_2$. The vertices v_0, v_1, ..., v_n and v and the path P be defined as in the proof of Theorem 9.2.1.

We color all vertices of P and $v = (0, 1, 0, 0, ...)$ white, the others black. Then each single vertex of P is fixed by any color preserving automorphism α by the same arguments as in part (a) of the last proof.

Furthermore we define the vertex u_{i_0} for every index i_0 with $P_{i_0} = K_3$ as follows: u_{i_0} is the vertex with $i_0 - 1$ entries 1 in the first $i_0 - 1$ coordinates, 2 in the i_0-th and 0 in the other coordinates. u_{i_0} is fixed, because it is the only common neighbor of v_{i_0-1} and v_{i_0}.

Consider two different vertices x, y of the given product that are not on the path P. Suppose they differ in coordinate i. W.l.o.g. we assume $x(i) = 2 \neq 0 = y(i)$. If they have different distance to u_i, x cannot be mapped on y by α. If they have equal distance to u_i, we know that $d(x, v_{i-1}) = d(x, u_i) + 1 = d(y, u_i) + 1 = d(y, v_{i-1}) + 2$, which means that x and y have different distance to v_{i-1}. Therefore we infer that x cannot be mapped onto y by α. Since x and y were arbitrarily chosen, all vertices of the product are fixed by α. $\square$

Using the fact that connected Cartesian products and nonbipartite connected cardinal products have the same automorphism group (see Imrich [19], Theorem 3) we can formulate one more corollary. The proof of the last corollary works here, too.

Corollary 9.2.3 *$D(\prod_{i \in S} K_3) = 2$ if S is a finite set with $|S| > 3$.*

Now to the main result of the section. It states that the distinguishing number of the weak Cartesian product of K_2-s and K_3-s is 2. The proof extends the preceding ideas.

Theorem 9.2.4 *$D(\prod_{i \in \mathbb{N}}^{\square\, v_0} P_i) = 2$ for $V(K_i) = \{0, 1, ..., i-1\}$, $P_i \in \{K_2, K_3\}$ and $v_0 = (0, 0, ...)$.*

Proof. Given $H = \prod_{i \in \mathbb{N}}^{\square\, v_0} P_i$ as in the statement. The vertex set of H is the set of all sequences with finitely many entries different from 0, where the entries in the coordinates i with $P_i = K_3$ are from the set $\{0, 1, 2\}$ and the other entries are in $\{0, 1\}$. Let the vertices v_1, v_2, ... be defined as in the proof of Theorem 9.2.1 and P be the one-sided infinite path $v_0 v_1 v_2$....

We color all vertices of P white, the others black, and claim that this is a distinguishing coloring.

Every color-preserving automorphism α of H stabilizes P. Since v_0 is the only vertex of degree 1 in P, considered as a subgraph of H, it is fixed by α. But then v_1, as the only neighbor of v_0 in P, is also fixed. In general, each vertex v_i $(i > 0)$ is the only white vertex of distance i to v_0. Thus every v_i must be fixed.

The proof is completed analogously to the proof of Corollary 9.2.2. $\square$

9.3 Products of relatively prime graphs

In this section I list some results of [21] about Cartesian products of relatively prime graphs and adapt it for the cardinal product. The main results of this section, Theorem 9.3.2 and Theorem 9.3.3, assert that the distinguishing number of such products is small provided that the sizes of the factors do not differ too much. The theorems depend essentially on the descriptions of $Aut(G)$ in Theorem 6.1.3 and Theorem 6.1.4.

Lemma 9.3.1 (Imrich, Jerebic and Klavžar [21]) *Let $k \geq 2$, $d \geq 2$, G a connected graph on k vertices, and H a connected graph on $d^k - k + 1$ vertices that is relatively prime to G. Then $D(G\square H) \leq d$.*

Proof. Since G and H are relatively prime every automorphism maps G-layers into G-layers and H-layers into H-layers (see 6.1.3).

Denote the set of vectors of length k with integer entries between 1 and d by $\mathbb{N}_d^k$, and let S be the set of the following $k-1$ vectors from $\mathbb{N}_d^k$:

$$
\begin{array}{l}
(1,1,1,....,1,1,1,2)\\
(1,1,1,....,1,1,2,2)\\
(1,1,1,....,1,2,2,2)\\
\quad\vdots\\
(1,2,2,....,2,2,2,2)
\end{array}
$$

Consider the $d^k - k + 1$ vectors from $\mathbb{N}_d^k \backslash S$ and label the G-layers with them. Then the number of 1's in the H-layers is $d^{k-1} - k + 1$, ..., $d^{k-1} - 1$, d^{k-1}. Hence any label preserving automorphism φ of $G\square H$ preserves these layers individually, so φ can only permute G-layers. But since they are all different, it follows that φ is the identity. Hence, the described labeling is d-distinguishing. $\square$

Theorem 9.3.2 (Imrich, Jerebic and Klavžar [21]) *Let be $k \geq 2$, $d \geq 2$, G and H connected, relatively prime graphs with $k \leq |G| \leq |H| \leq d^k - k + 1$. Then $D(G \Box H) \leq d$.*

Since the automorphism group of directed cardinal products has under certain conditions (Theorem 6.1.4) the same structure as $Aut(G)$ of a Cartesian product graph G we can formulate some analogous theorem:

Theorem 9.3.3 *Let be $k \geq 3$, $d \geq 2$, G and H R^+-thin, N^+-connected, relatively prime, directed graphs with $k \leq |G| \leq |H| \leq d^k - k + 1$. Then $D(G \times H) \leq d$.*

9.4 The distinguishing chromatic number

One interesting variation of the distinguishing number is the *distinguishing chromatic number* $\chi_D(G)$ of a graph G, which was introduced just in 2006 by Collins and Trenk [7]. They determined $\chi_D(G)$ of paths and cycles, but found also upper bounds of $\chi_D(G)$ depending on $\Delta(G)$ for trees and connected graphs in general.

Definition 9.4.1 *Let $G = (V, E)$ be a graph. A chromatic distinguishing coloring on n colors ($n \in \mathbb{N}$) is a distinguishing coloring of G using n colors, s.t. no two adjacent vertices have equal colors.*

Definition 9.4.2 *The distinguishing chromatic number $\chi_D(G)$ of a graph $G = (V, E)$ is the minimum of all $n \in \mathbb{N}$, s.t. G has a chromatic distinguishing coloring on n colors.*

Investigating products of complete graphs Choi, Hartke and Kaul proved in [6] that $\chi_D(Q_n) = 3$ for $5 \leq n < \aleph_0$ and $\chi_D(Q_3) = 4$. In this section we show, completing the investigation of finite and countably infinite hypercubes with respect to the distinguishing chromatic number, $\chi_D(Q_4) = 4$ and $\chi_D(Q_n) = 3$ for $8 \leq n \leq \aleph_0$.

Theorem 9.4.3 *The distinguishing chromatic number of the hypercube of dimension 4 is 4.*

Proof. We label the vertices of the Q_4 with the subsets of the set $\{1, 2, 3, 4\}$ in such a way that adjacent vertices have labels that differ in exactly one of the elements 1, 2, 3, 4. For example the vertices $\{1, 2\}$ and $\{1, 2, 3\}$ are

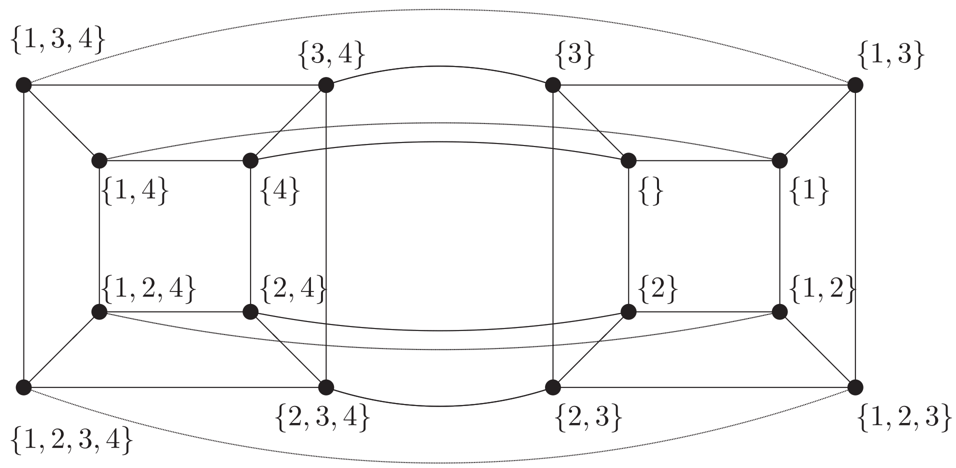

Figure 9.1: My Q_4

adjacent, but not $\{1,2,3\}$ and $\{1,2,4\}$, because they can be distinguished by 3 and 4, see Figure 9.1.

The distance between two vertices in Q_4 is the cardinality of the symmetric difference of their labels. Thus $\{\}$ and $\{1,2,3,4\}$ are antipodal vertices just as $\{1,3\}$ and $\{2,4\}$. The set of vertices of distance i $(0 \leq i \leq 4)$ from $\{\}$ will be called level i and denoted L_i.

It is nice to see that the interchange of two digits, for example 2 and 3, in each label defines an automorphism on Q_4. Such automorphisms are denoted by $\alpha_{(ij)}$ $(1 \leq i < j \leq 4)$, where the digits i and j are interchanged. Similarly $\alpha_{(ij)(kl)}$ denotes the product of $\alpha_{(ij)}$ and $\alpha_{(kl)}$. All those automorphisms preserve all L_i.

It is useful to see that $V_1 \cup V_2$ is the bipartition of Q_4, where $V_1 = \bigcup_{i\in\{1,3\}} L_i$ and $V_2 = \bigcup_{i\in\{0,2,4\}} L_i$. Further we sometimes need that the neighborhood intersection of two vertices in V_i consists of zero or two vertices, which implies that the union of their neighborhoods covers at least six vertices. Furthermore the union of the neighborhoods of three vertices in V_1 or V_2, respectively, covers at least seven vertices in V_2 or V_1, respectively, by the following argument. We can assume by symmetry that $\{\}$ is one of these vertices. A second vertex of V_2 covers at least two additional vertices in L_3 and no two vertices in L_2 have the same neighbors in L_3.

We show that there is no chromatic distinguishing coloring on three colors.

Suppose there is a chromatic distinguishing coloring on three colors, say white, black and green.

At first we wish to show that there is no three-coloring of Q_4, where both parts of the above bipartition consist of three colors: Assume the coloring has this property. No part of the partition can include more than four vertices of one color, because otherwise there would be no place for a vertex of this color in the other part. Clearly there must be one color, say green, with three or four vertices in V_1, but this implies that V_2 contains at most one green vertex. Hence we can assume without loss of generality that there are four white and three black vertices in V_2. Thus there can be at most one white and one black vertex in V_1, contrary to the fact that V_1 consists of eight vertices.

Since it is not possible that one part consists of vertices of three colors and all vertices in the other part have the same color, we always can assume that one part has exactly two colors, say V_2 is colored white and green. Now we just have to check the cases, where (a) V_1 is two- or three-chromatic and (b) V_1 is monochromatic.

(a) Assume $\{\}$ to be green. For symmetry reasons it is sufficient to consider the cases $1 \leq g_2 \leq 4$, where g_i denotes the number of green vertices in V_i ($i \in \{1, 2\}$).

Subcase (i) $g_2 = 1$.

If $g_1 = 0$, all vertices in V_1 must be black, which will be considered in (b).

$1 \leq g_1 \leq 4$:

The green vertices of V_1 must be in L_3. If $g_1 < 3$ we can interchange two white vertices of L_3, otherwise two green vertices, where all colors and levels are preserved.

In detail: If $g_1 = 1$, we can assume that $\{1, 2, 3\}$ is green and $\alpha_{(12)}$ is color preserving. If $g_1 = 2$, we can assume by using some level preserving automorphisms that $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are green, thus $\alpha_{(12)}$ is color preserving again. If $g_1 = 3$, we can assume that $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 3, 4\}$ are green and $\alpha_{(34)}$ is color preserving in this case. If all vertices in L_3 are green, any level preserving automorphism works.

Subcase (ii) $g_2 = 2$.

Then $0 \leq g_1 \leq 2$. $g_1 = 0$ is considered in (b). If $g_1 > 0$, the second green vertex of V_2 must be in level 2 and we can assume that it is $\{1, 2\}$. The green vertices of V_1 are in L_3 and in any case we can find some color preserving automorphism analogously to subcase (i).

Subcase (iii) $g_2 = 3$.

If $\{1,2,3,4\}$ is green, $g_1 = 0$, which will be considered in (b). If there are two green vertices in L_2, two things are possible: They can have distance two as $\{1,2\}$ and $\{1,3\}$. In this case $\{2,3,4\}$ can be green, too, but then $\alpha_{(23)}$ is color preserving.

They can be antipodal as $\{1,2\}$ and $\{3,4\}$, but then $g_1 = 0$.

Subcase (iv) $g_2 = 4$. If there is an antipodal pair of green vertices in V_2, there must be also a white antipodal pair in V_2, but then all vertices in V_1 are black, which will be considered in (b). If there is no antipodal pair of green vertices in V_2, we can assume that $\{1,2\}$, $\{1,3\}$ and $\{1,4\}$ are green. In this case $\{2,3,4\}$ can be green and $\{1\}$ can be white. All other vertices of L_1 are in any case black. Thus $\alpha_{(34)}$ works.

(b) All vertices in V_1 are black.

All subcases ($g_2 = 1,2,3,4$) are analogue to the subcases of (a).

Subcase (i) $g_2 = 1$. We can assume $\{\}$ is the green vertex in L_2. Each $\alpha_{(ij)}$ works then.

Subcase (ii) $g_2 = 2$. If the green vertices are not antipodal we can assume that $\{\}$ and $\{i,j\}$ are green. $\alpha_{(ij)}$ works then. Otherwise we can assume that $\{\}$ and $\{1,2,3,4\}$ are green. Each $\alpha_{(ij)}$ works in this case.

Subcase (iii) $g_2 = 3$. If no two of the three green vertices are antipodal, we can assume $\{\}$, $\{1,2\}$ and $\{1,3\}$ are green. $\alpha_{(23)}$ works then.

If there is an antipodal green pair, we can assume $\{\}$, $\{1,2\}$ and $\{1,2,3,4\}$ are green. $\alpha_{(12)}$ works then.

Subcase (iv) $g_2 = 4$. If no two of the four green vertices are antipodal, we can assume $\{\}$, $\{1,2\}$, $\{1,3\}$ and $\{1,4\}$ are green. $\alpha_{(23)}$ works then.

If there is one antipodal green pair, we can assume $\{\}$, $\{1,2\}$, $\{1,3\}$ and $\{1,2,3,4\}$ are green. $\alpha_{(23)}$ works then.

If there are two antipodal green pairs, we can assume $\{\}$, $\{1,2\}$, $\{3,4\}$ and $\{1,2,3,4\}$ are green. $\alpha_{(12)}$ works then.

Now we know $\chi_D(Q_4) > 3$. To show that $\chi_D(Q_4) = 4$ we draw a chromatic distinguishing coloring on 4 colors, see Figure 9.2.

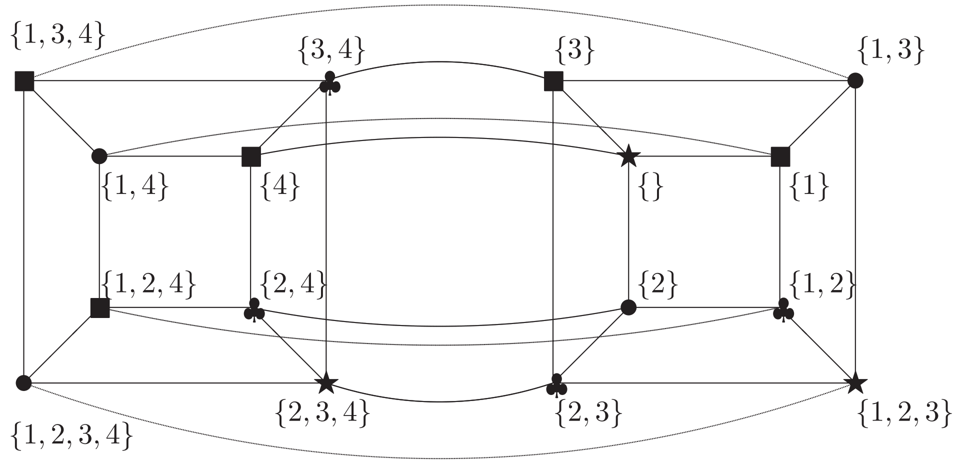

Figure 9.2: $\chi_D(Q_4) \leq 4$

$\{\}$ is the only ★ vertex with no ♣ neighbor, $\{1, 2, 3\}$ is the only ★ vertex with exactly two ♣ neighbors and $\{2, 3, 4\}$ is the only ★ vertex with three ♣ neighbors. Hence the ★ vertices are fixed. Their antipodal vertices $\{4\}$, $\{1\}$ and $\{1, 2, 3, 4\}$ are fixed, too. $\{2\}$ is the only • neighbor of $\{\}$, thus it is fixed as $\{1, 3, 4\}$, its antipode. $\{\}$ fixed implies: Neighbors of $\{\}$ must be mapped on neighbors of $\{\}$. $\{1\}$, $\{2\}$ and $\{4\}$ fixed implies $\{3\}$ and its antipode $\{1, 2, 4\}$ are fixed. Different vertices have different neighborhoods and the neighborhoods of the vertices in level two consist of vertices in level one and three, which are already fixed. From this we conclude that all vertices in level two are fixed, too. □

Lemma 9.4.4 (Choi, Hartke and Kaul [6]) *The distinguishing chromatic number of the 3-cube Q is 4.*

Proof. We label the vertices of the Q_3 $(= Q)$ with the subsets of the set $\{1, 2, 3\}$ analogously to the proof of Lemma 9.4.3.

The distance between two vertices in Q_3 is the cardinality of the symmetric difference of their labels. Thus $\{\}$ and $\{1, 2, 3\}$ are antipodal vertices just as $\{1, 3\}$ and $\{2\}$. The set of vertices of distance i $(0 \leq i \leq 3)$ from $\{\}$ will be called level i and denoted L_i.

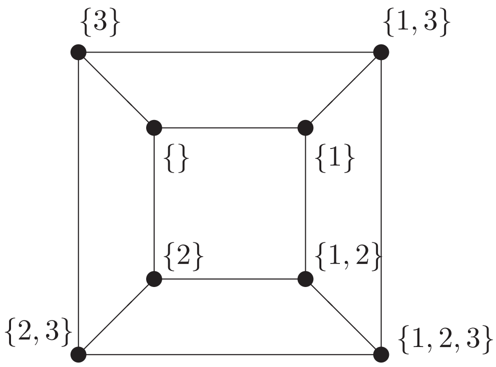

Figure 9.3: My Q_3

The automorphisms $\alpha_{(ij)}$ $(1 \leq i < j \leq 3)$ be defined as in the proof of Theorem 9.4.3. Note that such automorphisms preserve all L_i.

It is useful to see that $V_1 \cup V_2$ is the bipartition of Q, where $V_1 = \bigcup_{i \in \{1,3\}} L_i$ and $V_2 = \bigcup_{i \in \{0,2\}} L_i$.

We show that there is no chromatic distinguishing coloring on three colors.

Suppose there is a chromatic distinguishing coloring on three colors, say white, black and green. Let w, b and g be the numbers of white, black and green vertices, respectively.

Since we have a chromatic coloring w, b and g are less or equal 4. In the case where also $w, b, g < 4$ holds, we can assume without loss of generality $g = b = 3 = w + 1$. One green vertex in V_1 leaves just one place for a green vertex in V_2. Thus all three vertices of one color must be either in V_1 or in V_2 and we can suppose $\{\}$, $\{1,2\}$ and $\{1,3\}$ are green. From $b = 3$ we conclude $\{2,3\}$ is white, hence $\{1\}$ is also white and the other vertices are black. $\alpha_{(23)}$ is color preserving.

If $g = 4$, it is sufficient to consider the cases $b = 1$ and $b = 2$, but both can be checked easily.

To show that $\chi_D(Q) = 4$ we draw a chromatic distinguishing coloring on 4 colors.

$\{\}$ is the only ■ vertex with two ★ neighbors, hence the ■ vertices and their antipodal vertices $\{3\}$ and $\{1,2,3\}$ are fixed. Thus all ★ and • vertices are fixed, but then the ♣ vertices are fixed, too. □

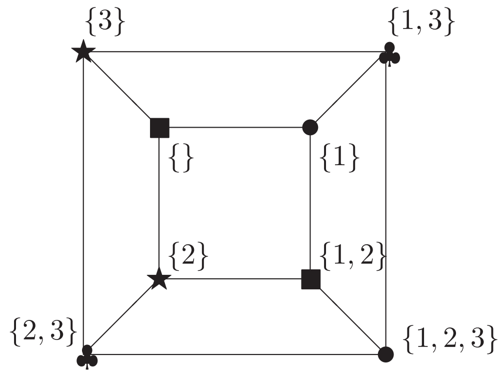

Figure 9.4: $\chi_D(Q_3) \leq 4$

The next theorem is for finite graphs an immediate consequence of a theorem of Choi, Hartke and Kaul, but we give a simple independent proof that works also for the countable infinite hypercube and uses in finite dimension n only $O(n/2)$ vertices of one color.

Theorem 9.4.5 *The distinguishing chromatic number of the hypercube Q_n with $8 \leq n \leq \aleph_0$ is three. There is one color we need only for $O(n/2)$ vertices.*

Proof. (a) n is finite.

We label the vertices with the subsets of $\{1, 2, ..., n\}$. The vertices v_i $(0 \leq i \leq n)$ are defined as $\{1, 2, ..., i\}$ and $v_0, v_1, ..., v_n$ be the path P. The idea is to fix this path as in Lemma 9.2.1. When we have done this, we are ready, because the rest is analogous to part (b) of the proof of Lemma 9.2.1.

V_1 be the set of vertices in Q_n with odd distance to v_0, V_2 the set of those with even distance to v_0 and L_i the set of vertices with distance i to v_0, clearly $V_1 \cup V_2$ is the bipartition of Q_n. The vertices v_i' $(i \in \mathbb{N})$ be defined as $\{1, 2, ..., i-1, i+1\}$.

We color all vertices of P that are in V_2 green ($O(n/2)$), the remaining vertices of V_2 black. Next we color the vertices p and q green, where p is defined as $\{2, 4, 6\}$ and q as $\{4, 6, 8\}$ if $n < 10$, for bigger n as $\{6, 8, 10, ..., 2 * [n/2]\}$ if $[n/2]$ is odd and as $\{8, 10, ..., 2 * [n/2]\}$ if $[n/2]$ is even. This ensures that both, p and q, are in V_1. The other vertices of V_1 be colored white. Neither p nor q has a green neighbor in V_2, so we have a chromatic three-coloring.

The vertex v_0 must be mapped on itself by any color preserving automorphism α, because v_0 and $v_{2*[n/2]}$ are the only green vertices in V_2 that have distance two to exactly one green vertex and there is no green vertex x in V_1 with $d(x, v_{2*[n/2]}) = d(p, v_0) = 3$. But then it is not hard to see that all green vertices of P are fixed by α. Since $d(p, v_2) < d(q, v_2)$, p and q are also fixed .

For odd i we know that v_i and v_i' are the only common neighbors of v_{i-1} and v_{i+1}, hence α maps $\{v_i, v_i'\}$ on itself. The vertices v_i and v_i' have different distance to at least one of the fixed vertices p or q, thus they are fixed by α and therefore the complete path P.

(b) $n = \aleph_0$.

The vertex set of $Q_{\aleph_0}$ be the set of all finite subsets of $\mathbb{N}$. The vertices v_i, v_j' and the vertex sets V_1, V_2 be defined as in (a), the one-sided infinite path $v_0v_1v_2$... will be called P.

We color all vertices of P that are in V_2 green, the other vertices of V_2 black. In V_1 we color the vertices $\{2, 4, 6\}$, $\{8, 10, 12, 14, 16\}$, $\{18, 20, 22, 24, 26, 28, 30\}$, ... green, the remaining vertices white.

It is not hard to see that no two green vertices are adjacent. Since $V_1 \cup V_2$ is the bipartition of $Q_{\aleph_0}$, this is a chromatic three-coloring. The vertex v_0 is the only green vertex to which only one green vertex has distance two, hence it is fixed by any color preserving automorphism α and therefore all green vertices of P. The green vertices of V_1 have pairwise different distance to v_0, thus they are also fixed by α.

The white vertices v_i of P (those with odd index) are fixed, because v_i and v_i' have different distance to one green vertex in V_1 and they are the only common neighbors of v_{i-1} and v_{i+1}, the remaining vertices of G are fixed by the same arguments as in the proof of Lemma 9.2.1. □

9.5 The local distinguishing number

An interesting generalization of the distinguishing number, introduced by Cheng and Cowen [5], is the *ith* (i integer greater zero) *local distinguishing number* of a graph G.

Definition 9.5.1 *Given a graph G. N_v^i is the induced subgraph of G with the vertex set $V(N_v^i) = \{u \in V(G) \mid 0 \le d(u, v) \le i\}$.*

Hence, N_v^1 is the closed neighborhood of the vertex v.

Definition 9.5.2 *Given a graph $G = (V, E)$. A labeling of the vertex set of G, $\Phi : V(G) \to \{1, 2, ..., r\}$ is said to be i-local distinguishing if $\forall u, v \in V(G)$, $u \neq v, N_v^i$ is not isomorphic to N_u^i.*

Definition 9.5.3 *The ith local distinguishing number of a graph G, $LD^i(G)$ is the minimum r s. t. G has an i-local distinguishing labeling that uses r colors.*

Cheng and Cowen proved also the following theorem:

Theorem 9.5.4 *Given C_n, let $k \in \mathbb{R}$ s.t. $n = (k^2(k+1))/2$. Let $r = \lceil k \rceil > 2$.*

(i) If r is odd, $LD^1(C_n) = r$

(ii) If r is even $n \leq (r^2(r+1))/2 - r$, then $LD^1(C_n) = r$; otherwise $LD^1(C_n) = r + 1$

Lemma 9.5.5 $LD^1(C_4 \Box C_4) = 3$

Proof. Note at first that $C_4 \Box C_4 = Q_4$. In $C_4 \Box C_4$ every vertex v has exactly four neighbors. Thus there are five pairwise nonisomorphic labelings consisting of two colors of N_v^1, where the color of v is fixed. (The number of black neighbors of v is between zero and four.) Since two colors can be chosen for v, the total number of pairwise nonisomorphic labelings consisting of two colors of N_v^1 is ten. But $C_4 \Box C_4$ has 16 vertices, hence $LD^1(C_4 \Box C_4) \geq 3$.

The proof is completed by showing that there exists a 1-local distinguishing labeling that uses three colors.

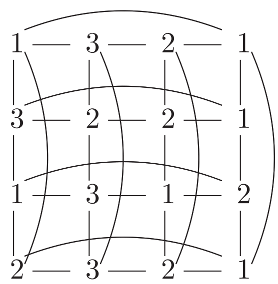

Figure 9.5: A 1-local distinguishing labeling of C_4^2

Chapter 10

Strong graph products

10.1 A local PFD algorithm

We mention just for the sake of completeness that a graph is prime with respect to the strong product, analogous to the other considered products, if it cannot be written as a strong product of two nontrivial graphs. The first, and up to now the only polynomial algorithm to decompose graphs with respect to the strong graph product, is due to Feigenbaum and Schäffer, compare also Theorem 4.2.9:

Theorem 10.1.1 (Feigenbaum, Schäffer [12]) *The PFD ($\boxtimes$) of every finite, connected, undirected graph without loops is unique. Furthermore it can be computed in polynomial time.*

Its main idea is analogous to the idea in our Cartesian skeleton chapter. It uses that G, the given graph, without non-Cartesian edges ($Sk(G)$) has the same or in the worst case a finer PFD with respect to the Cartesian product than G with respect to the strong product. Finer means here that every product factor is the product of some Cartesian factors. More precisely: Say $Sk(G) = \prod_{i\in I}^{\Box} H_i$. Then there exists a partition $I = \dot{\bigcup}_{k=1}^{n} J_k$ of I such that $G = \prod_{k=1}^{\boxtimes\, n} A_k$, where $A_k = \prod_{i\in J_k}^{\boxtimes} H_i$.

Since the decomposition of the skeleton can be computed in linear time now [23], it is quiet clear that the most complex part of the algorithm is the recognition of all non-Cartesian edges. Its effort is bounded by $O(n^5)$ in the number of vertices n. This is a rough bound, because at least one loop in the algorithm runs through all neighbors of some vertex, and in general the number of neighbors cannot be better bounded than by n.

The goal of our investigations is to speed up Feigenbaum's algorithm by some local application of itself, hence, the basic idea is simply to cover the given graph by smaller subgraphs, decompose the subgraphs using the algorithm of [12] into its prime factors and try to suggest from the factors of the subgraphs to global factors. This is possible if the chosen subgraphs are indeed *subproducts*, i.e. they are products of subgraphs of the factors. Appropriate subgraphs are balls:

Given $G = (V, E)$, $v \in V$. The *ball* with radius n and center v, denoted $B_n(v)$, is the subgraph of G induced by the vertices $\{x \in V \mid d(x, v) \leq n\}$.

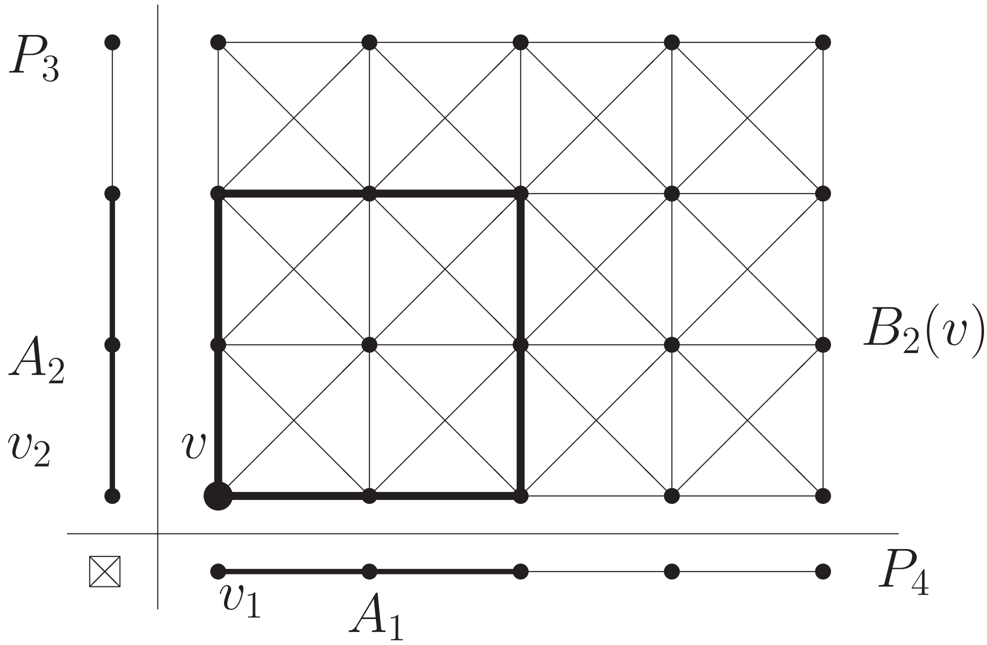

Figure 10.1: $B_2(v) = A_1 \boxtimes A_2$

Figure 10.1 shows that balls in strong products are the product of their projections to the factors and therefore subproducts . The question is, how can we use the factors of balls that are used to cover the given graph G, to suggest to global factors. After investigating a lot of approaches, it seems that working with edge classes is the best way of applying the information about prime factors of balls for an algorithm.

Suppose now that $G = A_1 \boxtimes A_2 \boxtimes ... \boxtimes A_n$ is the PFD of G. The edge class A'_1 is then defined as follows: $A'_1 = \{e \in E(B_1(v)) \mid e \in \text{copy of } A_1\}$, the classes A'_2, ..., A'_n analogously. Computing A'_1, A'_2, ..., A'_n from G in the algorithm means computing the edge classes of G related to its prime factors.

Furthermore we need some thinness condition: Two adjacent vertices x, y are in relation S if they have the same neighborhood and a graph is S-*thin* if no two different vertices are in relation S.

Algorithm 3

Choose $v \in V(G)$ with S-thin $B_1(v)$, $W = V(B_1(v))$
Compute A'_1, A'_2, ..., A'_n from $B_1(v)$ using the algorithm of [12].
While $(W \neq V(G))$
    $x \in W$ with $d(x, V(G) \setminus W) = 1$
    If $B_1(x)$ is S-thin
        $W = W \cup V(B_1(x))$
        Compute A'_{n+1}, A'_{n+2}, ..., A'_{n+m} from $B_1(x)$
        For $k = 1 : m$
            For $j = 1 : n$ {
                If $((A'_j \cap A'_{n+k}) \neq \emptyset)$ $A'_j = A'_j \cup A'_{n+k}$ }
            $A'_{n+k} = \emptyset$
If every single edge class A'_i $(i = 1, 2, \; ...n)$ induces connected subgraphs that are pairwise isomorphic
    If I is a maximal index set such that $A'_i \neq A'_j$ for $i \neq j$.
        $G = \prod_{i \in I}^{\boxtimes} A^*_i$, where A^*_i is a component of A_i.
else
    G is prime.

Description of the algorithm: The algorithm starts with the choice of some arbitrary vertex v and the computation of $B_1(v)$. Thinness of the chosen ball is important, because it ensures uniqueness of the coordinatization within the ball. Then the vertex set of $B_1(v)$ is stored on the set W that is the set of the currently covered vertices. Using Feigenbaum's and Schäffer's algorithm of [12] we compute the prime factors A_1, A_2, ..., A_n of $B_1(v)$ and related edge classes $A'_1 = \{e \in E(B_1(v)) \mid e \in \text{copy of } A_1\}$, A'_2, ..., A'_n, which are by thinness unique up to the order, compare Figure 10.3. It is clear that G admits at most n prime factors, because if G is a product of n factors, then every ball in G has to be also the product of at least n graphs. When the algorithm finishes, which means that the whole graph can be covered by thin balls of radius one, the A'_i will be the edge classes related to a global factor of G.

The algorithm works while $W \neq V(G)$, which means until all vertices are covered. Next we choose some covered vertex x of distance 1 to $V(G) \setminus W$, compute its ball of radius one, decompose it using the algorithm of [12] and determine the edge classes A'_{n+1}, ..., A'_{n+m}, see Figure 10.4. Within the for loops edge classes are compared: To have nonempty intersection means that they are induced by the same global factor. Therefore we merge them in this case and save the result on the class with the lower index as drawn in Figure 10.5 and Figure 10.6. Before increasing the counter of the outer for loop we set the class A'_{n+k} equal $\emptyset$.

In the last part we check for every single A'_i, whether the components of the graph that is induced by the edges in A'_i are pairwise isomorphic. If this holds, then $G = \prod^{\boxtimes}_{i \in I} A^*_i$, where A^*_i is a component of A_i and I a maximal index set such that $A'_i \neq A'_j$ for $i \neq j$. Otherwise G must be prime. These isomorphism tests are less expensive as it seems to be. Suppose A and B are two components of the graph induced by edges in A'_i. Then we only check the mapping that maps the vertex $x \in A$ to $y \in B$ that has the same coordinates with respect to all edge classes A'_j different from A'_i.

Note that $|I|$ is the number of prime factors of G, which can be of course less than n. Therefore it is still possible that G is prime.

Remark: The algorithm can be adapted for the direct product if all balls are products of balls in the factors, which means exactly that all balls are subproducts. In this case we need R-thinness instead of S-thinness, but then we can apply the algorithm of Imrich [19] locally.

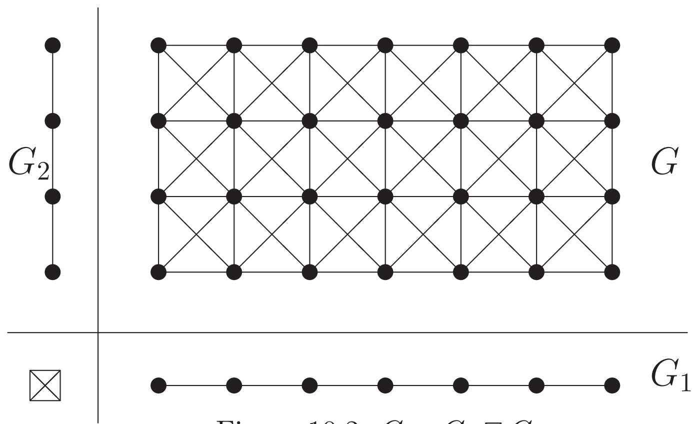

Figure 10.2: $G = G_1 \boxtimes G_2$

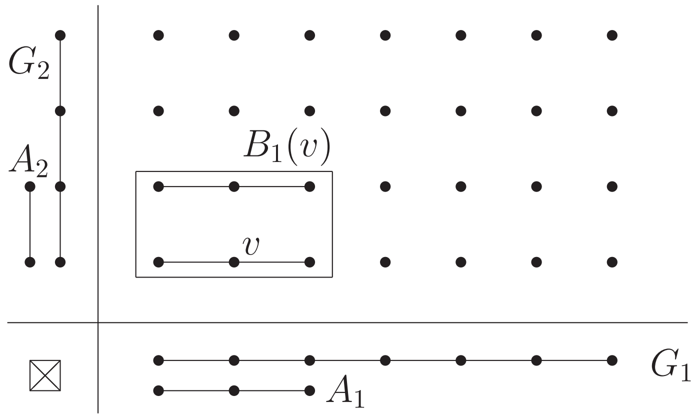

Figure 10.3: Only the edges of A_1' are drawn.

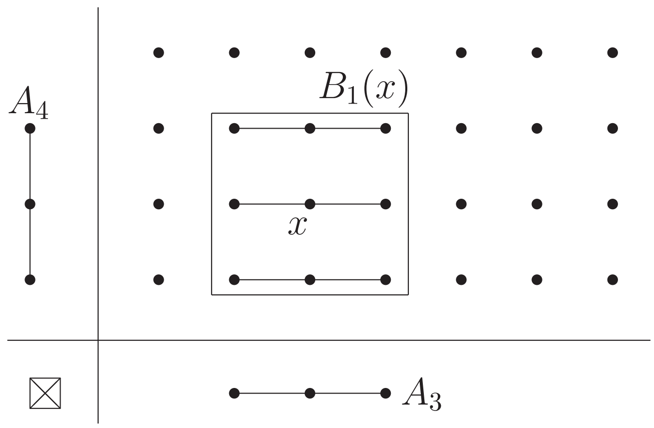

Figure 10.4: The second ball and the edges of A_3'.

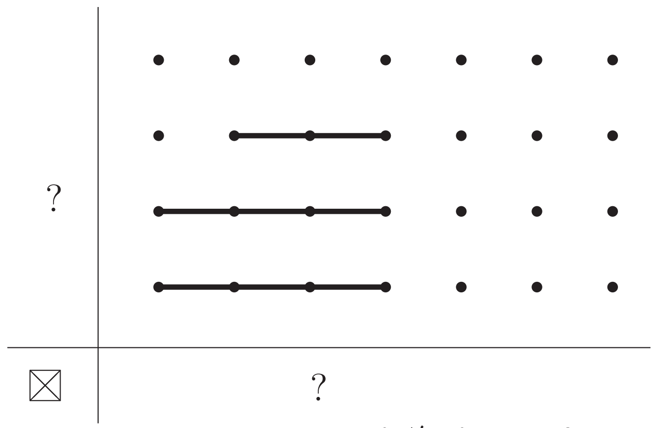

Figure 10.5: The edges of A_1' after the first union.

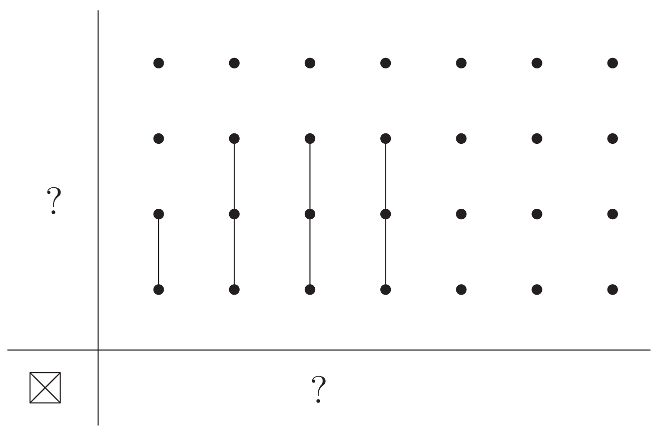

Figure 10.6: The edges of A_2' after the first run through the while loop.

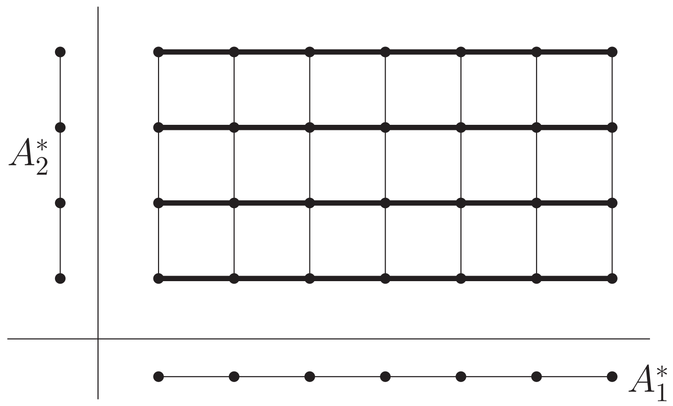

Figure 10.7: When the while loop is finished, A'_1 contains the thick edges, A'_2 the thin ones. Connected components are factors of the given graph.

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