Ziffernsysteme, Tilings und seminumerische Algorithmen

Dissertation[1]

Paul Surer

Betreuer:
Ao.Univ.-Prof. Dipl.-Ing. Dr.techn. Jörg Thuswaldner

Lehrstuhl für Mathematik und Statistik
Department Mathematik und Informationstechnologie
Montanuniversität Leoben

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich diese Arbeit selbstständig verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt und mich auch sonst keiner unerlaubten Hilfsmittel bedient habe.

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Introduction

In a first intention the term *numeration systems* is associated to representing numbers. Indeed, it is already taught in school how to represent numbers in bases alternative to 10. In view of the increasing importance of computers the focus is concentrated on base 2. The idea of numeration systems gives rise to several generalisations. Rényi (see [45]) introduced the so called β-expansions which describe a representation of real numbers by the use of a non-integer base (but still with non-negative integers as digits). After their introduction in 1957 the β-expansions gained interest of other mathematicians and have been investigated since then from many points of view.

Another way of generalisation of the concept of numeration systems was given by Knuth in 1969 (see [38]). He considered the complex plane and showed that $-1+i$ acts as a base for representing complex numbers using digits 0 and 1. This result seems to have started off a great number of research concerning this topic. Kátai and Szabó (see [36]), for example, gave a necessary and sufficient condition for Gaussian integers to be a base for representing complex numbers. Kátai and Kovács (see [34, 35]) added analogous results for real and imaginary quadratic fields. These types of numeration systems were known as canonical number systems. In 1991 Pethő (see [44]) gave a generalised definition by referring to a canonical number system as a numeration system in a residue class ring of a polynomial ring.

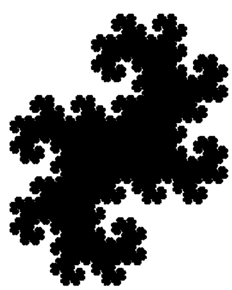

Figure 1: Knuth's Twin Dragon

Although β-expansions and canonical number systems seem to be completely different notions of numeration systems Akiyama *et al.* (see [3]) succeeded in unifying them in 2005 by introducing so-called shift radix systems. It was the initial paper of a series of research papers concerning shift radix systems (*cf.* [6, 7, 8]). But also other mathematicians were interested in shift radix systems and dealt with them.

The present thesis is more or less fully devoted to shift radix systems. At first we give a survey of what is known about them and we are going to improve several results. We will carefully investigate their relation to β-expansions and canonical number systems.

We will also treat fractal tiles induced by β-expansions, canonical number systems and shift radix systems and show some connections between them. As an example, in order to get a rough idea of such fractal tiles, we go back to Knuth's numeration system of complex numbers. Consider the complex numbers that only have a "fractional part" (and therefore "integer part" 0). Represented in the complex plane these numbers give the fractal tile shown in Figure 1. It is known as Twin Dragon and we will meet it again in an example concerning tiles induced by canonical number systems. The two-dimensional real space can be completely covered by the integer translates of the twin dragon such that two different translates do not overlap. Such types of coverings are called tilings. Tiles induced by canonical number systems are known to have this property. For the other two types of tiles this is only conjectured. We will give a lot of examples of tiles that confirm this conjecture.

The notion of shift radix systems can be generalised in several ways. Akiyama and Scheicher (see [12]), for example, defined symmetric shift radix systems. In their definition they differ from shift radix systems only slightly but symmetric shift radix systems behave in some sense much nicer than the original shift radix systems. In the last part of the thesis we will deal with this and other generalisations of shift radix systems.

Lots of the results presented in this theses are contained in papers co-authored by the author of the present thesis. Some of them have already been accepted for publication in international journals (*cf.* [30, 52, 53]). Others will be submitted later (*cf.* [17, 46]).

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Chapter 1

Basic notations and results

In the first chapter we are going to define shift radix systems, canonical number systems and β-expansions. We will give a brief historical background and present results that are of interest for us. Furthermore we are going to outline several problems and open questions and try to explain what new results will be presented in Chapter 2 to Chapter 4 in this context.

1.1 Shift radix systems

1.1.1 Definition

The concept of shift radix systems is rather new, more precisely, it was introduced in 2005 by Akiyama, Borbély, Brunotte, Pethő and Thuswaldner [3] in the following way:

Definition 1.1.1. For an $\mathbf{r} \in \mathbb{R}^d$ let

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} = (x_1, \dots, x_d) \mapsto (x_2, \dots, x_d, -\lfloor \mathbf{r}\mathbf{x} \rfloor).$$

$\tau_{\mathbf{r}}$ is called a shift radix system, SRS for short, if

$$\forall \mathbf{x} \in \mathbb{Z}^d \, \exists k \in \mathbb{N} : \tau_{\mathbf{r}}^k(\mathbf{x}) = \mathbf{0}.$$

The basic idea of the mapping $\tau_{\mathbf{r}}$ is due to Hollander [29] without explicitly defining SRS. He used it for analysing β-expansions. But it turned out that SRS can also be used to describe canonical number systems. We will treat with both, β-expansions and canonical number systems, later.

Define the sets

$$\begin{aligned}
\mathcal{D}_d &:= \left\{ \mathbf{r} \in \mathbb{R}^d \,\middle|\, \forall \mathbf{x} \in \mathbb{Z}^d \, \exists n, l \in \mathbb{N} : \tau_{\mathbf{r}}^k(\mathbf{x}) = \tau_{\mathbf{r}}^{k+l}(\mathbf{x}) \ \forall k \geq n \right\} \text{ and} \\
\mathcal{D}_d^0 &:= \left\{ \mathbf{r} \in \mathbb{R}^d \,\middle|\, \tau_{\mathbf{r}} \text{ is an SRS} \right\}.
\end{aligned}$$

The set $\mathcal{D}_d^0$ is the set that consists of those d-dimensional real vectors $\mathbf{r}$ such that $\tau_{\mathbf{r}}$ is a shift radix system while for a vector $\mathbf{r} \in \mathcal{D}_d$ the sequence $(\tau_{\mathbf{r}}^n(\mathbf{x}))_{n \in \mathbb{N}}$ ends up periodically for all $\mathbf{x} \in \mathbb{Z}^d$. It is easy to see that $\mathcal{D}_d^0 \subseteq \mathcal{D}_d$ holds. Additionally we have

$$\begin{aligned}
(r_0, \dots, r_{d-1}) \in \mathcal{D}_d &\Leftrightarrow (0, r_0, \dots, r_{d-1}) \in \mathcal{D}_{d+1}, \\
(r_0, \dots, r_{d-1}) \in \mathcal{D}_d^0 &\Leftrightarrow (0, r_0, \dots, r_{d-1}) \in \mathcal{D}_{d+1}^0.
\end{aligned} \tag{1.1.1}$$

In Section 3.1 we will show how the map $\tau_{\mathbf{r}}$ can be used to define a radix representation of d-dimensional integer vectors. Furthermore, for $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$, we will associate each $\mathbf{z} \in \mathbb{Z}^d$ to a tile.

The interior of the set $\mathcal{D}_d$ is quite easy to describe (problems occur only for the boundary), while the set $\mathcal{D}_d^0$ has a much more difficult shape even for $d = 2$. We will give a survey in the following subsections.

For an $\mathbf{r} = \{r_0, \dots, r_{d-1}\} \in \mathbb{R}^d$ let

$$R(\mathbf{r}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -r_0 & -r_1 & \cdots & -r_{d-2} & -r_{d-1} \end{pmatrix}.$$

The matrix $R(\mathbf{r})$ plays a very important role for describing SRS. This can easily be understood by observing that for all $\mathbf{x} \in \mathbb{Z}^d$ the point $R(\mathbf{r})\mathbf{x}$ is located "near" $\tau_{\mathbf{r}}(\mathbf{x})$, more precisely, $\tau_{\mathbf{r}}(\mathbf{x}) - R(\mathbf{r})\mathbf{x} = (0, \dots, 0, v)$ with $0 \leq v < 1$. Note that the characteristic polynomial of $R(\mathbf{r})$ is $\chi_{R(\mathbf{r})}(x) = x^d + r_{d-1}x^{d-1} + \cdots + r_1 x + r_0$.

For $\mathbf{r} \in \mathcal{D}_d$ denote by $C(\mathbf{r})$ the set of all points $\mathbf{x} \in \mathbb{Z}$ with $\tau_{\mathbf{r}}^l(\mathbf{x}) = \mathbf{x}$ for some $l > 0$. Note that $C(\mathbf{r}) = \tau_{\mathbf{r}}(C(\mathbf{r}))$ and define $\mathcal{O}(C(\mathbf{r}))$ the set $C(\mathbf{r})/\tau_{\mathbf{r}}$ of orbits of $\tau_{\mathbf{r}}$. $\mathcal{O}(C(\mathbf{r}))$ is obviously a partition of $C(\mathbf{r})$. We will denote elements of $\mathcal{O}(C(\mathbf{r}))$ by small, greek letters. Furthermore define

$$\Pi_d := \bigcup_{\mathbf{r} \in \mathcal{D}_d} \mathcal{O}(C(\mathbf{r})).$$

Definition 1.1.2. Let $\mathbf{r} \in \mathcal{D}_d$. The points of $C(\mathbf{r})$ are called *purely periodic* (with respect to $\tau_{\mathbf{r}}$). An orbit $\pi \in \mathcal{O}(C(\mathbf{r}))$ with $|\pi| = l$ is called a *cycle of period l of $\tau_{\mathbf{r}}$*. We refer to an element of Π_d more generally as a *cycle of $\mathcal{D}_d$*.

A cycle $\pi \in \mathcal{O}(C(\mathbf{r}))$ of period l is a subset of $C(\mathbf{r}) \subset \mathbb{Z}^d$ with l elements which is equipped with a chronological order. We can enumerate its elements as $\mathbf{x}_0, \dots, \mathbf{x}_{l-1}$ such that

$$\mathbf{x}_0 \overset{\tau_{\mathbf{r}}}{\mapsto} \mathbf{x}_1 \overset{\tau_{\mathbf{r}}}{\mapsto} \cdots \overset{\tau_{\mathbf{r}}}{\mapsto} \mathbf{x}_{l-1} \overset{\tau_{\mathbf{r}}}{\mapsto} \mathbf{x}_0.$$

Note that the $\mathbf{x}_i$ are pairwise disjoint. For $j \in \{0, \dots, l-1\}$ let x_j the first component of $\mathbf{x}_j$. By the definition of $\tau_{\mathbf{r}}$ we have

$$\mathbf{x}_j = (x_j, x_{j+1}, \dots, x_{j+d-1})$$

with the indices taken modulo l. Since π is completely determined by the finite integer sequence $(x_0, \dots, x_{l-1})$ we will identify π with this sequence and denote it by $\langle x_0, \dots, x_{l-1} \rangle$. Note that we may rotate the x_j cyclically without changing the orbit, *i.e.*, for all $j \in \{1, \dots, l\}$ we have

$$\langle x_0, \dots, x_{l-1} \rangle = \langle x_j, \dots, x_{l-1}, x_0, \dots, x_{j-1} \rangle.$$

Orbits are marked by the brackets $\langle \cdot \rangle$. Note that $\langle 0 \rangle$ is a cycle for any $\mathbf{r} \in \mathcal{D}_d$. We will refer to it as trivial cycle. By the above mentioned pairwise disjointness of the vectors $\mathbf{x}_0, \dots, \mathbf{x}_{l-1}$ we immediately get

Lemma 1.1.3. *Let $\mathbf{r} \in \mathcal{D}_d$ and $\langle x_0, \dots, x_{l-1} \rangle \in \mathcal{O}(C(\mathbf{r}))$. Set $\mathbf{x}_k = (x_k, x_{k+1}, \dots, x_{k+d-1})^T$ (indices are taken modulo l). Then $\mathbf{x}_i \neq \mathbf{x}_j$ for $i \not\equiv j \bmod l$.*

1.1.2 The set $\mathcal{D}_d$

In the following we summarise what is known about $\mathcal{D}_d$. A full description of its interior was given in [3]. Denote by $\varrho(A)$ the spectral radius of a matrix A.

Theorem 1.1.4 (*cf.* [3]). *Let $d \in \mathbb{N}$. If $\mathbf{r} \in \mathcal{D}_d$ then $\varrho(R(\mathbf{r})) \leq 1$. On the other hand $\varrho(R(\mathbf{r})) < 1$ for some $\mathbf{r} \in \mathbb{R}^d$ implies $\mathbf{r} \in \mathcal{D}_d$. Moreover, for the boundary of $\mathcal{D}_d$, we have*

$$\partial\mathcal{D}_d = \{\mathbf{r} \in \mathbb{R}^d | \, \varrho(R(\mathbf{r})) = 1\}.$$

It is now obvious to ask for the structure of the set of all $\mathbf{r}$ with $\varrho(R(\mathbf{r})) < 1$, more precisely, for a characterisation of the set

$$\mathcal{E}_d := \{\mathbf{r} \in \mathbb{R}^d | \, \varrho(R(\mathbf{r})) < 1\}.$$

[3, Proposition 4.9] provides such a characterisation. It is based on results of Schur [50] and Takagi [54]. For that reason $\mathcal{E}_d$ is often referred to as Schur-Takagi-region.

Lemma 1.1.5 (Schur-Takagi). *Let $d \in \mathbb{N}$. For $0 \le k < d$ define*

$$\delta_k(x_0, \dots, x_{d-1}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & x_0 & \cdots & \cdots & x_k \\ x_{d-1} & \ddots & \ddots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ x_{d-k} & \cdots & x_{d-1} & 1 & 0 & \cdots & 0 & x_0 \\ x_0 & 0 & \cdots & 0 & 1 & x_{d-1} & \cdots & x_{d-k} \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & \vdots & \ddots & \ddots & x_{d-1} \\ x_k & \cdots & \cdots & x_0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{2(k+1)\times 2(k+1)}.$$

Then

$$\mathcal{E}_d = \left\{(x_0, \dots, x_{d-1}) \in \mathbb{R}^d \,|\, \forall k \in \{0, \dots, d\} : \det\left(\delta_k(x_0, \dots, x_{d-1})\right) > 0 \, \forall k \in \{0, \dots, d\}\right\}.$$

For $\mathbf{r} \in \mathbb{R}^d$ denote by $\lambda_1, \dots, \lambda_d$ the d (not necessarily distinct) roots of $\chi_{R(\mathbf{r})}$. It is easy to see that $\mathbf{r} \in \mathcal{E}_d$ if and only if $|\lambda_i| < 1$ for all $i \in \{1, \dots, d\}$ and $\mathbf{r} \in \partial(\mathcal{E}_d)$ if and only if $|\lambda_i| \le 1$ for all $i \in \{1, \dots, d\}$ and equality holds for at least one index.

For small d we have

$$\begin{aligned} \mathcal{E}_1 =& \{x \in \mathbb{R} | \; |x| < 1\}, \\ \mathcal{E}_2 =& \{(x, y) \in \mathbb{R}^2 | \; |x| < 1, |y| < x + 1\}, \\ \mathcal{E}_3 =& \{(x, y, z) \in \mathbb{R}^3 | \; |x| < 1, |y - xz| < 1 - x^2, |x + z| < y + 1\}. \end{aligned} \tag{1.1.2}$$

$\mathcal{E}_1$ equals the interval $(-1, 1)$ and $\mathcal{E}_2$ is an open triangle. Figure 1.1 shows the shape of $\mathcal{E}_3$. Huszti, Scheicher, Surer and Thuswaldner [30] observed that some problems occur with the closure of $\mathcal{E}_d$. The first intention to obtain $\overline{\mathcal{E}_d}$ would be to change all the strict inequalities to not strict ones. This obviously works for $d = 1$ and $d = 2$ but it is definitely wrong for $d = 3$. Denote by E_3 the set that we get by exchanging all "$<$" by "$\le$" in the definition of $\mathcal{E}_3$. Then all points of the shape $(1, y, y)$ and $(-1, y, -y)$ for $y \in \mathbb{R}$ are elements of E_3. E_3 can therefore not be equal to $\overline{\mathcal{E}_3}$ since $\mathcal{E}_3$ is bounded. We only have $\overline{\mathcal{E}_3} \subset E_3$. The authors also provided an explicit parametrisation of $\overline{\mathcal{E}_3}$ which we will present in Section 4.3.

According to Theorem 1.1.4 we can "estimate" $\mathcal{D}_d$ with the help of $\mathcal{E}_d$ in the following way:

$$\mathcal{E}_d \subset D_d \subset \overline{\mathcal{E}_d}. \tag{1.1.3}$$

Furthermore we have

Theorem 1.1.6 (*cf.* [3, Theorem 4.10]). *$\mathcal{D}_d$ is Lebesgue measurable and $\mu_d(\mathcal{D}_d) = \mu_d(\mathcal{E}_d)$ where μ_d denotes the d-dimensional Lebesgue measure.*

Remark 1.1.7. One can show that for $d = 1, \dots, 5$ $\mu_d(\mathcal{E}_d)$ equals $2, 4, \frac{16}{3}, \frac{64}{9}, \frac{1024}{135}$, respectively. A general formula is still outstanding.

We see that the above theory does not tell anything about the set $\mathcal{D}_d \cap \partial\mathcal{E}_d$, *i.e.*, we do not know which points of the boundary of $\mathcal{D}_d$ belong to $\mathcal{D}_d$. One easily checks that $\mathcal{D}_1 = [-1, 1]$ but

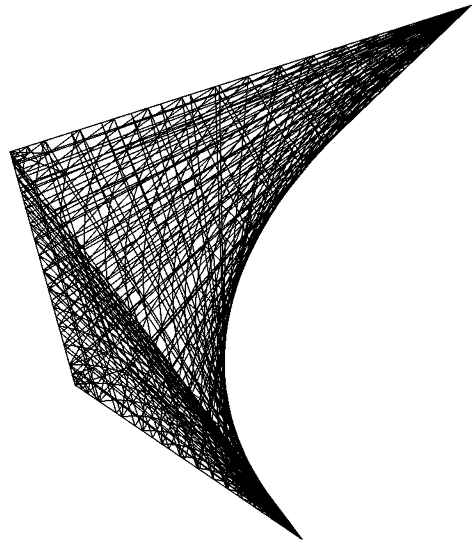


Figure 1.1: The set $\mathcal{E}_3$

for higher dimensions it turned out that this is a very hard problem and even $\mathcal{D}_2 \cap \partial\mathcal{E}_2$ is not fully characterised yet. There exist only partial results. Define

$$\begin{aligned} L_1 =& \{(x, 1+x) \,|\, x \in [0,1)\}, \\ L_2 =& \{(-x, -1+x) \,|\, x \in [0,1)\}, \\ L_3 =& \{(x, 1+x) \,|\, x \in [-1,0)\}, \\ L_4 =& \{(1, y) \,|\, y \in (-2,2)\}. \\ L_5 =& \{(x, -1-x) \,|\, x \in (0,1]\}. \end{aligned} \tag{1.1.4}$$

Note that $\bigcup_{i=1}^{5} L_i \cup \{(1,2)\} = \partial\mathcal{E}_2$. We then have

Theorem 1.1.8 (*cf.* [6, Theorem 2.1]).

$$L_1 \cup L_2 \cup L_3 \cup \{(1,1),(1,0),(1,-1)\} \subset \mathcal{D}_2$$

$$\big(L_5 \cup \{(1,2)\}\big) \cap \mathcal{D}_2 = \emptyset.$$

The second part of the theorem is quite easy to show (by giving a counterexample). In Subsection 2.1.1 we will give an alternative proof of $L_1, L_2 \subset \mathcal{D}_2$ which seems to be more elegant than the original one. For the line L_4 it is very difficult to verify which points belong to $\mathcal{D}_2$ (except for the integer points). It is conjectured that the whole line is a subset of $\mathcal{D}_2$. The problem was thoroughly studied in [4, 5]. We state the main results from these papers.

Theorem 1.1.9 (*cf.* [4, 5]). *$(1,\alpha) \in \mathcal{D}_2$ for $\alpha \in \{\frac{\pm 1 \pm \sqrt{5}}{2}, \pm\sqrt{2}, \pm\sqrt{3}\}$.*

Summarising the above results gives

Theorem 1.1.10. *Let $A = \{0, \pm 1, \frac{\pm 1 \pm \sqrt{5}}{2}, \pm\sqrt{2}, \pm\sqrt{3}\}$.*

$$\mathcal{E}_2 \cup L_1 \cup L_2 \cup L_3 \cup \{(1,\alpha) \,|\, \alpha \in A\} \subseteq \mathcal{D}_2 \subseteq \overline{\mathcal{E}_2} \setminus (L_5 \cup \{(1,2)\}).$$

Figure 1.2 shows the shape of $\mathcal{D}_2$. The grey parts of the right boundary are up to now only conjectured to be part of $\mathcal{D}_2$. Concerning higher dimensions the only result seems to be contained in Kirschenhofer *et al.* [37]. There the point $\mathbf{r} := \left(1, \frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right) \in \partial\mathcal{E}_3$ is analysed and it is shown that $\mathbf{r} \notin \mathcal{D}_3$.

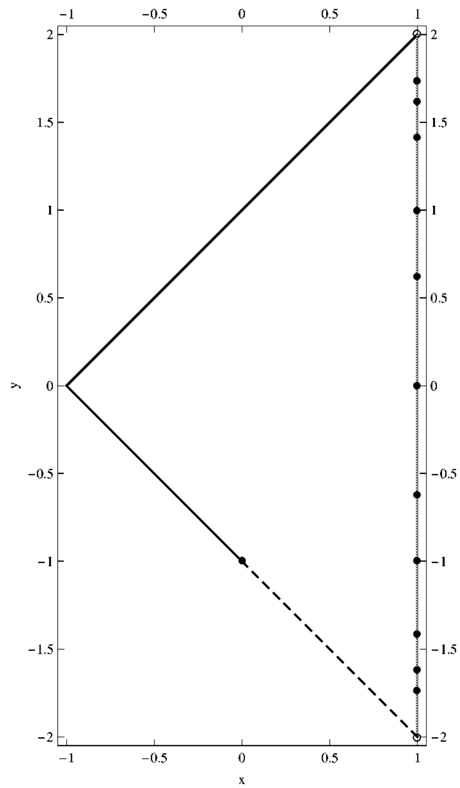


Figure 1.2: The set $\mathcal{D}_2$

1.1.3 The set $\mathcal{D}_d^0$

The characterisation of $\mathcal{D}_d^0$ is a quite difficult task, apart from $d = 1$ where it is easy to see that $\mathcal{D}_1^0 = [0, 1)$. We give an overview of the most important results on $\mathcal{D}_d^0$ in this subsection.

Lemma 1.1.11 (*cf.* [3, Lemma 4.2]). *Let* $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$. *Then* $C(\mathbf{r})$ *is a finite set.*

We obviously have $\mathbf{r} \in \mathcal{D}_d^0 \Leftrightarrow C(\mathbf{r}) = \{\mathbf{0}\}$. In order to analyse $\mathcal{D}_d^0$ we argue as follows: given some potential orbit $\pi := \langle x_0, \ldots, x_{l-1} \rangle$, $\pi \neq \langle 0 \rangle$, we ask whether π is a cycle of $\mathcal{D}_d$. In particular, we search for all $\mathbf{r} \in \mathcal{D}_d$ with $\pi \in \mathcal{O}(C(\mathbf{r}))$. By the definition of $\tau_{\mathbf{r}}$ such an $\mathbf{r} = (r_0, \ldots, r_{d-1})$ must satisfy the system of l double inequalities

$$\begin{array}{lccccccccc} 0 \leq & x_0 r_0 & + & x_1 r_1 & + & \cdots & + & x_{d-1} r_{d-1} & + & x_d & < 1, \\ \vdots & \vdots & & \vdots & & & & \vdots & & \vdots & \vdots \\ 0 \leq & x_{l-1} r_0 & + & x_l r_1 & + & \cdots & + & x_{l+d-2} r_{d-1} & + & x_{l+d-1} & < 1 \end{array} \qquad (1.1.5)$$

(with the indices of x taken modulo l). Set

$$P_d(\pi) := \{(r_0, \ldots, r_{d-1}) \in \mathbb{R}^d \,|\, r_0, \ldots, r_{d-1} \text{ satisfy } (1.1.5)\}.$$

In general $P_d(\pi)$ is a polyhedron. This polyhedron is not necessarily a d-dimensional solid. It may degenerate to some lower dimension or even be equal to the empty set. On the other hand $P_d(\pi)$ is not necessarily bounded. Obviously π is a cycle of $\mathcal{D}_d$ if and only if $P_d(\pi) \cap \mathcal{D}_d \neq \emptyset$. According to [3] we call π a *non-degenerated cycle* if π is a cycle and $P_d(\pi)$ is a non-degenerated d-dimensional polyhedron. Observe that $\mathbf{r} \in P_d(\pi) \cap \mathcal{D}_d$ if and only if $\pi \in \mathcal{O}(C(\mathbf{r}))$. Hence we have the identity

$$\mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi \in \Pi_d \setminus \{\langle 0 \rangle\}} P_d(\pi)$$

which shows that $\mathcal{D}_d^0$ can be obtained by cutting out polyhedra from $\mathcal{D}_d$. For this reason we often refer to $P_d(\pi)$ for some cycle π as *cutout polyhedron.* The difficulty of the characterisation of $\mathcal{D}_d^0$ is now clear: the set of all cycles is a priori infinite. In [3] the authors found an infinite family of cycles of $\mathcal{D}_2$ providing pairwise disjoint non-empty cutout polyhedra. In [52] a second family of cycles with the same property was discovered. Together with (1.1.1) this shows that the set Π_d definitely is infinite for $d \geq 2$. We will give a full analysis of both of these families in Subsection 2.2.4. So the above representation is only a theoretical one. But it shows together with Theorem 1.1.6 the Lebesgue measurability of $\mathcal{D}_d^0$. We also have a connection between $P_d(\pi)$ and $P_{d+1}(\pi)$.

Theorem 1.1.12 (Lifting Theorem, *cf.* [3, Theorem 6.2])**.** *Let π a non-degenerated cycle of $\mathcal{D}_d$. Then π is a non-degenerated cycle of $\mathcal{D}_{d+1}$.*

One can see that it is much easier to prove that a point does not belong to $\mathcal{D}_d^0$ than that it does. A solution of this problem was presented in [3]. The authors stated an algorithm that returns a finite set of cycles Π_Q for a sufficiently small $Q \subset \mathcal{D}_d$ such that

$$Q \cap \mathcal{D}_d^0 = Q \setminus \bigcup_{\pi \in \Pi_Q} P_d(\pi).$$

It is based on an idea of Brunotte [22] and therefore often referred to as Brunotte Algorithm. With the aid of the Brunotte Algorithm it was possible to show prove for several sets that they are subsets of $\mathcal{D}_d^0$. Concretely we have

Theorem 1.1.13 (*cf.* [6, Theorem 3.3])**.** $(r_0, \dots, r_{d-1}) \in \mathcal{D}_d^0$ *if* $r_i \geq 0$ *for* $i \in \{0, \dots, d-1\}$ *and* $\sum_{i=0}^{d-1} r_i < 1$.

Theorem 1.1.14 (*cf.* [6, Theorem 3.4])**.** $(r_0, \dots, r_{d-1}) \in \mathcal{D}_d^0$ *if* $\sum_{i=0}^{d-1} |r_i| < 1$, *there exists exactly one* $k \in \{1, \dots, d\}$ *with* $r_{d-k} < 0$ *and* $\sum_{i=1}^{\lfloor \frac{d}{k} \rfloor} r_{d-ki} \geq 0$.

Theorem 1.1.15 (*cf.* [6, Theorem 3.5])**.** $(r_0, \dots, r_{d-1}) \in \mathcal{D}_d^0$ *if* $0 \leq r_1 \leq \dots, \leq r_d < 1$.

The Brunotte Algorithm was improved in [52]. We will state and discuss this whole theory in Subsection 2.1.2. In Subsection 2.1.4 we are going to present ideas of a computational implementation of the Brunotte algorithm.

1.1.4 About $\mathcal{D}_2^0$

In order to get a first rough idea where $\mathcal{D}_2^0$ is located inside $\mathcal{D}_2$ we first note that

$$\begin{aligned} P_2(\langle 1,0\rangle) =& [-1,0) \times [0,1), \\ P_2(\langle 1\rangle) =& \{(x,y) \in \mathbb{R}^2 | \, x \in \mathbb{R}, -x-1 \leq y < -x\}. \end{aligned}$$

This was already observed in [3]. Akiyama *et al.* [6] started to analyse $\mathcal{D}_2^0$ explicitly. First they showed that $\mathcal{D}_2^0$ has empty intersection with $\partial\mathcal{E}_2$.

Theorem 1.1.16 (*cf.* [6, Corollary 2.5])**.**

$$\mathcal{D}_2^0 \subset \mathcal{E}_2.$$

It would be interesting if this holds for higher dimensions, too. Hereafter wide areas of $\mathcal{D}_2$ were investigated in order to decide whether they belong to $\mathcal{D}_2^0$. The result is depicted in Figure 1.3. The dark grey areas do not belong to $\mathcal{D}_2^0$ where the sets

$$\begin{aligned} E_1 =& \{(x,y) \in \mathbb{R}^2 | \, x \leq 1, y < 2x, 2y+3 \leq 3y\}, \\ E_2 =& \{(x,y) \in \mathbb{R}^2 | \, x \leq 1, x+2 < 2y, y < 2x, 3y < 2x+3\}, \\ E_3 =& \{(x,y) \in \mathbb{R}^2 | \, x \leq 1, 1 \leq y+2x, 2y+x < 0\}. \end{aligned} \tag{1.1.6}$$

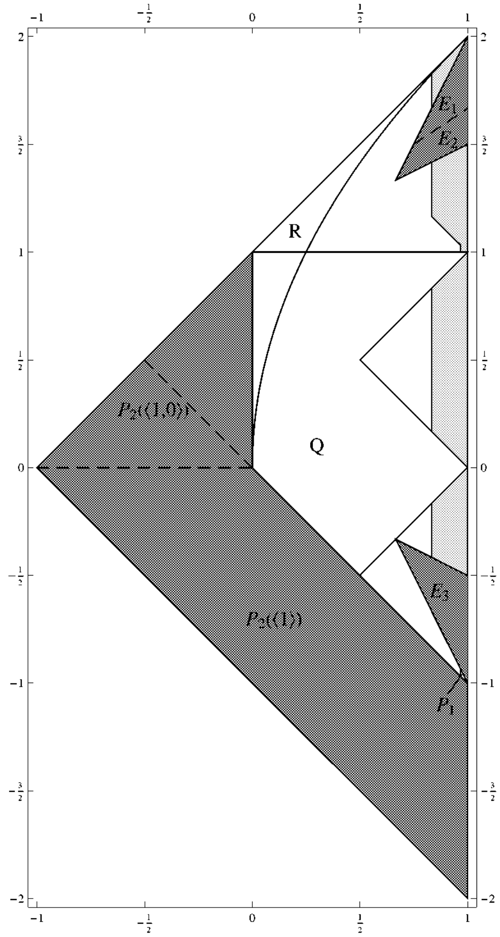


Figure 1.3: Several subsets of $\mathcal{D}_2^0$

correspond to several cycles. In particular

$$\begin{aligned}
E_1 =& P_2(\langle 1,-2,3,-3,3,-2,1\rangle) \cap \overline{\mathcal{D}_2},\\
E_2 =& P_2(\langle 3,-2,1,1,-2\rangle) \cap \overline{\mathcal{D}_2},\\
E_2 =& P_2(\langle 2,1,-2,-2,1\rangle) \cap \overline{\mathcal{D}_2}.
\end{aligned}$$

The white areas do belong to $\mathcal{D}_2^0$. For the area labelled by Q in Figure 1.3 this is an immediate consequence of Theorem 1.1.13 - Theorem 1.1.15. The other white areas, except for P_1 and R, have been treated with the Brunotte Algorithm. As we will see this algorithm does not work properly for sets near the boundary of $\mathcal{D}_d$. Thus the authors developed another algorithm for analysing the set

$$R = \left\{(x,y) \in \mathbb{R}^2 | 0 < x < y^2/4, 0 < y < x+1\right\} \tag{1.1.7}$$

which is located near the upper boundary of $\mathcal{D}_2$. Note that in [6] R was not fully characterised. An area on the right, which is that small that it cannot be recognised properly in the figure, has been left uninvestigated. Finally the set

$$P_1 = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{29}{30} \le x < 1, -x \le y < -2x+1\right\}$$

was shown to be a subset of $\mathcal{D}_2^0$ by a direct observation of the behaviour of the orbits. The proof is quite long and technical. Altogether the authors fully characterised $\mathcal{D}_2^0$ for $x \le \frac{5}{6}$.

In [52] the strategies presented in [6] were improved and most of the up to this point uninvestigated areas (depicted in light grey) have been treated. We will present these methods and results in Section 2.2.

1.1.5 Topological properties of $\mathcal{D}_d^0$

Akiyama *et al.* [3] gave the following definition:

Definition 1.1.17. Let $\mathbf{r} \in \overline{\mathcal{D}_d} = \overline{\mathcal{E}_d}$.

- $\mathbf{r}$ is called a *regular point* if there exists an open neighbourhood U of $\mathbf{r}$ such that $U \cap \mathcal{D}_d$ intersects with only finitely many cutout polyhedra.
- $\mathbf{r}$ is called a *weak critical point* if for each open neighbourhood U of $\mathbf{r}$ the set $U \cap \mathcal{D}_d$ intersects with infinitely many cutout polyhedra.
- $\mathbf{r}$ is called a *critical point* if for each open neighbourhood U of $\mathbf{r}$ the set $(U \cap \mathcal{D}_d) \setminus \mathcal{D}_d^0$ cannot be covered by finitely many cutout polyhedra.

It is easy to see that each critical point is a weak critical point. With the aid of the previously mentioned family of cycles and the Lifting Theorem 1.1.12 the authors of [3] were able to show the point $K_d^{(1)} := (0, \dots, 0, 1, 0) \in \partial\mathcal{D}_d$ for $d \geq 2$ to be critical. The existence of critical points immediately implies that $\mathcal{D}_d^0$ cannot be obtained by cutting only finitely many polyhedra from $\mathcal{D}_d$. In Subsection 2.2.5 we will show the point $K_d^{(2)} := (0, \dots, 0, 1, 1) \in \partial\mathcal{D}_d$ to be critical. It is conjectured that these two points are the only critical points in the two dimensional case. A general characterisation of the critical points is still outstanding. It is not even known whether there exists only finitely many of them.

Up to now it is unknown whether $\mathcal{D}_d^0$ is connected for $d \geq 2$. In Subsection 2.2.5 we will show that $\left(\frac{40}{41}, \frac{30}{41}\right)$ is a cutpoint of $\mathcal{D}_2^0$.

1.1.6 Modifications of shift radix systems

Akiyama and Scheicher [12] presented a modification of SRS.

Definition 1.1.18 (*cf.* [12])**.** Let $d \geq 1$ be an integer, $\mathbf{r} \in \mathbb{R}^d$ and define

$$\tilde{\tau}_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \quad \mathbf{a} = (a_1, \dots, a_d) \mapsto \left(a_2, \dots, a_d, -\left\lfloor \mathbf{ra} + \frac{1}{2} \right\rfloor\right). \tag{1.1.8}$$

Then $\tilde{\tau}_{\mathbf{r}}$ is called a *symmetric shift radix system* (SSRS for short), if

$$\forall \mathbf{a} \in \mathbb{Z}^d \quad \exists n \in \mathbb{N} : \tilde{\tau}_{\mathbf{r}}^n(\mathbf{a}) = \mathbf{0}.$$

Observe that the only difference between $\tau_{\mathbf{r}}$ and $\tilde{\tau}_{\mathbf{r}}$ is just the additional summand "$+\frac{1}{2}$" inside the floor function. Analogously the sets

$$\begin{aligned} \tilde{\mathcal{D}}_d &:= \left\{\mathbf{r} \in \mathbb{R}^d \,\middle|\, \forall \mathbf{a} \in \mathbb{Z}^d \, \exists n, l \in \mathbb{N} : \tilde{\tau}_{\mathbf{r}}^k(\mathbf{a}) = \tilde{\tau}_{\mathbf{r}}^{k+l}(\mathbf{a}) \; \forall k \geq n\right\} \text{ and} \\ \tilde{\mathcal{D}}_d^0 &:= \left\{\mathbf{r} \in \mathbb{R}^d \,\middle|\, \tilde{\tau}_{\mathbf{r}} \text{ is an SSRS}\right\} \end{aligned}$$

where defined. The authors found a lot of analogies between SSRS and SRS. Most of the definitions, notations and results can be adapted without any difficulties. We also have

$$\mathcal{E}_d \subset \tilde{\mathcal{D}}_d \subset \overline{\mathcal{E}_d}.$$

Note that $\tilde{\mathcal{D}}_d \cap \partial\mathcal{E}_d$ is not expected to be equal to $\mathcal{D}_d \cap \partial\mathcal{E}_d$. However, $\tilde{\mathcal{D}}_d \cap \partial\mathcal{E}_d$ has not been analysed yet. It is very remarkable that it was possible to give a full characterisation of $\tilde{\mathcal{D}}_2^0$. Let

$$D = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x \leq \frac{1}{2}, -x - \frac{1}{2} \leq y \leq x + \frac{1}{2}\right\} = \frac{1}{2}\overline{\mathcal{E}_2}$$

and

$$J_1 = \left\{ \left(t, -t - \frac{1}{2} \right) \middle| -\frac{1}{2} \leq t \leq \frac{1}{2} \right\},$$
$$J_2 = \left\{ \left(\frac{1}{2}, y \right) \middle| \frac{1}{2} < y < 1 \right\}.$$

Theorem 1.1.19 (*cf.* [12, Theorem 5.2])**.** $\tilde{\mathcal{D}}_2^0 = D \setminus (J_1 \cup J_2)$.

Figure 1.4 shows the set $\tilde{\mathcal{D}}_2^0$.

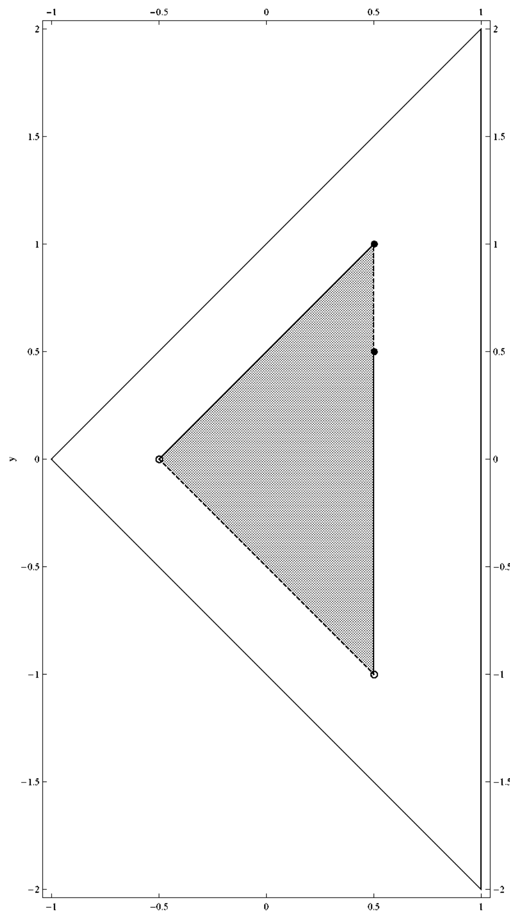


Figure 1.4: The set $\tilde{\mathcal{D}}_2^0$

In [30] a full characterisation of $\tilde{\mathcal{D}}_3^0$ was given. This set turned out to be the composition of three convex polyhedra where some parts of the boundary are excluded. We will present it in Section 4.3.

In [53] a further generalisation, so called ε-Shift Radix Systems, was given which unifies both, SRS and SSRS.

Definition 1.1.20. For an $\varepsilon \in [0, 1)$ and an $\mathbf{r} \in \mathbb{R}^d$ let

$$\tau_{\mathbf{r},\varepsilon} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{z} = (z_0, \ldots, z_{d-1}) \mapsto (z_1, \ldots, z_{d-1}, -\lfloor \mathbf{r}\mathbf{z} + \varepsilon \rfloor).$$

$\tau_{\mathbf{r},\varepsilon}$ is called an *ε-shift radix system* (ε-SRS for short) if for each $\mathbf{z} \in \mathbb{Z}^d$ there exists a $k \in \mathbb{N}$ such that $\tau^k_{\mathbf{r},\varepsilon}(\mathbf{r}) = \mathbf{0}$. Further define

$$\mathcal{D}_{d,\varepsilon} := \left\{\mathbf{r} \in \mathbb{R}^d \,\middle|\, \tau_{\mathbf{r},\varepsilon}(\mathbf{z}) \text{ is ultimately periodic for all } \mathbf{z} \in \mathbb{Z}^d\right\} \text{ and}$$
$$\mathcal{D}^0_{d,\varepsilon} := \left\{\mathbf{r} \in \mathbb{R}^d \,\middle|\, \tau_{\mathbf{r},\varepsilon} \text{ is an } \varepsilon\text{-SRS}\right\}.$$

We immediately see that 0-SRS correspond to the usual SRS while $\frac{1}{2}$-SRS correspond to symmetric shift radix systems. Note that defining modified ε-shift radix systems $\tau^*_{\mathbf{r},\varepsilon}$ with a ceiling function instead of the floor function does not yield a new dynamical system since for each integer vector $\mathbf{z} \in \mathbb{Z}^d$ we had $\tau^*_{\mathbf{r},\varepsilon}(\mathbf{z}) = -\tau_{\mathbf{r},\varepsilon}(-\mathbf{z})$. An ε-SRS for $\varepsilon \notin [0,1)$ would not be meaningful any more since then $\tau_{\mathbf{r},\varepsilon}(\mathbf{0}) \neq \mathbf{0}$ and problems occur in defining finiteness.

In the first part of Chapter 4 we investigate these ε-SRS more closely and show that $\mathcal{D}^0_{2,\varepsilon}$ can be completely characterised for $\varepsilon \in (0,1)$. We give a full analysis of $\mathcal{D}^0_{2,\varepsilon}$ for two exemplary values of ε.

1.2 Canonical Number Systems

1.2.1 Definition of canonical number systems

Consider the Gaussian integers $\mathbb{Z}[i]$. Knuth [39] showed that, for $q = -1 + i$ and $\mathcal{N} \in \{0,1\}$, each $a \in \mathbb{Z}[i]$ can be represented uniquely as

$$a = \sum_{j=0}^{n} e_j q^j, \quad e_j \in \mathcal{N} \tag{1.2.1}$$

for some $n \in \mathbb{N}$. This observation gave rise to the first definition of canonical number system by Kátai and Szabó (see [36]). For some $q \in \mathbb{Z}[i]$ they referred to a pair $(q, \mathcal{N})$, with $\mathcal{N} = \{0, \dots, N(q) - 1\}$, $N(\cdot)$ denoting the algebraic norm, as number system when each $a \in \mathbb{Z}[i]$ admits a representation as in (1.2.1). The authors found out that all Gaussian integers inducing a canonical number system are given by $A \pm i$ with negative A.

Later Kátai and Kovács [34, 35] extended the investigation to real and imaginary quadratic fields. More precisely, given such a field, the authors asked for bases q such that, analogously to above, each element of the ring of integers can be represented uniquely as in (1.2.1) with digit set $\mathcal{N} = \{0, \dots, N(q) - 1\}$. The pair $(p, \mathcal{N})$ was also called canonical number system.

Finally, in 1991, Pethő [44] gave a unifying generalisation.

Definition 1.2.1 (*cf.* [44])**.** For a polynomial $P(x) = x^d + \ldots + p_1 x + p_0 \in \mathbb{Z}[x]$ with $|p_0| \geq 2$ let $\mathcal{R} := \mathbb{Z}[x]/(P)$ be the residue ring. The pair $(P, \mathcal{N})$ with $\mathcal{N} := \{0, \dots, |p_0| - 1\}$ is called a *canonical number system* (CNS) if each element $A \in \mathcal{R}$ can be uniquely represented as

$$A = \sum_{j=0}^{n} e_j X^j, \quad e_j \in \mathcal{N}$$

where X is the image of x under the canonical epimorphism. If $(P, \mathcal{N})$ is a CNS we call P a CNS polynomial.

Consider the maps

$$m_{\mathcal{N}} : \mathcal{R} \to \mathcal{N}, A \mapsto a \text{ with } a \in \mathcal{N} \text{ such that } Q \equiv a \bmod X,$$
$$T_P : \mathcal{R} \to \mathcal{R}, A \mapsto \frac{A - m_{\mathcal{N}}(A)}{X}.$$

Note that the definition is meaningful since a is uniquely determined. Set

$$X_P : \mathcal{R} \to \mathcal{N}^{\mathbb{N}}, A \mapsto (m_{\mathcal{N}}(T^n_P(A)))_{n \in \mathbb{N}}. \tag{1.2.2}$$

We will call $X_P(A)$ for an $A \in \mathcal{R}$ the X-ary representation of A.

Notation 1.2.2. Let $X_P(A) = (e_n)_{n\in\mathbb{N}}$ be the X-ary representation of $A \in \mathcal{R}$. We call $X_P(A)$ periodic if there exists an index k and a positive integer l such that $a_i = a_{i+l}$ for all $i \geq k$. This will be written as $(a_k, \dots, a_{k-1}, (a_k, \dots, a_{k+l-1})^\infty)$. We will call $X_P(A)$ finite if $X_P(A) = (e_0, \dots, e_k, (0)^\infty)$ for some $k \in \mathbb{N}$ and denote this by $(e_0, \dots, e_k)$.

Now, if $X_P(A) = (e_0, \dots, e_k)$, we have

$$A = \sum_{j=0}^{k} e_j X^j.$$

With this notation $(P, \mathcal{N})$ is a CNS if and only if for all $A \in \mathcal{R}$ there exists a $k \in \mathbb{N}$ such that $T_P^k(A) = 0$ if and only if $X_P(A)$ is finite for all $A \in \mathcal{R}$.

Concretely, if $A \in \mathcal{R}$ is represented by

$$A = \sum_{j=0}^{n} a_j X^j, \quad a_j \in \mathbb{Z},$$

we have

$$T_P(A) = \sum_{j=1}^{n} a_j X^{j-1} - q \sum_{j=1}^{d} p_j X^{j-1}$$

where $q = \left\lfloor \frac{q_0}{p_0} \right\rfloor$. Furthermore, if p_0 is positive, $m_{\mathcal{N}}(A) = a_0 - p_0 q$. The successive application of T_P is, due to [3], often referred to as *backward division algorithm.*

Example 1.2.3. Let $P(x) = x^2 + 2x + 2$ and $\mathcal{N} = \{0, 1\}$. This example obviously corresponds to Knuth's $q = -1 + i$ number system cited above. We want to calculate the CNS representation of $A = 3X^2 - 2X + 5$. We have $q := \lfloor \frac{5}{2} \rfloor = 2$ and thus $T_P(A) = 3X - 2 - 2(X + 2) = X - 6$. Continuing in the same way gives

$$\begin{array}{llll} T_P^2(A) = 3X + 7, & T_P^3(A) = -3X - 3, & T_P^4(A) = 2X + 1, & T_P^5(A) = 2, \\ T_P^6(A) = -X - 2, & T_P^7(A) = X + 1, & T_P^8(A) = 1, & T_P^9(A) = 0. \end{array}$$

We have $m_{\mathcal{N}}(A) = 5 - 2 \cdot 2 = 1$. Further

$$\begin{array}{llll} m_{\mathcal{N}}\left(T_P(A)\right) = 0, & m_{\mathcal{N}}\left(T_P^2(A)\right) = 1, & m_{\mathcal{N}}\left(T_P^3(A)\right) = 1, & m_{\mathcal{N}}\left(T_P^4(A)\right) = 1, \\ m_{\mathcal{N}}\left(T_P^5(A)\right) = 0, & m_{\mathcal{N}}\left(T_P^6(A)\right) = 0, & m_{\mathcal{N}}\left(T_P^5(A)\right) = 1, & m_{\mathcal{N}}\left(T_P^7(A)\right) = 1. \end{array}$$

Therefore $X_P(A) = (1, 0, 1, 1, 1, 0, 0, 1, 1)$ is the X-ary representation of A and $A = 1 + X^2 + X^3 + X^4 + X^7 + X^8$.

In Subsection 3.2.1 we will show how we can generalise CNS to non-monic polynomials P (*cf.* [46]).

Brunotte [22] characterised all quadratic polynomials that give raise to a CNS.

Theorem 1.2.4. *Let $P(x) = x^2 + p_1 x + p_0$, $\mathcal{N} = \{0, \dots, |p_0| - 1\}$. $(P, \mathcal{N})$ is a CNS if and only if $p_0 \geq 2$ and $-1 \leq p_1 \leq p_0$.*

For polynomials of higher degree there exists only partial results. For instance, Kovács [40] showed

Theorem 1.2.5. *Let $P(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_0$. If*

$$2 \leq p_0 \geq p_1 \geq \cdots \geq p_{n-1} > 0$$

then P is a CNS polynomial.

For further results concerning the characterisation of CNS polynomials we want to refer to [9, 10, 21, 22, 23, 48].

Very remarkable and of special interest for us is a result that was given in [3]. It was shown that the problem of characterising CNS polynomials is closely related to verifying that $\tau_{\mathbf{r}}$ is a SRS for special choices of $\mathbf{r}$. In particular we have

Theorem 1.2.6. *$x^n+p_{n-1}x^{n-1}+\cdots+p_0$ is a CNS polynomial if and only if $\left(\frac{1}{p_0},\frac{p_{d-1}}{p_0},\ldots,\frac{p_1}{p_0}\right)\in\mathcal{D}_d^0$.*

We already mentioned that we will generalise the concept of CNS to non-monic polynomials and in this context we will state and prove a generalisation of the above theorem.

Analogously to the sets $\mathcal{D}_d$ and $\mathcal{D}_d^0$ we can define

$$\mathcal{C}_d := \{(p_0,\ldots,p_{d-1})\in\mathbb{Z}^d | X_{x^d+p_{d-1}x^{d-1}+\cdots+p_0}(A) \text{ is periodic for all } A\in\mathcal{R}\},$$
$$\mathcal{C}_d^0 := \{(p_0,\ldots,p_{d-1})\in\mathbb{Z}^d |\, x^d+p_{d-1}x^{d-1}+\cdots+p_0 \text{ induces a CNS}\}.$$

By Theorem 1.2.6 we have the relation

$$(p_0,\ldots,p_{d-1})\in\mathcal{C}_d^0 \Leftrightarrow \left(\frac{1}{p_0},\frac{p_{d-1}}{p_0},\ldots,\frac{p_1}{p_0}\right)\in\mathcal{D}_d^0$$

and analogously for $\mathcal{C}_d$ and $\mathcal{D}_d$. $\frac{1}{p_0}$ is close to zero for large p_0 and by (1.1.1) we may conjecture that the sets

$$\mathcal{C}_d(M) := \left\{\left(\frac{p_{d-1}}{M},\ldots,\frac{p_1}{M}\right) | \, (M,p_1,\ldots,p_{d-1})\in\mathcal{C}_d\right\},$$
$$\mathcal{C}_d^0(M) := \left\{\left(\frac{p_{d-1}}{M},\ldots,\frac{p_1}{M}\right) \Big| \, (M,p_1,\ldots,p_{d-1})\in\mathcal{C}_d^0\right\}.$$

give good approximations of $\mathcal{D}_{d-1}$ and $\mathcal{D}_{d-1}^0$, respectively, for large M. Indeed, Akiyama *et al.* [7] showed that

$$\lim_{M\to\infty}\mathcal{C}_d(M)=\overline{\mathcal{D}_{d-1}}$$

([7, Theorem 4.11]) and

$$\lim_{M\to\infty}\frac{\left|\mathcal{C}_d^0(M)\right|}{M^{d-1}}=\mu_{d-1}(\mathcal{D}_{d-1}^0)$$

where μ_{d-1} denotes the $d-1$ dimensional Lebesgue measure ([7, Theorem 7.1]). It is still an open question whether $\lim_{M\to\infty}\mathcal{C}_d^0(M)=\overline{\mathcal{D}_{d-1}^0}$.

1.2.2 Tiles associated to expanding polynomials

Due to Kátai and Kőrnyei [33] (see also [47]) we can associate a CNS to a self-affine tile. Actually this works for any expanding polynomial.

Definition 1.2.7. Let $P(x)=x^d+p_{d-1}x^{d-1}+\cdots+p_1x+p_0\in\mathbb{Z}[x]$ be an expanding polynomial. We call

$$\mathcal{F} := \left\{\mathbf{z}\in\mathbb{R}^d \,\middle|\, \mathbf{z}=\sum_{i=0}^{\infty}B^{-i}(c_i,0,\ldots,0)^T, c_i\in\mathcal{N}\right\} \tag{1.2.3}$$

with $\mathcal{N}=\{0,\ldots,|p_0|-1\}$ and

$$B := \begin{pmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{d-1} \end{pmatrix}$$

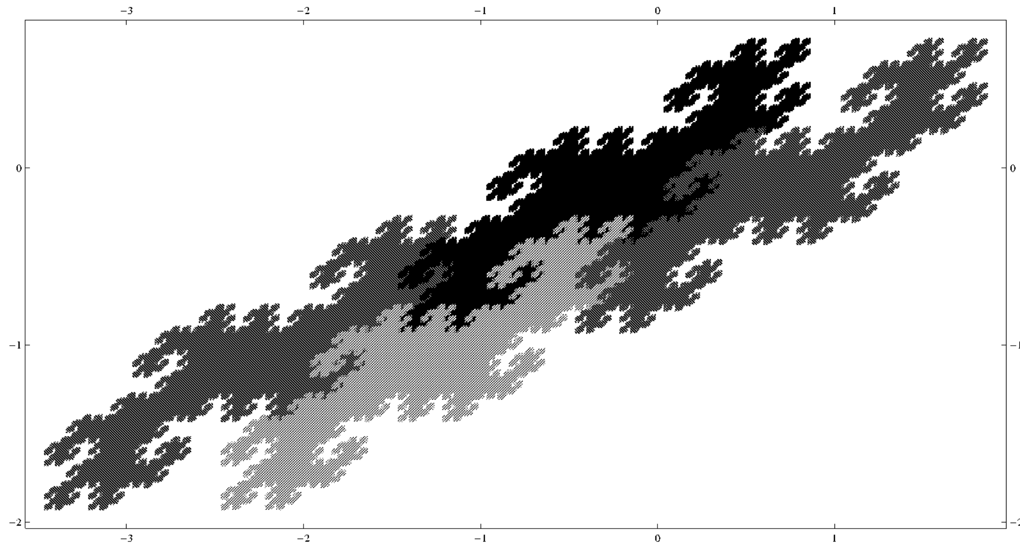


Figure 1.5: Self affine tile associated to $x^2 + 2x + 2$

the *self-affine tile associated to* P. For $(P, \mathcal{N})$ being a CNS, $\mathcal{F}$ is called the *(central) CNS tile* associated to $(P, \mathcal{N})$. It is compact, self-affine and defines a tiling of the n-dimensional real vector space.

P is the characteristic polynomial of B. Since B is expanding it is easy to see that the series in (1.2.3) always converges. Note that $\mathcal{F}$ is a self-affine tile as it obeys the functional equation

$$B\mathcal{F} = \bigcup_{c \in \mathcal{N}} (\mathcal{F} + (c, 0 \ldots, 0)). \tag{1.2.4}$$

Indeed, it is the unique nonempty compact set satisfying this equation (*cf. e.g.* Hutchinson [31]). Self-affine tiles have been studied extensively in a very general context in literature. We refer the reader to the surveys by Vince [56] and Wang [57]. $\mathbf{0}$ is an inner point of $\mathcal{F}$ if and only if $(P, \mathcal{N})$ is a CNS.

Figure 1.5 shows the self-affine tile associated to x^2+2x+2 (in black) and three of its translates. It is Knuth's Twin Dragon that we already heard about in the introduction.

In [17] the relation between tiles associated to expanding polynomials and SRS-tiles was analysed. We will discuss these results in Subsection 3.2.3.

1.2.3 Symmetric CNS and ε-CNS

Akiyama and Scheicher [12] introduced an alternative version of canonical number systems. These symmetric canonical number systems are defined similarly to CNS. Only the used digit set differs.

Definition 1.2.8 (*cf.* [12])**.** For a $P(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_0$ let $\mathcal{R} = \mathbb{Z}[x]/(P)$, $\mathcal{N} = \left[-\frac{|p_0|}{2}, \frac{|p_0|}{2}\right) \cap \mathbb{Z}$ and X the image of x under the canonical epimorohism. $(P, \mathcal{N})$ is called a *symmetric canonical number system* (SCNS) if each $A \in \mathcal{R}$ can be represented as

$$A = \sum_{j=0}^{n} e_j X^j, \quad e_j \in \mathcal{N}.$$

The authors fully characterised all polynomials of degree 1 and 2 that give rise to a SCNS: $x + a$ induces a SCNS if and only if $a \geq 2$ or $a < 2$ and $x^2 + ax + b$ if and only if either $|a| < 1 + \frac{b}{2}$ and $|b| \geq 2$ or $a = 1 + \frac{b}{2}$ and $|b| \geq 2$. The most interesting connection in this context is

Theorem 1.2.9 (*cf.* [12, Theorem 2.1]). *Let* $P(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_0$ *and* $\mathcal{N} = \left[-\frac{|p_0|}{2}, \frac{|p_0|}{2}\right) \cap \mathbb{Z}$. $(P, \mathcal{N})$ *is a SCNS if and only if* $\tau_{\mathbf{r}}$ *is a SSRS for*

$$\mathbf{r} = \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}\right).$$

Therefore SCNS are similarly related to SSRS as CNS to SRS. Later, in Chapter 4, we will treat with another generalisation of CNS involving both, CNS and SCNS. It was presented in [53].

Definition 1.2.10. Let $\varepsilon \in [0,1)$, $P(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_0 \in \mathbb{Z}[x]$ with $|p_0| \geq 2$, $\mathcal{R} := \mathbb{Z}[x]/(P)$ the factor ring, $X \in \mathcal{R}$ the image of x under the canonical epimorphism and the set of digits $\mathcal{N} := [-\varepsilon |p_0|, (1-\varepsilon)|p_0|) \cap \mathbb{Z}$. The pair $(P, \mathcal{N})$ is called an *ε-Canonical Number System* (ε-CNS) if each $P \in \mathcal{R}$ allows a representation as

$$P = \sum_{i=0}^{n} e_i X^i \text{ with } e_i \in \mathcal{N}.$$

It is easy to see that the case $\varepsilon = 0$ corresponds to CNS while $\varepsilon = \frac{1}{2}$ corresponds to the SCNS. It will turn out that these ε-CNS are analogously related to ε-SRS.

1.3 β-expansion

1.3.1 Definition and basic properties

The second type of numeration system we are going to deal with is the so-called β expansion. It was introduced in [43, 45].

Definition 1.3.1. Let $\beta > 1$ be a real number. For some $\gamma \in \mathbb{R} \cap [0, \infty)$ a representation of the shape

$$\gamma = \sum_{j=-m}^{\infty} b_j \beta^{-j}, \quad b_j \in \mathcal{N} \tag{1.3.1}$$

with $\mathcal{N} = [0, \lceil \beta \rceil - 1)$ is called the *β-expansion* of γ if the *greedy condition*

$$0 \leq \sum_{j=-n}^{\infty} b_j \beta^{-j} < \beta^{n+1}$$

holds for all $n \leq m$.

Define the maps

$$T_\beta : \gamma \mapsto \beta\gamma - \lfloor \beta\gamma \rfloor \quad (\beta \text{ transformation})$$

and

$$d_\beta : \gamma \mapsto \left(\left\lfloor \beta T_\beta^{j-1}(\gamma) \right\rfloor\right)_{j \in \mathbb{N}^*} = \left(\beta T_\beta^{j-1}(\gamma) - T_\beta^j(\gamma)\right)_{j \in \mathbb{N}^*}.$$

Then, for $\gamma \in [0,1)$, the β-expansion of γ equals

$$\gamma = \sum_{j \geq 1} e_j \beta^{-j} \text{ with } d_\beta(\gamma) = (e_j)_{j \in \mathbb{N}^*}. \tag{1.3.2}$$

For the sequence $d_\beta(\gamma)$ we can adopt the rules from Notation 1.2.2.

For some $\gamma \geq 1$ the β-expansion can easily be computed by multiplying γ with β^{-k} for a suitable $k \in \mathbb{N}$ such that $\beta^{-k}\gamma \in [0,1)$ (such a k exists since $\beta > 1$). Suppose $d_\beta(\beta^{-k}\gamma) = (a_1, a_2, a_3, \cdots)$. Then

$$\gamma = \sum_{j \geq -k+1} a_{j+k} \beta^{-j}$$

is the β-expansion of γ. Therefore we can restrict our investigations to the interval $[0,1)$.

Define

$$\begin{aligned}\mathrm{Per}(\beta) :=& \{\gamma \in [0,1) \,|\, d_\beta(\gamma) \text{ is periodic}\}\\ \mathrm{Fin}(\beta) :=& \{\gamma \in [0,1) \,|\, d_\beta(\gamma) \text{ is finite}\}\end{aligned}$$

We obviously have $\mathrm{Fin}(\beta) \subset \mathrm{Per}(\beta)$. It is also easy to see that for an algebraic number β we have

$$\begin{aligned}\mathrm{Per}(\beta) \subseteq& \mathbb{Q}(\beta) \cap [0,1) \text{ and}\\ \mathrm{Fin}(\beta) \subseteq& \mathbb{Z}\left[\beta^{-1}\right] \cap [0,1).\end{aligned}$$

Definition 1.3.2 (*cf.* [28])**.** An algebraic number $\beta > 1$ is said to have property (F) if $\mathrm{Fin}(\beta) = \mathbb{Z}\left[\beta^{-1}\right] \cap [0,1)$.

Frougny and Solomyak [28] proved that (F) implies β that β is a Pisot number (*i.e.*, an algebraic integer greater than 1 such that all its Galois conjugates have modulus less than 1). It was also shown that (F) holds for all quadratic Pisot numbers. For Pisot numbers of higher degree the authors presented the following result:

Theorem 1.3.3 (*cf.* [28])**.** *Let β be a Pisot number with minimal polynomial $x^{d+1} - p_d x^d - \cdots - p_1 x - p_0$. If*

$$1 \leq p_0 \leq \cdots \leq p_d$$

then β has property (F).

A more detailled characterisation of Pisot numbers satisfying (F), apart from quadratic ones, exists for the cubic case. Akiyama [1] characterised all cubic Pisot units satisfying (F).

Theorem 1.3.4 (*cf.* [1])**.** *Let β a cubic Pisot unit. β satifies property (F) if and only if the minimal polynomial of β is of the form*

$$x^3 - p_2 x^2 - p_1 x - 1$$

with $-1 \leq p_1 \leq p_2 + 1$.

In this context we also want to mention the papers [11, 13, 29] here. Of special interest for us is the relation between the β-expansion and SRS. Let β a algebraic number of degree $d+1$, which we can assume to be a Pisot number, with minimal polynomial $P(x) = x^{d+1} + p_d x^d + \cdots + p_1 x + p_0 \in \mathbb{Z}[x]$. Let

$$\mathbf{r} = (r_0, \ldots, r_{d-1}) \quad \text{with} \quad \begin{aligned} r_d &= 1, \\ r_j &= a_{j+1} + \beta r_{j+1} \qquad (0 \leq j \leq d-1) \end{aligned}. \tag{1.3.3}$$

Note that $r_0 = -\frac{a_0}{\beta}$. One can easily verify that $P(x) = (x-\beta)(r_d x^d - r_{d-1} x^{d-1} - \cdots - r_0)$.

Theorem 1.3.5 (*cf.* [3, Theorem 2.1] (*cf.* also [29]))**.** *For an algebraic number $\beta > 1$ and $\mathbf{r}$ defined as in* (1.3.3) *we have $\mathbf{r} \in \mathcal{D}_d^0$ if and only if β has property (F).*

The relation between β-expansions and SRS was excessively studied in [17]. We will present these results in Section 3.3 including the proof of the previous theorem.

Concerning $\mathrm{Per}(\beta)$ it was shown in [18, 49] that $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,1)$ if β is a Pisot number. On the other hand, following [49], if $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,1)$ then β is a Pisot or Salem number, where a Salem number is an algebraic integer such that all of its conjugates have modulus less or equal to 1 and equality holds for at least one of them.

From the combinatorial point of view the sequence $d_\beta(1)$ is of special interest. A sequence $(e_j)_{j\in\mathbb{N}}$ is called *admissible* if $(e_j)_{j\in\mathbb{N}} = d_\beta(\gamma)$ for some $\gamma \in [0,1)$. Of course, $d_\beta(1)$ is not admissible.

Definition 1.3.6. β is called a *Parry number* if $d_\beta(1)$ is periodic and it is called a *simple Parry number* if $d_\beta(1)$ is finite.

Note that in some papers Parry numbers are referred to as β-numbers. For Parry numbers define

$$d^*_\beta(1) = \begin{cases} \big((t_1, \ldots, t_{n-1}, t_n - 1)^\infty\big) \text{ if } d_\beta(1) = (t_1, t_2, \ldots, t_n), t_n \neq 0 \\ d_\beta(1) \text{ otherwise} \end{cases} .$$

With this definition a sequence $a = (a_i)_{i\in\mathbb{N}^*} \in \mathbb{N}^\infty$ is admissible if and only if

$$(a_{i+k})_{i\in\mathbb{N}^*} <_{\text{lex}} d^*_\beta(\beta) \text{ for all } k \geq 0. \tag{1.3.4}$$

This condition is often called the lexicographical order condition.

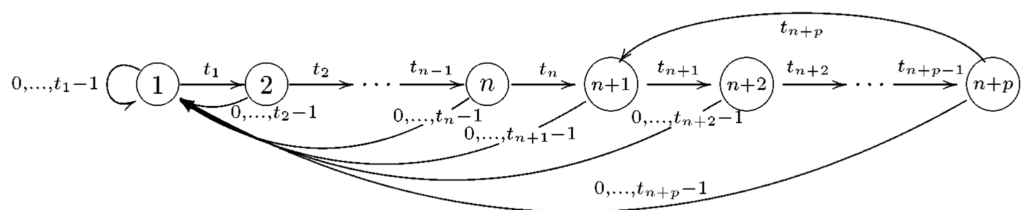
Set

$$D_\beta = \{d_\beta(\gamma) \mid \gamma \in [0, 1)\}.$$

Note that $\mathcal{D}_\beta$ is invariant with respect to the left shift σ, *i.e.*,

$$D_\beta = \sigma(D_\beta) = \{(a_{i+1})_{i\in\mathbb{N}^*} \mid (a_i)_{i\in\mathbb{N}^*} \in \mathcal{D}_\beta\},$$

making (D_β, σ) a subshift. Suppose $d^*_\beta(1) = (t_1, \ldots, t_n, (t_{n+1}, \ldots, t_{n+p})^\infty)$ for a Parry number β. For simple β-numbers we have $n = 0$. Then the finite factors of the sequences of D_β can be recognised by a finite automaton with $n + p$ states.



(1.3.5)

Hence, for β a Parry numbers, (D_β, σ) is sofic. (D_β, σ) is of finite type if β is a simple Parry number.

The problem of generally characterising Parry numbers is far from being solved. Of course, a Parry number is always algebraic and it is known that it is necessarily a Perron number (*i.e.*, an algebraic integer that is the dominant root of its minimal polynomial) with no real conjugate greater than 1. Conversely we obviously have that each Pisot number as a Parry number. Note that the dominant root of the polynomal $x^4 - 3x^3 - 2x^2 - 3$ is an example of a simple Parry number which is a Perron number and neither a Pisot nor a Salem number.

Concerning simple Parry-numbers of small degree we have the following characterisation results:

Theorem 1.3.7 ([28]). *The simple Parry numbers of degree 2 are exactly the quadratic Pisot numbers without a positive real conjugate, hence, the positive roots of the polynomials $x^2 - ax - b$ with $a \geq b \geq 1$. Then we have $d_\beta(1) = (a, b, 0, 0, \ldots)$.*

Bassino [14] characterised all cubic Pisot numbers that are simple Parry numbers.

Theorem 1.3.8 ([14]). *A Pisot number with minimal polynomial $x^3 - ax^2 - bx - c$ is a simple Parry number if and only if one of the following conditions holds:*

- $b \geq 0$ *and* $c > 0$,
- $-a < b < 0$ *and* $b + c \geq 0$,
- $b \leq -a$ *and* $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$ *with* $k \in \{2, \ldots, a-2\}$ *such that*

$$1 - a + \frac{a-2}{k} \leq b + c < 1 - a + \frac{a-2}{k-1}.$$

We already noticed the close relation between the β-expansion and SRS. In Subsection 3.3.2 we will treat Bassino's result in the point of view of SRS. For more details concerning combinatorial aspects of β-expansions we refer to [16, 19, 27, 28, 32]. We also want to mention that, for example in [26], the problem of addition in base β was studied. We will not go into that here.

Let

$$\begin{aligned}\mathcal{B}_d &:= \{(b_1,\dots,b_d)\in\mathbb{Z}^d | X^d+b_1X^{d-1}+\dots+b_d \\ &\qquad\qquad \text{is the minimal polynomial of a Pisot or Salem number}\} \\ \mathcal{B}_d^0 &:= \{(b_1,\dots,b_d)\in\mathbb{Z}^d | X^d+b_1X^{d-1}+\dots+b_d \\ &\qquad\qquad \text{is the minimal polynomial of a Pisot number having property (F)}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_d(M) &:= \left\{(b_2,\dots,b_d)\in\mathbb{Z}^{d-1} \,:\, (M,b_2,\dots,b_d)\in\mathcal{B}_d\right\} \\ \mathcal{B}_d^0(M) &:= \left\{(b_2,\dots,b_d)\in\mathbb{Z}^{d-1} \,:\, (M,b_2,\dots,b_d)\in\mathcal{B}_d^0\right\}.\end{aligned}$$

In the fourth part of a series of papers concerning SRS Akiyama *et al.* [8] showed that for $d\geq 2$

$$\left|\frac{|\mathcal{B}_d(M)|}{M^{d-1}}-\mu_{d-1}(\mathcal{D}_{d-1})\right| = O\left(M^{-1/(d-1)}\right)$$

and

$$\lim_{M\to\infty}\frac{|\mathcal{B}_d^0(M)|}{M^{d-1}} = \mu_{d-1}(\mathcal{D}_{d-1}^0),$$

where μ_{d-1} denotes the $d-1$ dimensional Lebesgue measure, holds. Note that we already stated analogous results for CNS.

1.3.2 Tiles associated to Pisot numbers

In the following assume β to be a Pisot number of degree $d+1$. Let $\beta_1,\dots,\beta_d$ be the $d=r+2s$ Galois conjugates of β, such that $\beta_1,\dots,\beta_r\in\mathbb{R}$ and $\beta_{r+1},\dots,\beta_{r+2s}\in\mathbb{C}$ with

$$\beta_{r+1}=\overline{\beta_{r+s+1}},\dots,\beta_{r+s}=\overline{\beta_{r+2s}}.$$

For a $\gamma\in\mathbb{Q}(\beta)$ and $i\in\{1,\dots,d\}$ denote by $\gamma^{(i)}\in\mathbb{Q}(\beta_i)$ the corresponding conjugate of γ. Define the mapping

$$\begin{aligned}\Phi:\ \mathbb{Q}(\beta) &\rightarrow \mathbb{R}^d, \\ \gamma &\mapsto \left(\gamma^{(1)},\dots,\gamma^{(r)},\Re(\gamma^{(r+1)}),\Im(\gamma^{(r+1)}),\dots,\Re(\gamma^{(r+s)}),\Im(\gamma^{(r+s)})\right)^T.\end{aligned}$$

Following Akiyama [2] we give

Definition 1.3.9. For the Pisot number β and $\omega\in\mathbb{Z}[\beta]\cap[0,1)$ let

$$S_{\beta,n}(\omega) := \left\{\gamma\in\mathbb{Z}\left[\beta^{-1}\right]\cap[0,1)\ \middle|\ T_\beta^n(\gamma)=\omega\right\}.$$

Then the set

$$S_\beta(\omega) := \overline{\lim_{n\to\infty}\Phi(\beta^n S_{\beta,n}(\omega))}\subset\mathbb{R}^d$$

(the limit is taken with respect to the Hausdorff metric) is called a *β-tile*. The tile $S_\beta(0)$ will be called the *central β-tile.*

We will give a short summary of what is known about β-tiles. Following [2] we have

$$\bigcup_{\omega\in\mathbb{Z}[\beta]\cap[0,1)} S_\beta(\omega)=\mathbb{R}^d.$$

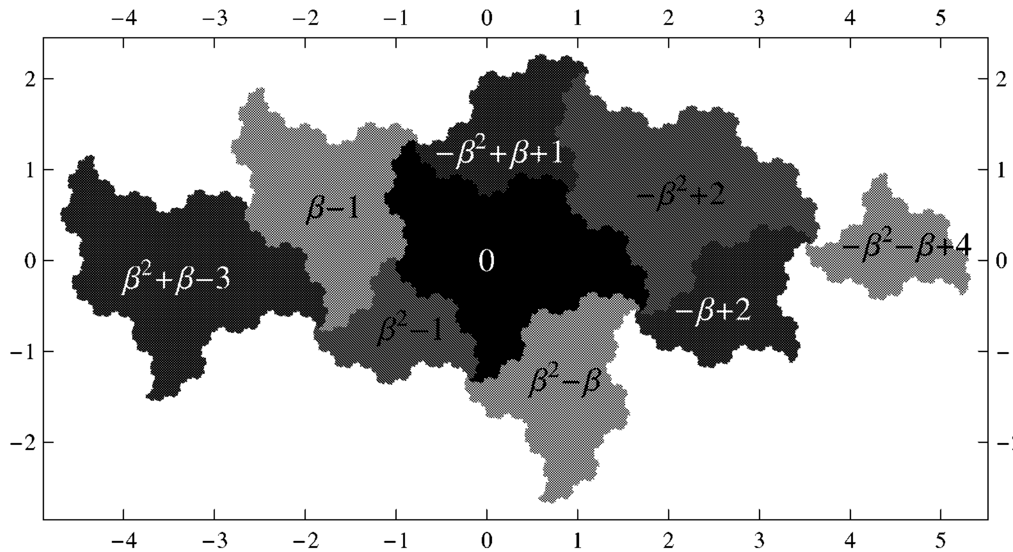


Figure 1.6: β-tiles corresponding to the smallest Pisot number

For non unit Pisot numbers the tiles can easily be seen to overlap and therefore they do not provide a tiling. For Pisot units the previously mentioned covering is conjectured to be a tiling. β-tiles are graph directed self-affine sets in the sense of Mauldin and Williams [42]. They are strongly related to Rauzy fractals associated to unimodular Pisot substitutions (see for example [16]). Especially we have

Theorem 1.3.10 (*cf.* [2, Lemma 5]). *The number of different β-tiles up to translation induced by a Pisot unit equals the number of states of the automaton 1.3.5 induced by β, i.e.,*

$$\left|\{T_\beta^k(1) \,|\, k \geq 1\}\right| + \begin{cases} 0 & \text{for simple Parry numbers} \\ 1 & \text{otherwise} \end{cases}.$$

Note that this number is finite since β is a Pisot number.

Example 1.3.11. Consider β to be the dominant root of the polynomial $P(x) = x^3 - x - 1$. $\beta = 1.32472\ldots$ is the smallest Pisot number. It can easily be verified that $d_\beta(1) = (1,0,0,0,1)$. Thus β is a Parry number and $d_\beta^*(1) = \left((1,0,0,0,0)^\infty\right)$. The corresponding automaton is

$$\begin{tikzcd} 1 \arrow[loop left, "0"] \arrow[r, "1"] & 2 \arrow[r, "0"] & 3 \arrow[r, "0"] & 4 \arrow[r, "0"] & 5 \arrow[llll, bend right, "0"'] \end{tikzcd}.$$

Therefore β induces 5 different tiles up to translation. Figure 1.6 shows the central tile and some of its neighbours. Each of them is labelled with the corresponding $\gamma \in \mathbb{Z}[\beta] \cap [0,1)$. The central tile is called Hokkaido fractal its shape reminds of the Japanese island Hokkaido. β satisfies the property (F) and the corresponding SRS vector is $\mathbf{r} := (\beta^2 - 1, \beta)$.

In [17] an alternative definition of β-tiles was given:

Definition 1.3.12. For the Pisot number β and $\omega \in \mathbb{Z}[\beta] \cap [0,1)$ let

$$\tilde{S}_{\beta,n}(\omega) := \{\gamma \in \mathbb{Z}[\beta] \cap [0,1) \mid T_\beta^n(\gamma) = \omega\}.$$

Then the set

$$\tilde{S}_\beta(\omega) := \overline{\lim_{n\to\infty} \Phi(\beta^n \tilde{S}_{\beta,n}(\omega))},$$

(again with respect to the Hausdorff metric) is called a *new β-tile*. The tile $S_\beta(0)$ will be called *central new β-tile*.

Note that

$$S_\beta(\omega) \supseteq \tilde{S}_\beta(\omega)$$

where equality holds exactly if β is a unit. Contrary to β-tiles associated to non-unit Pisot numbers new β-tiles are conjectured not to overlap and therefore provide a tiling of the Euclidean space. We will deal with new β-tiles in Section 3.3.

1.3.3 Symmetric β-expansions and ε-β-expansions

Analogously to symmetric canonical number systems Akiyama and Scheicher [12] also defined a symmetric β-expansion.

Definition 1.3.13 (*cf.* [12])**.** For a real $\beta > 1$ a representation of some $\gamma \in [-\frac{1}{2}, \frac{1}{2})$ of the shape

$$\sum_{j\geq 1} e_j \beta^{-j}, \quad e_j \in \mathcal{N} := \left(-\frac{\beta+1}{2}, \frac{\beta+1}{2}\right) \cap \mathbb{Z}$$

is called the symmetric β expansion of γ if

$$-\frac{1}{2\beta^{n-1}} \leq \sum_{i\geq n} < \frac{1}{2\beta^{n-1}}$$

holds for all $n \geq 1$.

The last inequality is analogous to the greedy condition. The symmetric β-expansion can be obtained by applying the symmetric β-shift

$$\tilde{T}_\beta : \left[-\frac{1}{2}, \frac{1}{2}\right) \to \left[-\frac{1}{2}, \frac{1}{2}\right), \gamma \mapsto \beta\gamma - \left\lfloor \beta\gamma + \frac{1}{2} \right\rfloor.$$

In a similar way as for the β-expansion we define

$$\tilde{d}_\beta(\gamma) := \left(\beta \tilde{T}_\beta^{n-1}(\gamma) - \tilde{T}_\beta^{n}(\gamma)\right)_{n\in\mathbb{N}^*}$$

and call a sequence $(e_n)_{n\in\mathbb{N}^*} \in \mathcal{N}^\infty$ admissible if $(e_n)_{n\in\mathbb{N}^*}$ equals $\tilde{d}_\beta(\gamma)$ for some $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right)$.

Theorem 1.3.14 (*cf.* [12, Theorem 3.1])**.** *A sequence $(e_n)_{n\in\mathbb{N}^*} \in \mathcal{N}^\infty$ is admissible if and only if*

$$\tilde{d}_\beta\left(-\frac{1}{2}\right) \leq_{\text{lex}} (e_{n+k})_{n\in\mathbb{N}^*} <_{\text{lex}} -\tilde{d}_\beta\left(-\frac{1}{2}\right)$$

holds for all $k \in \mathbb{N}$.

An algebraic number β is said to have the symmetric finiteness property (SF) if each $\gamma \in \mathbb{Z}[\beta^{-1}] \cap \left[-\frac{1}{2}, \frac{1}{2}\right)$ has a finite symmetric β-expansion. The property (SF) is shown to be connected to SSRS.

Theorem 1.3.15 (*cf.* [12, Theorem 3.7])**.** *Let β a Pisot number with minimal polynomial $x^{d+1} + p_d x^d + \cdots + p_0$ and define $\mathbf{r}$ as in (1.3.3). β has property (SF) if and only if $\mathbf{r} \in \tilde{\mathcal{D}}_d^0$.*

In the same spirit as for ε-CNS in [53] a definition of ε-β-expansions was given.

Definition 1.3.16. Let $\varepsilon \in [0, 1)$, $\beta > 1$ a real number and $\mathcal{N} = (-1+\varepsilon(1-\beta), \beta+\varepsilon(1-\beta)) \cap \mathbb{Z}$. For a $\gamma \in \mathbb{R}$ a representation of the shape

$$\gamma = \sum_{j\geq m} e_j \beta^{-j}, \quad e_j \in \mathcal{N}$$

is called the *ε-β-expansion* of γ when it satisfies

$$\sum_{j\geq n} e_j \beta^{-j} \in [-\varepsilon, 1-\varepsilon)\beta^{n+1}. \tag{1.3.6}$$

Formula (1.3.6) is the generalisation of the β-expansion's greedy condition and its analogue for the symmetric β-expansion. We will treat ε-β-expansions in Section 4.1 and show their relation to ε-SRS.

Chapter 2

Shift Radix Systems

This chapter is fully devoted to the analysis of $\mathcal{D}_d$ and $\mathcal{D}_d^0$. Concerning D_d we present in Theorem 2.1.3 a new idea for obtaining characterisation results for $\mathcal{D}_d \cap \partial\mathcal{E}_d$. In the rest of Section 2.1 we thoroughly discuss the Brunotte Algorithm, present several improvements and state ideas for a computational implementation. In Section 2.2 we are going to specialise to the two dimensional case and use the previously mentioned implementation in order improve the characterisation of $\mathcal{D}_2^0$. Since areas near the boundary of $\mathcal{D}_2^0$ are not really applicable for an analysis by the Brunotte Algorithm we will go alternative ways: in Subsection 2.2.1 we discuss an idea presented in [6] for analysing the set R (see (1.1.7)) which is located near the upper boundary of $\mathcal{D}_2$. In Subsection 2.2.3 we will investigate a small set near the point $(1,1) \in \overline{\mathcal{D}_2}$ by directly looking at the orbits of $\tau_{\mathbf{r}}$. In Subsection 1.1.3 we noted the existence of an infinite family of cycles. We will present a second family and give a full analysis of the cutout polyhedra that correspond to these families in Section 2.2.4. Altogether this will yield a very good approximation of $\mathcal{D}_2^0$. Finally, in Subsection 2.2.5, we will be able to prove the existence of a second critical point (Theorem 2.2.29) and show that $\mathcal{D}_2^0$ has a cutpoint (Theorem 2.2.30).

2.1 Results concerning $\mathcal{D}_d$ and $\mathcal{D}_d^0$

2.1.1 The set $\mathcal{D}_d$

In the following we will deal with the set $\mathcal{D}_d$ and present a new idea in order to analyse $\mathcal{D}_d \cap \partial\mathcal{E}_d$. Define for a $\lambda \in \mathbb{R}$ the mappings W_λ for vectors and Y_λ for infinite sequences by

$$\begin{aligned} W_\lambda : \mathbb{R}^n \to \mathbb{R}^{n+1}, &(a_0, a_1, a_2, \dots, a_{n-1}) \mapsto (\lambda a_0, \lambda a_1 + a_0, \dots, \lambda a_{n-1} + a_{n-2}, \lambda + a_{n-1}), \\ Y_\lambda : \mathbb{R}^\infty \to \mathbb{R}^\infty, &(a_0, a_1, a_2, \dots) \mapsto (\lambda a_0 + a_1, \lambda a_1 + a_2, \dots). \end{aligned} \tag{2.1.1}$$

Note that $\chi_{R(W_\lambda(\mathbf{r}))}(x) = (x - \lambda)\chi_{R(\mathbf{r})}(x)$. Therefore we have

$$\varrho\left(R\left(W_\lambda(\mathbf{r})\right)\right) = \max\left(\varrho\left(R\left(\mathbf{r}\right)\right), |\lambda|\right) \tag{2.1.2}$$

and thus $W_{\pm 1}(\mathbf{r}) \in \partial\mathcal{E}_{d+1}$ for $\mathbf{r} \in \overline{\mathcal{E}_d}$. Our strategy now is to investigate how $\tau_{\mathbf{r}}$ and $\tau_{W_\lambda(\mathbf{r})}$ are related. The next theorem shows that for $\lambda \in \mathbb{Z}$ this connection can be described by the map Y_λ.

Theorem 2.1.1. *Let $\lambda \in \mathbb{Z}$, $\mathbf{r} := (r_0, \dots, r_{d-1}) \in \mathbb{R}^d$, $(x_i)_{i\in\mathbb{N}} \in \mathbb{Z}^\infty$ and $(y_i)_{i\in\mathbb{N}} := Y_\lambda((x_i)_{i\in\mathbb{N}}) \in \mathbb{Z}^\infty$. Then the following point are equivalent:*

1.

$$\tau^n_{W_\lambda(\mathbf{r})}\big((x_0, \dots, x_d)\big) = (x_n, \dots, x_{n+d}) \quad (\forall n \in \mathbb{N}),$$

2.

$$\tau^n_{\mathbf{r}}\big((y_0, \dots, y_{d-1})\big) = (y_n, \dots, y_{n+d-1}) \quad (\forall n \in \mathbb{N}).$$

Proof. 1. $\Rightarrow$ 2.: Suppose

$$\tau^n_{W_\lambda(\mathbf{r})}\big((x_0,\dots,x_d)\big) = (x_n,\dots,x_{n+d})$$

for all $n \in \mathbb{N}$. We go on by induction on n. For $n = 0$ the statement is trivially true. Now suppose we already know that $\tau^{n-1}_{\mathbf{r}}\big((y_0,\dots,y_{d-1})\big) = \big(y_{n-1},\dots,y_{n+d-2}\big)$. Then we have

$$\tau^n_{\mathbf{r}}\big((y_0,\dots,y_{d-1})\big) = \tau_{\mathbf{r}}\big((y_{n-1},\dots,y_{n+d-2})\big) = (y_n,\dots,y_{n+d-2},\tilde{y})$$

with

$$\begin{aligned}
\tilde{y} =& -\left\lfloor \sum_{i=0}^{d-1} r_i y_{n-1+i} \right\rfloor = -\left\lfloor \sum_{i=0}^{d-1} r_i(\lambda x_{n-1+i} + x_{n+i}) \right\rfloor \\
=& -\left\lfloor \lambda r_0 x_{n-1} + \sum_{i=0}^{d-2} (\lambda r_{i+1} + r_i) x_{n+i} + (\lambda + r_{d-1}) x_{n+d-1} - \lambda x_{n+d-1} \right\rfloor \\
=& -\lfloor W_\lambda(\mathbf{r})(x_{n-1},\dots,x_{n+d-1})\rfloor + \lambda x_{n+d-1} \\
=& x_{n+d} + \lambda x_{n+d-1} = y_{n+d-1}.
\end{aligned}$$

and hence $\tau^n_{\mathbf{r}}\big((y_0,\dots,y_{d-1})\big) = (y_n,\dots,y_{n+d-2},y_{n+d-1})$. 2. $\Rightarrow$ 1.: This can be shown analogously. $\square$

Observe that (1.1.1) is an immediate consequence of Theorem 2.1.1 with $\lambda = 0$. Since we are mainly interested in vectors $\mathbf{r}$ with $R(\mathbf{r})$ having spectral radius 1 at most we can from now on restrict ourselves to $\lambda \in [-1,1]$.

Corollary 2.1.2. *Let $\lambda \in \{-1,1\}$ and $\mathbf{r} \in \mathbb{R}^d$. If $W_\lambda(\mathbf{r}) \in \mathcal{D}_{d+1}$ then $\mathbf{r} \in \mathcal{D}_d$.*

Proof. Just consider Theorem 2.1.1 and note that for an ultimately periodic sequence $(x_n)_{n\in\mathbb{N}}$ the sequence $W_\lambda\left((x_n)_{n\in\mathbb{N}}\right)$ ends up periodically, too. $\square$

We already mentioned the analysis of Kirschenhofer *et al.* [37] concerning the point $\mathbf{r} := \left(1, \frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right) \in \partial\mathcal{E}_3$ which turned out not to belong to $\mathcal{D}_3$. Since $\mathbf{r} = W_1\left(\left(1, \frac{1+\sqrt{5}}{2}\right)\right)$ and $\left(1, \frac{1+\sqrt{5}}{2}\right)$ is known to be element of $\mathcal{D}_2$ (see Theorem 1.1.10) we immediately see that the inverse of Corollary 2.1.2 does not hold in general.

We can now state a result that shows a connection between $\mathcal{D}^0_d$ and $\mathcal{D}_{d+1} \cap \partial\mathcal{E}_{d+1}$

Theorem 2.1.3. *Let $\mathbf{r} \in \mathcal{D}^0_d$ and $\lambda \in \{1,-1\}$. Then $W_\lambda(\mathbf{r}) \in \mathcal{D}_{d+1} \cap \partial\mathcal{E}_{d+1}$.*

Proof. By (2.1.2) we have

$$\varrho\left(R\left(W_\lambda(\mathbf{r})\right)\right) = \max\left(\varrho\left(R\left(\mathbf{r}\right)\right), |\lambda|\right) = 1$$

and hence $W_\lambda(\mathbf{r}) \in \partial\mathcal{E}_{d+1}$. Choose an arbitrary $\mathbf{x} = (x_0,\dots,x_{d-1},x_d) \in \mathbb{Z}^{d+1}$ and let $(x_i)_{i\in\mathbb{N}}$ such that $\tau^n_{W_\lambda(\mathbf{r})}\big((x_0,\dots,x_{d-1},x_d)\big) = (x_n,\dots,x_{n+d})$. We claim that $\left(\tau^n_{W_{-1}(\mathbf{r})}(\mathbf{x})\right)_{n\in\mathbb{N}}$ is eventually periodic. By Theorem 2.1.1 the sequence $(y_i)_{i\in\mathbb{N}} := Y_\lambda\big((x_i)_{i\in\mathbb{N}}\big)$ has the property that $\tau^n_{\mathbf{r}}\big((y_0,\dots,y_{d-1})\big) = (y_n,\dots,y_{d+n-1})$. Now, as $\mathbf{r} \in \mathcal{D}^0_d$, there exists an n_0 with $y_n = 0$ for $n \geq n_0$. By the definition of Y_λ we have $y_i = \lambda x_i + x_{i+1}$. If $\lambda = -1$ this implies $0 = y_{n_0} = -1x_{n_0} + x_{n_0+1}$ and therefore $x_{n_0+1} = x_{n_0}$. With the same argument we find $x_{n_0} = x_{n_0+1} = x_{n_0+2} = \cdots$ showing the ultimate periodicity of $\left(\tau^n_{W_{-1}(\mathbf{r})}(\mathbf{x})\right)_{n\in\mathbb{N}}$. Since this works for any $\mathbf{x} \in \mathbb{Z}^{d+1}$ we easily see $W_{-1}(\mathbf{r}) \in \mathcal{D}_{d+1}$. If $\lambda = 1$ we use the same argumentation with the only difference that we have an eventually alternating sequence $x_{n_0+i} = (-1)^i x_{n_0}$ for all $i \geq 1$. $\square$

Remember the lines L_1 and L_2 defined in (1.1.4). They are subsets of $\partial\mathcal{E}_2$ and proved in [6, Theorem 2.1] to belong to $\mathcal{D}_2$. Note that this result is a corollary to Theorem 2.1.3.

Corollary 2.1.4.

$$L_1, L_2 \subset \mathcal{D}_2.$$

Proof. The set $\mathcal{D}_1^0$ equal the interval $[0,1)$. Then Theorem 2.1.3 immediately yields the result since $L_1 = W_1([0,1))$ and $L_2 = W_{-1}([0,1))$. □

In Section 2.2 we will give an approximation of $\mathcal{D}_2^0$. With this one easily obtain partly results for $\mathcal{D}_3 \cap \partial\mathcal{E}_3$ by using Theorem 2.1.3.

2.1.2 An algorithmic way to characterise $\mathcal{D}_d^0$

In the following we present an algorithmic way to analyse $\mathcal{D}_d^0$. It is based on an idea of Brunotte (see [22]). Its adaption for applying it within the SRS framework was presented in [3, Theorem 6.2]. We state these results using the formalism which was presented in [52].

We start with some basic definitions we will use. For $Q \subset \mathbb{R}^d, V \subset \mathbb{Z}^d, \mathbf{x} \in \mathbb{Z}^d$ let

$$\tau_Q(\mathbf{x}) = \{\tau_{\mathbf{r}}(\mathbf{x}) \mid \mathbf{r} \in Q\},$$
$$\tau_Q(V) = \{\tau_{\mathbf{r}}(\mathbf{v}) \mid \mathbf{r} \in Q, \mathbf{v} \in V\}.$$

Definition 2.1.5. Let $\mathbf{r} \in \mathcal{D}_d$. A set $\mathcal{V} \subset \mathbb{Z}^d$ that satisfies

1. $\forall \mathbf{x} \in \mathbb{Z}^d \exists k \in \mathbb{N}, (\mathbf{b}_1, \dots, \mathbf{b}_k) \in \mathcal{V}^k : \mathbf{x} = \sum_{j=1}^k \mathbf{b}_j$,
2. $\mathbf{x} \in \mathcal{V} \Rightarrow \tau_{\mathbf{r}}(\mathbf{x}), -\tau_{\mathbf{r}}(-\mathbf{x}) \in \mathcal{V}$

is called a *set of witnesses* of $\mathbf{r}$.

A set of witnesses has nice properties concerning $\tau_{\mathbf{r}}$. We will see this in the next theorem.

Theorem 2.1.6 (*cf.* [3, Theorem 5.1])**.** *Let $\mathbf{r} \in \mathcal{D}_d$ and $\mathcal{V}$ a set of witnesses of $\mathbf{r}$. $\mathbf{r} \in \mathcal{D}_d^0$ if and only if $\mathcal{V}$ does not contain purely periodic points with respect to $\tau_{\mathbf{r}}$ except $\mathbf{0}$.*

Proof. "$\Rightarrow$": This can be directly seen by the definition of $\mathcal{D}_d^0$. "$\Leftarrow$": This direction is a little more tricky. It is based on the observation that for $a, b \in \mathbb{R}$ the floor function satisfies as

$$\lfloor a+b \rfloor \in \{\lfloor a \rfloor + \lfloor b \rfloor, \lfloor a \rfloor + \lceil b \rceil = \lfloor a \rfloor - \lfloor -b \rfloor\}.$$

This implies that for any $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ we have

$$\tau_{\mathbf{r}}(\mathbf{a}+\mathbf{b}) \in \{\tau_{\mathbf{r}}(\mathbf{a}) + \tau_{\mathbf{r}}(\mathbf{b}), \tau_{\mathbf{r}}(\mathbf{a}) + (-\tau_{\mathbf{r}}(-\mathbf{b}))\}.$$

Now suppose that $\mathbf{b} \in \mathcal{V}$. Then there exists a $\mathbf{c} \in \mathcal{V}$ such that

$$\tau_{\mathbf{r}}(\mathbf{a}+\mathbf{b}) = \tau_{\mathbf{r}}(\mathbf{a}) + \mathbf{c}$$

by the definition of $\mathcal{V}$. Thus, if there is an $n \in \mathbb{N}$ such that $\tau_{\mathbf{r}}^n(\mathbf{a}) = \mathbf{0}$, we have that

$$\tau_{\mathbf{r}}^n(\mathbf{a}+\mathbf{b}) \in \mathcal{V}.$$

The fact that each element of $\mathbb{Z}^d$ can be represented as a finite sum of elements of $\mathcal{V}$ shows the desired result. □

It is also possible to define a set of witnesses for some $Q \subset \mathcal{D}_d$.

Definition 2.1.7. Let $Q \subset \mathcal{D}_d$. A set $\mathcal{V} \subset \mathbb{Z}^d$ that satisfies

1. $\forall \mathbf{x} \in \mathbb{Z}^d \exists k \in \mathbb{N}, (\mathbf{b}_1, \dots, \mathbf{b}_k) \in \mathcal{V}^k : \mathbf{x} = \sum_{j=1}^k \mathbf{b}_j$,
2. $\tau_Q(\mathcal{V}) \cup -\tau_Q(-\mathcal{V}) \subset \mathcal{V}$

is called a *set of witnesses* of Q.

It is easy to see that if $\mathcal{V}$ is a set of witnesses of Q then $\mathcal{V}$ is a set of witnesses of each $\mathbf{r} \in Q$.

Definition 2.1.8. For a finite set $\mathcal{W} \subset \mathbb{Z}^d$ and a set $Q \subset \mathcal{D}_d$, we define $G(\mathcal{W}, Q) = V \times E$ to be the smallest directed graph with vertices $V \subset \mathbb{Z}^d$ and edges $E \subset \mathbb{Z}^d \times \mathbb{Z}^d$ such that

1. $\mathcal{W} \subseteq V$,
2. $\tau_Q(V) \subset V$,
3. $E = \{(\mathbf{x}, \tau_{\mathbf{r}}(\mathbf{x})) | \mathbf{x} \in V, \mathbf{r} \in Q\}$.

We are interested in the (directed, simple) cycles of such a graph, *i.e.*, the paths with coinciding starting vertex and end vertex that visit each of the other vertices once at most. To avoid confusion we will refer to cycles of graphs as graph-cycles. Although, we will see that the two types of cycles are closely related in this context. A graph-cycle of length l consists of l d-dimensional integer vectors. By the definition of the edges a graph-cycle has the shape

$$(x_0, \dots, x_{d-1}) \to (x_1, \dots, x_d) \to \dots \to (x_{l-1}, \dots, x_{d-2}) \to (x_0, \dots, x_{d-1}).$$

Similar to cycles, a graph-cycle is uniquely determined by the l integers $x_0, \dots, x_{l-1}$. Again the elements are chronologically ordered but it is irrelevant which element is the initial one. By this consideration let us make the convention that, when we speak of a graph-cycle of $G(\mathcal{V}, Q)$ for some sets $\mathcal{V}$ and Q, we mean the integer sequence of corresponding length that determines it and denote it also by $\langle x_0, \dots, x_{l-1} \rangle$. By the definition of the graph-cycles such an integer sequence also fulfils the conditions of Lemma 1.1.3.

We now generalise Theorem 2.1.6 for an application on a $Q \subset \mathcal{D}_d$.

Theorem 2.1.9 (Brunotte Algorithm, *cf.* [3, Theorem 5.2]). *Let $Q \subset \mathcal{D}_d$ and suppose $\mathcal{V}$ is a finite set of witnesses of Q. Furthermore denote by Π_Q the set of graph-cycles of $G(\mathcal{V}, Q)$ without the trivial one $\langle 0 \rangle$. Then*

$$\mathcal{D}_d^0 \cap Q = Q \setminus \bigcup_{\pi \in \Pi_Q} P_d(\pi).$$

Proof. First note that $G(\mathcal{V}, Q)$ is finite since $\mathcal{V}$ is closed under an application of τ_Q and thus the set of vertices V equals $\mathcal{V}$ which is supposed to be a finite set. Let $\mathbf{r} \in Q$. Then $\mathcal{V}$ also includes a set of witnesses $\mathcal{V}_{\mathbf{r}}$ of $\mathbf{r}$. Let $\mathbf{z} \in \mathcal{V}_{\mathbf{r}}$. Then, by the construction of $G(\mathcal{V}, Q)$, there exists an edge $(\mathbf{z}, \tau_{\mathbf{r}}(\mathbf{z}))$ of $G(\mathcal{V}, Q)$. Hence, if $\mathbf{r} \notin \mathcal{D}_d^0$, there exists a graph-cycle $\pi \in \Pi_Q$ which is a cycle of $\tau_{\mathbf{r}}$ by Theorem 2.1.6 and therefore $\mathbf{r} \in P_d(\pi)$. This is true for all $\mathbf{r} \in Q$ which proves the theorem. $\square$

We do not know yet whether there exists a finite set of witnesses for a given $Q \subset \mathcal{D}_d$ anyway. We show later that such a set always exists, provided that Q is small enough, and explicitly construct a very small one.

Observe that not all of the graph-cycles obtained by the use of Theorem 2.1.9 are necessarily cycles. This is only true if Q is a single point.

Suppose π is a cycle such that $P_d(\pi) \cap Q \neq \emptyset$. Note that it is possible that π does not correspond to a graph-cycle of $G(\mathcal{V}, Q)$ for some set of witnesses $\mathcal{V}$. This effect does not cause any troubles in so far as in this case Theorem 2.1.9 provides a cycle ζ such that $P_d(\zeta) \supset P_d(\pi)$. Hence Theorem 2.1.9 provides enough cycles for characterising $Q \cap \mathcal{D}_d^0$. A more exact analysis of this phenomenon does not exist up to now. We can only mention that it depends on the chosen set of witnesses $\mathcal{V}$. We will later (in Theorem 2.1.13) deal with a set of witnesses that really induces all cycles.

We now turn to the question how to obtain a set of witnesses. Of course, $\mathbb{Z}^d$ itself is a set of witnesses but in the view of Theorem 2.1.9 we are mainly interested in finite ones. At first note that it is convenient to require Q to be simply connected. Then, by the linearity of the scalar product, we have that a set of witnesses of some set Q is also a set of witnesses of the convex hull of Q. We will present a way to construct a very small set of witnesses for a set Q and give a sufficient condition for Q to prove its finiteness. We will call this set of witnesses $\mathcal{V}(Q)$. By the above considerations we can suppose Q to be convex. Let $Q \subset \mathcal{D}_d$ and calculate the sequence $\mathcal{V}(Q)_0, \mathcal{V}(Q)_1, \ldots$ inductively by

$$\begin{aligned} \mathcal{V}(Q)_0 &:= \{(0,\ldots,0,1),(0,\ldots,0,-1)\}, \\ \mathcal{V}(Q)_{j+1} &:= \tau_Q(\mathcal{V}(Q)_j) \cup -\tau_Q(-\mathcal{V}(Q)_j) \cup \mathcal{V}(Q)_j \quad (j \geq 1). \end{aligned} \tag{2.1.3}$$

We obviously have $\mathcal{V}(Q)_{j+1} \supseteq \mathcal{V}(Q)_j$. Now set $\mathcal{V}(Q) = \lim_{n\to\infty} V(Q)_n$.

Lemma 2.1.10. *$\mathcal{V}(Q)$ is a set of witnesses of Q.*

Proof. Choose some $\mathbf{r} \in Q$. Since $(0,\ldots,0,1)$ and $(0,\ldots,0,-1)$ are in $\mathcal{V}(Q)_0$ we also have that $\{\tau_{\mathbf{r}}^i((0,\ldots,0,a)) \mid a = \pm 1, 0 \leq i < d\} \subset \mathcal{V}_j$ if $j \geq d-1$. The elements of this set have the shape

$$\begin{aligned} \tau_{\mathbf{r}}^i((0,\ldots,0,1)) &= (\underbrace{0,\ldots,0}_{d-1-i},1,\underbrace{\star,\ldots,\star}_{i}), \\ \tau_{\mathbf{r}}^i((0,\ldots,0,-1)) &= (\underbrace{0,\ldots,0}_{d-1-i},-1,\underbrace{\star,\ldots,\star}_{i}), \end{aligned}$$

where $\star$ denotes some integers. These $2d$ integer vectors are elements of $\mathcal{V}(Q)$ and therefore $\mathcal{V}(Q)$ fulfils item 1. of Definition 2.1.7. Item 2. of Definition 2.1.7 can be easily seen to be satisfied by the construction of $\mathcal{V}(Q)$. ☐

For this special type of set of witnesses we are now able to give a finiteness condition. For a matrix A with $\varrho(A) < 1$ and a $\delta \in \mathbb{R}$ with $1 > \delta > \varrho(A)$, denote by $\|\cdot\|_{A,\delta}$ a vector norm with

$$\forall \mathbf{x} \in \mathbb{R}^d : \|A\mathbf{x}\|_{A,\delta} \leq \delta \|\mathbf{x}\|_{\mathbf{A},\delta}. \tag{2.1.4}$$

One example of such a norm can be found in [41, Formula (3.2)]. It is given by

$$\|\mathbf{x}\|_{A,\delta} := \sum_{i=0}^{\infty} \delta^i \|A^{-i}\mathbf{x}\|_2,$$

but also [12, Formula 4.1] provides a norm with the desired property. Denote also by $\|\cdot\|_{A,\delta}$ a compatible matrix norm, *i.e.*,

$$\forall \mathbf{x} \in \mathbb{R}^d \forall B \in \mathbb{R}^{d\times d} : \|B\mathbf{x}\|_{A,\delta} \leq \|B\|_{A,\delta}\|\mathbf{x}\|_{A,\delta}.$$

Theorem 2.1.11. *Let $Q \subset \mathcal{E}_d$ convex. $\mathcal{V}(Q)$ is finite if there exist $\mathbf{r} \in Q$, $\delta, \delta' \in \mathbb{R}$ with $1 > \delta > \rho(R(\mathbf{r}))$ and $0 \leq \delta' < 1 - \delta$ such that $\|R(\mathbf{r}) - R(\mathbf{s})\|_{R(\mathbf{r}),\delta} \leq \delta'$ holds for all $\mathbf{s} \in Q$.*

Proof. Since $\mathcal{V}(Q)_j \subset \mathcal{V}(Q)_{j+1}$ for all $j \in \mathbb{N}$ it suffices to show the existence of a finite set $\mathcal{V}(Q)'$ such that $\mathcal{V}(Q)_k \subseteq \mathcal{V}(Q)'$ for all $k \geq 0$. Let

$$\begin{aligned} K &:= \|(0,\ldots,0,1)\|_{R(\mathbf{r}),\delta}, \\ \mathcal{V}(Q)' &:= \left\{ \mathbf{x} \in \mathbb{Z}^d \middle| \|\mathbf{x}\|_{R(\mathbf{r}),\delta} \leq \frac{K}{1-\delta-\delta'} \right\}. \end{aligned}$$

We will prove the claim by induction on k. For $k = 0$ we obviously have $\mathcal{V}(Q)_0 \subseteq V(Q)'$, since $0 < 1-\delta-\delta' < 1$ and therefore $K < \frac{K}{1-\delta-\delta'}$. Now suppose we already know that $\mathcal{V}(Q)_k \subseteq \mathcal{V}(Q)'$. Then we have

$$\mathcal{V}(Q)_{k+1} = \tau_Q(\mathcal{V}(Q)_k) \cup -\tau_Q(-\mathcal{V}(Q)_k) \cup \mathcal{V}(Q)_k \subset \tau_Q(\mathcal{V}(Q)') \cup -\tau_Q(-\mathcal{V}(Q)') \cup \mathcal{V}(Q)'. \tag{2.1.5}$$

For $\mathbf{x} \in \mathcal{V}'(Q), \mathbf{s} \in Q$ the functions $\tau_{\mathbf{s}}(\mathbf{x})$ and $-\tau_{\mathbf{s}}(-\mathbf{x})$ can be written as $R(\mathbf{s})\mathbf{x} + (0,\ldots,0,\nu)$ with $|\nu| < 1$. Thus

$$\begin{aligned} \|R(\mathbf{s})\mathbf{x} + (0,\ldots,0,\nu)\|_{R(\mathbf{r}),\delta} &\leq \|\left(R(\mathbf{s}) - R(\mathbf{r}) + R(\mathbf{r})\right)\mathbf{x}\|_{R(\mathbf{r}),\delta} + \|(0,\ldots,0,\nu)\|_{R(\mathbf{r}),\delta} \\ &< \|R(\mathbf{s}) - R(\mathbf{r})\|_{R(\mathbf{r}),\delta}\|\mathbf{x}\|_{R(\mathbf{r}),\delta} + \|R(\mathbf{r})\mathbf{x}\|_{R(\mathbf{r}),\delta} + K \\ &\leq \delta'\|\mathbf{x}\|_{R(\mathbf{r}),\delta} + \delta\|\mathbf{x}\|_{R(\mathbf{r}),\delta} + K \leq \frac{K(\delta'+\delta)}{1-\delta-\delta'} + K = \frac{K}{1-\delta-\delta'} \end{aligned}$$

shows that $\tau_{\mathbf{s}}(\mathbf{x}), -\tau_{\mathbf{s}}(-\mathbf{x}) \in \mathcal{V}'(Q)$. Note that this observation makes $\mathcal{V}(Q)'$ itself a set of witnesses. Hence (2.1.5) reduces to

$$\mathcal{V}(Q)_{k+1} \subset \tau_Q(\mathcal{V}(Q)') \cup -\tau_Q(-\mathcal{V}(Q)') \cup \mathcal{V}(Q)' = \mathcal{V}(Q)'$$

proving the theorem. □

This theorem shows that $\mathcal{V}(Q)$ is finite for sufficiently small $Q \subset \mathcal{E}_d$. We cannot expect $\mathcal{V}(Q)$ to be finite if $Q \cap \partial\mathcal{D}_d \neq \emptyset$. On the other hand we see that $\mathcal{V}(Q)$ for $Q = \{\mathbf{r}\}$ a single point is finite whenever $\mathbf{r} \in \mathcal{E}_d$. When we successively calculate $\mathcal{V}(Q)_0, \mathcal{V}(Q)_1, \ldots$ in order to obtain $\mathcal{V}(Q)$ then, for a sufficiently small $Q \subset \mathcal{E}_d$, there will exist an n such that $\mathcal{V}(Q)_n = \mathcal{V}(Q)_{n+1} =: \mathcal{V}(Q)$. For some $\mathcal{V}(Q)_j$ the calculation of $\mathcal{V}(Q)_{j+1}$ as it is described in (2.1.3) is only a theoretical one. In practice we proceed in the following way:

Definition 2.1.12. For a closed Q and an $\mathbf{x} \in \mathbb{Z}^d$ let $Q_{\mathbf{x}} \subset Q$ the set of those $\mathbf{r} \in Q$, where $\mathbf{rx}$ is extreme.

Because $\mathbf{rx}$ is linear and Q is closed, we have $Q_{\mathbf{x}} \subset \partial Q$ for each $\mathbf{x}$. The easiest case is when Q is a polygon. Then $Q_{\mathbf{x}}$ consists of its vertices, independently of $\mathbf{x}$. But also for non-polygonial Q with differentiable curves as boundaries it should be no problem to calculate $Q_{\mathbf{x}}$. With the usage of $Q_{\mathbf{x}}$, the rule of calculating $\mathcal{V}(Q)_{j+1}$ from $\mathcal{V}(Q)_j$ changes to

$$\mathcal{V}(Q)_{j+1} := \bigcup_{\mathbf{x} \in \mathcal{V}(Q)_j} \left\{ (x_2, \ldots, x_d, j) \middle| j = \min_{\mathbf{r} \in Q_{\mathbf{x}}} \{\lfloor -\mathbf{rx} \rfloor\}, \ldots, \max_{\mathbf{r} \in Q_{\mathbf{x}}} \{-\lfloor \mathbf{rx} \rfloor\} \right\} \cup \mathcal{V}(Q)_j, \quad (2.1.6)$$

where $\mathbf{x} = (x_1, \ldots, x_d)$.

2.1.3 Critical points

We already introduced the concept of critical points and will now go deeper into it. At first we note that critical points can only occur on the boundary of $\mathcal{D}_d$.

Theorem 2.1.13 (*cf.* [3, Theorem 7.2]). *If* $\mathbf{r} \in \mathcal{E}_d$ *then* $\mathbf{r}$ *is a regular point.*

Proof. Since $\varrho(R(\mathbf{r})) < 1$ we can find $\delta, \delta' > 0$ with $\varrho(R(\mathbf{r})) < \delta < 1$ and $\delta' < 1 - \delta$. Let $Q := \{s \in \mathcal{E}_d | \, \|R(\mathbf{r}) - R(\mathbf{s})\|_{R(\mathbf{r}),\delta} \leq \delta'\}$. In the proof of Theorem 2.1.11 the set

$$\mathcal{V}(Q)' := \left\{ \mathbf{x} \in \mathbb{Z}^d | \|\mathbf{x}\|_{R(\mathbf{r}),\delta} \leq \frac{K}{1 - \delta - \delta'} \right\}$$

with $K := \|(0, \ldots, 0, 1)\|_{R(\mathbf{r}),\delta}$ was recognised to be a set of witnesses of Q. Thus $Q \cap \mathcal{D}_d^0$ can be characterised using only finitely many cutout polyhedra by Theorem 2.1.9 ($G(V(Q)', Q)$ is a finite graph and thus can have only a finite number of cycles). As already mentioned Theorem 2.1.9 does not necessarily provide all cycles. This depends on the chosen set of witnesses. We now have to prove that the usage of $\mathcal{V}(Q)'$ yields all cycles. We obtain for sure the cycles lying within $\mathcal{V}(Q)'$. It is now enough to prove that for any $\mathbf{s} \in Q$ and $\mathbf{x} \in \mathbb{Z}^d$ with $\mathbf{x} \notin \mathcal{V}(Q)'$ there exists an $n \in \mathbb{N}$ such that $\tau_{\mathbf{s}}^n(\mathbf{x}) \in \mathcal{V}(Q)'$. Note that

$$\tau_{\mathbf{r}}^n(\mathbf{x}) = R(\mathbf{r})^n \mathbf{x} + \sum_{i=1}^{n} R(\mathbf{r})^{n-i} \mathbf{v}_i$$

with $\mathbf{v}_i = (0, \dots, 0, v_i)^T$ and $-1 < v_i \le 0$. Similar calculations as in Theorem 2.1.11 now show

$$\begin{aligned}\|\tau_{\mathbf{s}}^n(\mathbf{x})\|_{R(\mathbf{r}),\delta} <& \|R(\mathbf{s})^n\|_{R(\mathbf{r}),\delta}\|\mathbf{x}\|_{R(\mathbf{r}),\delta} + K\sum_{i=0}^{n-1}\|R(\mathbf{s})^i\|_{R(\mathbf{r}),\delta} \\ \le& \|\mathbf{x}\|_{R(\mathbf{r}),\delta}\left(\|R(\mathbf{s}) - R(\mathbf{r})\|_{R(\mathbf{r}),\delta} + \|R(\mathbf{r})\|_{R(\mathbf{r}),\delta}\right)^n \\ &+ K\sum_{i=0}^{n-1}\left(\|R(\mathbf{s}) - R(\mathbf{r})\|_{R(\mathbf{r}),\delta} + \|R(\mathbf{r})\|_{R(\mathbf{r}),\delta}\right)^i \\ \le& \|\mathbf{x}\|_{R(\mathbf{r}),\delta}(\delta' + \delta)^n + K\frac{1 - (\delta' + \delta)^n}{1 - \delta - \delta'}.\end{aligned}$$

Since $\mathbf{x}$ can only obtain integer values and $\delta' + \delta < 1$ we can find an $n \in \mathbb{N}$ such that

$$\|\tau_{\mathbf{s}}^n(\mathbf{x})\|_{R(\mathbf{r}),\delta} \le \frac{K}{1 - \delta - \delta'}$$

and thus $\tau_{\mathbf{s}}^n(\mathbf{x}) \in \mathcal{V}(Q)'$. Therefore $\mathcal{V}(Q)'$ really yields all cycles and only finitely many cutout polyhedra intersect with Q. The same is true for any open neighbourhood U of $\mathbf{r}$ with $U \subset Q$ making $\mathbf{r}$ a regular point. □

For characterising $\mathcal{D}_d^0$ we usually prefer the use of small sets of witnesses. We may loose some unimportant cycles but small sets are easier to handle. We now see that $\mathcal{V}(Q)'$, as we defined it in the proof of Theorem 2.1.11, is a maximal set of witnesses of Q in the sense that it induces all cycles.

Corollary 2.1.14. *If* $\mathbf{r}$ *is a weak critical point then* $\mathbf{r} \in \partial\mathcal{D}_d$.

Corollary 2.1.15 (*cf.* [3, Corollary 5.4]). *Let D be a closed subset of $\mathcal{E}_d$. Then $\mathcal{D}_d^0 \cap D$ intersects with only finitely many cutout polyhedra.*

Proof. From Theorem 2.1.13 we know that each point of D is a regular point. We can therefore find an open neighbourhood $U_{\mathbf{r}}$ for each point $\mathbf{r} \in D$ that intersects with only finitely many cutout polyhedra. Since D is compact there must exist a finite number of points $\mathbf{r}_1, \dots, \mathbf{r}_n$ with $D = \bigcup_{i=1}^n U_{\mathbf{r}_i}$. This proves the claim. □

We already noticed that there exist critical points for $d = 2$. In dimension one we only have a weak critical point.

Proposition 2.1.16. *The point $-1 \in \partial\mathcal{D}_1$ is a weak critical point.*

Proof. Consider $P_1(\langle a\rangle)$ for the cycles $\langle a\rangle$ with $a \in \mathbb{N}^*$. The points of $P_1(\langle a\rangle)$ are characterised by the inequality $0 \le ra + a < 1$. Hence $P_1(\langle a\rangle) = [-1, \frac{1-a}{a})$. Any open neighbourhood U of the point -1 intersects with infinitely many of such intervals and therefore -1 is a weak critical point. -1 is no critical point since for any sufficiently small neighbourhood U the set $(U \cap \mathcal{D}_1) \setminus \mathcal{D}_1^0$ can be fully characterised by $P_1(\langle 1\rangle)$. □

Similar calculations show that for $d = 2$ each point on the line $\overline{L_2 \cup L_5}$ (see (1.1.4)) is a weak critical point. Generally, by the Lifting Theorem 1.1.12, it is easy to see that if $(r_0, \dots, r_{d-1}) \in \partial\mathcal{D}_d$ is a (weak) critical point then

$$(\underbrace{0, \dots, 0}_{j}, r_0, \dots, r_{d-1}) \in \partial\mathcal{D}_{d+j}$$

is also a (weak) critical point for any $j > 0$.

2.1.4 Computational implementation

In Subsection 2.1.2 we introduced the Brunotte Algorithm and discussed how it can be used to characterise $\mathcal{D}_d^0$. It is obvious that the corresponding graphs and sets are growing very fast. This suggests the use of computers. We will now present some ideas how to implement the algorithm (in pseudocode). They are taken from [52]. An implementation in Mathematica® for the two dimensional case can be downloaded from the author's homepage [51].

We start with an algorithm that calculates the set of witnesses $\mathcal{V}(Q)$ for some given set Q. According to the discussion in Subsection 2.1.2 we may restrict ourselves to closed and convex sets. Lemma 2.1.11 shows that $\mathcal{V}(Q)$ is finite for sufficiently small Q. For our further discussions it is sufficient to know this. Algorithm 1 shows how the calculation of the set of witnesses could look like. We will refer to an application of this algorithm with parameters Q and p as $\mathrm{SOW}(Q,p)$

Algorithm 1 $\mathrm{SOW}(Q,p)$, calculation of $\mathcal{V}(Q)$.

Input: Q, p
Output: The set of witnesses $\mathcal{V}(Q)$

$\mathcal{V}(Q) \leftarrow \{(0,\dots,0,1),(0,\dots,0,-1)\}$
$M \leftarrow \emptyset$
while $(\mathcal{V} \neq M) \wedge (\#\mathcal{V}(Q) < p)$ **do**
  $N \leftarrow \mathcal{V}(Q) \setminus M$
  $M \leftarrow \mathcal{V}(Q)$
  for all $\mathbf{x} = (x_1,\dots,x_d) \in N$ **do**
    calculate $Q_{\mathbf{x}}$
    $i \leftarrow \min_{(r_1,\dots,r_d)\in Q_{\mathbf{x}}} \left\{ \left\lfloor -\sum_{k=1}^{d} x_k r_k \right\rfloor \right\}$
    $j \leftarrow \max_{(r_1,\dots,r_d)\in Q_{\mathbf{x}}} \left\{ -\left\lfloor \sum_{k=1}^{d} x_k r_k \right\rfloor \right\}$
    $\mathcal{V}(Q) \leftarrow \mathcal{V}(Q) \cup \{(x_2,\dots,x_d,k) | k = i,\dots,j\}$
  end for
end while
if $\mathcal{V}(Q) \neq M$ **then**
  Return(overflow)
else
  Return$(\mathcal{V}(Q))$
end if

(set of witnesses). The algorithm starts with

$$\mathcal{V} := \{(0,\dots,0,1),(0,\dots,0,-1)\} \tag{2.1.7}$$

and successively applies (2.1.6). Hence, for a finite set $\mathcal{V}(Q)$, the process will stabilise yielding $\mathcal{V}(Q)$. To avoid problems with the possible infiniteness of the set, we use an additional input parameter $p \in \mathbb{R}^+ \cup \{\infty\}$. If the size of the set of witnesses reaches p, the process stops and the algorithm returns "overflow". Concrete choices of p depend on the particular setting. At the moment it is just important to assure that the algorithm terminates. We also allow $p = \infty$ if we know that $\mathcal{V}(Q)$ is finite, for example, if Q is a single point.

The construction of the graph $G(\mathcal{W},Q) = V \times E$ for an initial set $\mathcal{W} \subset \mathbb{Z}^d$ and a set $Q \subset \mathcal{E}_d$ runs analogously, at least the calculation of the set of vertices V. By the same argument as above we will concentrate on closed, convex sets Q. For this purpose we have to modify Algorithm 1 a little bit in order to obtain Algorithm 2.

Again the algorithm builds up the set of vertices V inductively by starting with

$$V_0 = \mathcal{W} \tag{2.1.8}$$

and applying the rule

$$V_{i+1} = \bigcup_{\mathbf{x}\in V_i} \left\{ (x_2,\dots,x_d,j) \,\middle|\, j = \min_{\mathbf{r}\in Q_{\mathbf{x}}} -\lfloor \mathbf{rx} \rfloor, \dots, \max_{\mathbf{r}\in Q_{\mathbf{x}}} -\lfloor \mathbf{rx} \rfloor \right\} \cup V_i \tag{2.1.9}$$

Algorithm 2 $\mathrm{Gr}(\mathcal{W}, Q, p)$, calculation of $G(\mathcal{W}, Q)$.

Input: $\mathcal{W}$, Q, p
Output: The graph $G(\mathcal{W}, Q)$ as $V \times E$

$V \leftarrow \mathcal{W}$
$E \leftarrow \emptyset$
$M \leftarrow \emptyset$
while $(V \neq M) \wedge (\#\mathcal{V} < p)$ **do**
  $N \leftarrow V \setminus M$
  $M \leftarrow V$
  for all $\mathbf{x} = (x_1, \ldots, x_d) \in N$ **do**
    calculate $Q_{\mathbf{x}}$
    $i \leftarrow \min_{(r_1,\ldots,r_d) \in Q_{\mathbf{x}}} \left\{ - \left\lfloor \sum_{k=1}^{d} x_k r_k \right\rfloor \right\}$
    $j \leftarrow \max_{(r_1,\ldots,r_d) \in Q_{\mathbf{x}}} \left\{ - \left\lfloor \sum_{k=1}^{d} x_k r_k \right\rfloor \right\}$
    $V \leftarrow V \cup \{(x_2, \ldots, x_d, k) | k = i, \ldots, j\}$
    $E \leftarrow E \cup \{((x_1, \ldots, x_d,), x_2, \ldots, x_d, k) | \, k = i, \ldots, j\}$
  end for
end while
if $V \neq M$ **then**
  Return(overflow)
else
  Return$(V \times E)$
end if

with $\mathbf{x}$ denoting the vector $(x_1, \ldots, x_d)$. As soon as $V_{i+1} = V_i$ we set $V = V_i$. Simultaneously we calculate the edges in an obvious way. It is easy to see that the graph returned by Algorithm 2 is $G(\mathcal{W}, Q)$ (see Definition 2.1.8), of course, provided that the algorithm terminates without returning "overflow". In Definition 2.1.12 we defined the set $Q_{\mathbf{x}}$ and showed how to calculate $\mathcal{V}(Q)$ with its aid afterwards. We can use the same strategy for the calculation of $G(\mathcal{W}, Q)$. But how can the set $Q_{\mathbf{x}}$ be obtained algorithmically? `Mathematica`®, for example, provides implemented procedures for maximising and minimising given functions. They make use of the *cylindrical algebraic decomposition algorithm* (see [25]). A detailed overview concerning this topic can be found in [24].

Let us make a few remarks on the finiteness of $G(\mathcal{W}, Q)$. It is easy to prove this finiteness in an analogous way as in Lemma 2.1.11 for a sufficiently small closed convex set $Q \subset \mathcal{E}_d$ and a finite set $\mathcal{W}$. As item 2. of in Definition 2.1.7 obviously is a stronger condition thanitem 2. of Definition 2.1.8, we can expect that there are weaker requirements for $G(\mathcal{W}, Q)$ to be finite. In Section 2.2 we are going to calculate this graph with $Q \subset \mathcal{D}_2$ and $Q \cap \partial\mathcal{D}_2 \neq \emptyset$ and it is finite there. It is an up to now unanswered question what the exact conditions are for its finiteness.

We now can state Algorithm 3 that returns a list of all cycles that describes a given closed convex set $Q \subset \mathcal{E}_d$ when we assume that $|\mathcal{V}(Q)| < p$ for some $p \in \mathbb{R}^+ \cup \{\infty\}$. We call this algorithm $\mathrm{Br}_1(Q, p)$ since it is a first implementation of the Brunotte Algorithm. Remember that in general we will not obtain all cycles inducing polyhedra intersecting with Q. We only get enough for characterising $\mathcal{D}_d^0 \cap Q$. Graph-cycles that do not correspond to cycles are removed. However, note that we may get cycles whose corresponding polyhedra do not intersect with Q (the polyhedra then will be located very close to Q). They can be removed if it is required by the particular setting. Since we are mainly interested in an as detailed as possible characterisation of $\mathcal{D}_d^0$ it does not seem to be useful to remove these cycles.

We have to make a few remarks on finding cycles in a directed graph. In general our graph will have only few edges compared to the number of vertices. Cycles can only occur within the strongly connected components. They can be found with the aid of an algorithm of Tarjan [55]. Its requirements in time and space is linear to the size of V and E. Once the strongly connected

Algorithm 3 $\mathrm{Br}_1(Q,p)$, search for a set of cycles describing $Q \cap \mathcal{D}_d^0$.

Input: Q, p
Output: Π_Q list of cycles
$\mathcal{V}(Q) \leftarrow \mathrm{SOW}(Q,p)$
if $\neg$(overflow) **then**
  $G(\mathcal{V}(Q),Q) \leftarrow \mathrm{Gr}((\mathcal{V}(Q),Q,\infty))$
  $\Pi_Q \leftarrow$ all graph-cycles of $G(\mathcal{V}(Q),Q)$
  $\Pi_Q \leftarrow \Pi_Q \setminus \{\langle 0 \rangle\}$
  $\Pi_Q \leftarrow \Pi_Q \setminus \{\pi \in \Pi_Q | P_d(\pi) = \emptyset\}$
  Return(Π_Q)
else
  Return($|\mathcal{V}(Q)| > p$ for the chosen Q. Enlarge p or shrink Q.)
end if

components are found, we can extract the cycles from each such component.

We now present an improvement of Algorithm 3. It makes use of Corollary 2.1.15. Suppose $\mathcal{V}(Q)$ to be infinite for some given closed, convex $Q \subset \mathcal{E}_d$. Then we can divide Q into sufficiently small Q_i, $i \in I$ with finite I, such that $Q = \bigcup_{i \in I} Q_i$ and $\mathcal{V}(Q_i)$ is finite for each $i \in I$. Afterwards we apply Algorithm 3 on each Q_i separately.

Algorithmically this can be realised in the following way (*cf.* [52]). Given some closed, convex, set $Q \subset \mathcal{E}_d$ which is not equal to a single point and some bound $p_1 \in \mathbb{R}^+$ we apply $\mathrm{SOW}(Q,p_1)$. If the algorithm terminates correctly we get a finite set of witnesses $\mathcal{V}(Q)$ on which we can apply Theorem 2.1.9 and we are done. If an overflow is returned then $|\mathcal{V}(Q)| > p_0$ (and is possibly infinite). Now we subdivide Q into two closed, convex, non punctual sets Q_1 and Q_2 each of which contains more than one point and apply $\mathrm{SOW}(Q_i,p_2)$ for each $i \in \{1,2\}$. If this yields finite sets of witnesses $\mathcal{V}(Q_1)$, $\mathcal{V}(Q_2)$ we are done. Otherwise we proceed as above with each of the sets Q_1 and Q_2 separately, *i.e.*, splitting each of them into two parts and applying Algorithm 1 on each part with some bound $p_3 \in \mathbb{R}^+$. This idea can be realised in a recursive algorithm. In order to prove its termination we use the following

Algorithm 4 $\mathrm{Br}_2(Q)$, search for a set of cycles describing $Q \cap \mathcal{D}_d^0$ (recursively).

Input: Q
Output: list of cycles Π_Q
$p \leftarrow p(Q)$ suitable bound
$\mathcal{V}(Q) \leftarrow \mathrm{SOW}(Q,p)$
if $\neg$(overflow) **then**
  $G(\mathcal{V}(Q),Q) \leftarrow \mathrm{Gr}(\mathcal{V}(Q),Q,\infty)$
  $\Pi_Q \leftarrow$ all cycles of $G(\mathcal{V}(Q),Q)$
  $\Pi_Q \leftarrow \Pi_Q \setminus \{\langle 0 \rangle\}$
  $\Pi_Q \leftarrow \Pi_Q \setminus \{\pi \in \Pi_Q | P_d(\pi) = \emptyset\}$
else
  Split Q into sets Q_1, Q_2
  $\Pi_Q \leftarrow \mathrm{Br}_2(Q_1) \cup \mathrm{Br}_2(Q_2)$
end if
Return(Π_Q)

Lemma 2.1.17. *For any sequences $(p_n)_{n \in \mathbb{N}^*}$ of increasing non negative reals with no upper bound and $(Q_n)_{n \in \mathbb{N}^*}$ of closed, convex sets contained in $\mathcal{E}_d$ with $Q_n \subset Q_{n-1}$ and $\bigcap_{n \geq 0} Q_n = \mathbf{r}$ for some $\mathbf{r} \in \mathcal{E}_d$ there exists an n_0 such that $\mathrm{SOW}(Q_{n_0}, p_{n_0})$ terminates without returning an overflow.*

Proof. Let $\delta > \varrho(R(\mathbf{r}))$ and $U = \{\mathbf{s} \in \mathcal{E}_d | \, \|R(\mathbf{s}) - R(\mathbf{r})\|_{R(\mathbf{r}),\delta} < 1 - \delta\}$. By the conditions made on the sequence $(Q_n)_{n \in \mathbb{N}^*}$ there exists an n_1 such that $Q_n \subset U$ for all $n \geq n_1$. Due to

Theorem 2.1.11 the set $\mathcal{V}(Q_{n_1})$ is finite and obviously $|\mathcal{V}(Q_{n+1})| \leq |\mathcal{V}(Q_n)|$ for $n \geq n_1$. Now, by the unboundedness of $(p_n)_{n\in\mathbb{N}^*}$, there must exist an $n_0 \geq n_1$ such that $|\mathcal{V}(Q_{n_0})| \leq p_{n_0}$ and thus $\mathrm{SOW}(Q_{n_0}, p_{n_0})$ terminates without returning an overflow. $\square$

How can we split Q and find a suitable value of p in order to ensure the termination of Algorithm 4? We will assume that p depends on Q. Thus $p = p(Q)$. We give a concrete example afterwards. For a given Q the splitting rule induces a binary tree where each vertex corresponds to a subset of Q. The root of this tree corresponds to Q itself. If a vertex corresponds to some set $\tilde{Q}$ then the two children correspond to the sets $\tilde{Q}_1$ and $\tilde{Q}_2$ gained by applying the splitting rule on $\tilde{Q}$ and therefore $\tilde{Q}_1 \cup \tilde{Q}_2 = \tilde{Q}$. Each set $\tilde{Q}$ induces a bound $p(\tilde{Q})$ and, additionally, each vertex induces a level in a canonical way: the root has level 1 and if a vertex has level l each of its two children has level $l+1$. We label each vertex with the corresponding pair $(\tilde{Q}, l)$ and call the tree $T(Q)$. Note that $T(Q)$ is infinite.

Theorem 2.1.18. *Let $Q \subset \mathcal{E}_d$ be closed and convex and $T(Q)$ the tree from above. Each simple infinite path of $T(Q)$ starting at the root induces a sequence $((Q_n, n))_{n\in\mathbb{N}^*}$. $\mathrm{Br}_2(Q)$ terminates when for all simple paths the corresponding sequences $(Q_i)_{i\in\mathbb{N}^*}$ and $(p(Q_i))_{i\in\mathbb{N}^*}$ satisfy the conditions of Lemma 2.1.17.*

Proof. This is an immediate consequence of Lemma 2.1.17. $\square$

What are concrete examples of splitting rules and bounds $p(Q)$ to assure the determination of Algorithm 4? The only way of splitting a general convex set into two convex sets is by a line. We will concentrate on the following simple and easily calculated splitting rule. For a given closed, convex set $Q \subset \mathcal{E}_d$, $\#Q > 1$, let

$$m_j = \min_{(r_1,\ldots,r_d)\in Q} \{r_j\} \quad (1 \leq j \leq d),$$
$$M_j = \max_{(r_1,\ldots,r_d)\in Q} \{r_j\} \quad (1 \leq j \leq d)$$

and $k \in \{1, \ldots, d\}$ the minimal index such that $M_k - m_k = \max_{j\in\{1,\ldots,d\}}\{M_j - m_j\}$. The fact that $\#Q > 1$ ensures that $M_k - m_k > 0$. Now Q is split into $Q_1(Q)$ and $Q_2(Q)$ with

$$Q_1(Q) = \left\{(r_1, \ldots, r_d) \in Q \,\middle|\, r_k \leq \frac{m_k + M_k}{2}\right\},$$
$$Q_2(Q) = \left\{(r_1, \ldots, r_d) \in Q \,\middle|\, r_k \geq \frac{m_k + M_k}{2}\right\}.$$

For the suitable bound we set $p(Q) = \frac{c}{M_k - m_k}$ with constant $c > 0$. It is easy to see that any sequence $(\tilde{Q}_n)_{n\in\mathbb{N}}$ of sets gained by successive application of that splitting rule, *i.e.*, $\tilde{Q}_n \in \{Q_1(\tilde{Q}_{n-1}), Q_2(\tilde{Q}_{n-1})\}$ for each $n > 1$, converges to a point and the sequence $(p(\tilde{Q}_n))_{n\in\mathbb{N}}$ is increasing and unbounded. Hence, with this splitting rule and this choice of $p(Q)$, Algorithm 4 will terminate for each convex, closed $Q \subset \mathcal{E}_d$, $\#Q > 1$. This setting will be used throughout the whole thesis. We will not (and cannot) give optimal values for the constant c. It strongly depends on the position of Q relative to the boundary of $\mathcal{D}_d$. We restricted ourselves to bounds p that do only depend on Q. In this context in seems to be useful, since we have to determine m_k and M_k for the splitting anyway. One may also consider p to be dependent on the level of recursion.

2.2 About the set $\mathcal{D}_2^0$

2.2.1 An algorithm for analysing areas near the upper boundary of $\mathcal{D}_2$

From Theorem 2.1.11 we know that the size of sets of witnesses "grows" when we move towards the boundary of $\mathcal{D}_d$. Therefore it is difficult to use algorithms based on Theorem 2.1.9 for such areas. Akiyama *et al.* presented in [6, Section 4] an algorithmic way to find subsets of $\mathcal{D}_2^0$ near the upper boundary of $\mathcal{D}_2$. Here we will present these ideas. Let

$$R = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, 0 < x < \frac{y^2}{4}, 0 < y < x + 1 \right\}.$$

Note that R is clearly a subset of the interior of $\mathcal{D}_2$ and that the the eigenvalues of $R(\mathbf{r})$ for $\mathbf{r} \in R$ are real and negative. Denote them by $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ and assume w.l.o.g. that $-1 < \alpha(\mathbf{r}) < \beta(\mathbf{r}) < 0$.

Lemma 2.2.1 (*cf.* [6]). *Let* $\mathbf{r} = (x, y) \in R$ *and* $\langle a_0, \ldots, a_{l-1} \rangle \in \mathcal{O}(C(\mathbf{r}))$. *Then for all* $i \in \{0, \ldots, l-1\}$ *we have*

$$\frac{\beta}{1 - \beta(\mathbf{r})^2} \leq a_{i+1} - \alpha(\mathbf{r}) a_i \leq \frac{1}{1 - \beta(\mathbf{r})^2}$$

with the indices taken modulo l.

Proof. Take the indices of a modulo l for the rest of the proof. By (1.1.5) we have for all $j \in \mathbb{Z}$

$$0 \leq x a_j + y a_{j+1} + a_{j+2} < 1.$$

Since $x = \alpha(\mathbf{r})\beta(\mathbf{r})$ and $-y = \alpha(\mathbf{r}) + \beta(\mathbf{r})$ this is equivalent to

$$0 \leq (a_{j+2} - \alpha(\mathbf{r}) a_{j+1}) - \beta(\mathbf{r})(a_{j+1} - \alpha(\mathbf{r}) a_j) < 1.$$

Denote this double inequality by I_j. Of course, $I_j = I_{j+l}$ for all $j \in \mathbb{N}$. Now choose some $i \in \{0, \ldots, l-1\}$ and note that $I_i + \beta(\mathbf{r}) I_{i-1}$ gives

$$\beta(\mathbf{r}) < (a_{i+2} - \alpha(\mathbf{r}) a_{i+1}) - \beta(\mathbf{r})^2 (a_i - \alpha(\mathbf{r}) a_{i-1}) < 1.$$

Thus $\sum_{j=0}^{n} \alpha(\mathbf{r})^j I_{i-j}$ gives

$$\sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \beta(\mathbf{r})^{2j-1} < (a_{i+1} - \alpha(\mathbf{r})(\mathbf{r}) a_{i+1}) - \beta(\mathbf{r})^{n+1} (a_{i-n+1} - \alpha(\mathbf{r}) a_{i-n}) < \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \beta(\mathbf{r})^{2j}.$$

Since $a_{i-n+1} - \beta(\mathbf{r}) a_{i-n}$ is bounded for all n (in fact this difference can obtain only l different values) we can take the limit $n \to \infty$ which shows the inequality. $\square$

Now consider the polynomial $P_q(t) = qt^3 + qt^2 - qt - q + 1$ for $q \in \mathbb{N}_0$. Denote by η_q the greatest (real) root of P_q.

Lemma 2.2.2 (*cf.* [6]). $\frac{1}{3} < \eta_q < 1$ *for all* $q \in \mathbb{N}_0$. *The other roots of* P_q *are real and not positive. Additionally the sequence* $(\eta_q)_{q \in \mathbb{N}_0}$ *is strictly increasing with* $\lim_{q \mapsto \infty} \eta_q = 1$.

Proof. An easy calculation shows that P_q has a local minimum at $\frac{1}{3}$ and a local maximum at -1, independently of q, with $P_q(\frac{1}{3}) = -q\frac{32}{27} + 1 < 0$ and $P_q(-1) = 1$. Thus, since the leading coefficient of P_q is $q > 0$ we have $\eta_q > \frac{1}{3}$ and another root is negative. Since the constant term of P_q equals $-q + 1 \leq 0$ the third root is smaller or equal to 0 (and, of course, greater than -1) where equality holds exactly for $q = 1$. Since $P_q(1) = 1$ for all q we also have $\eta_q < 1$.

Now consider the polynomial $P'(t) = t^3 + t^2 - t - 1$. It has a root at 1 and also a local maximum at -1 and a local minimum at $\frac{1}{3}$ with $P'(\frac{1}{3}) < 1$. Thus we must have $P'(t) < 0$ for $t \in [1/3, 1)$. Therefore $P_q(\eta_{q-1}) = P'(\eta_{q-1}) < 0$ which induces $\eta_{q-1} < \eta_q$ showing that $(\eta_q)_{q \in \mathbb{N}_0}$ is strictly increasing. Finally note that $\frac{1}{q} P_q(\eta_q) = P'(\eta_q) + \frac{1}{q} = 0$ holds for all $q \geq 1$. Consider the limit for $q \to \infty$. Since polynomial functions are continuous we have

$$\lim_{q \to \infty} \left(P'(\eta_q) + \frac{1}{q} \right) = P'(\lim_{q \to \infty} \eta_q) = 0$$

which shows that $\lim_{q \to \infty} \eta_q$ is a root of P' and by the above considerations the only questionable root is 1. $\square$

For $\kappa \in (0,1)$ define

$$R_\kappa = \left\{(x,y) \in R \,|\, x < \kappa y - \kappa^2\right\}.$$

Lemma 2.2.3 (*cf.* [6]). *Let $q \in \mathbb{N}_0$ and $\kappa \in (0, \eta_q]$. Then*

$$\alpha(\mathbf{r}) < -\kappa < \beta(\mathbf{r}) \text{ and } \frac{1}{(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)} < q$$

holds for all $\mathbf{r} = (x,y) \in R_\kappa$.

Proof. Since $\mathbf{r} \in R_\kappa$ we have $x < \kappa y - \kappa^2$ and therefore $(\alpha(\mathbf{r}) + \kappa)(\beta(\mathbf{r}) + \kappa) < 0$. Since $-1 < \alpha < \beta < 0$ this immediately implies $\alpha(\mathbf{r}) + \kappa < 0$ and $\beta(\mathbf{r}) + \kappa > 0$ and thus $\alpha(\mathbf{r}) < -\kappa < \beta(\mathbf{r})$.

In Lemma 2.2.2 we investigated the locations of the roots of P_q and found that the only positive root is η_q while the other roots are real and not positive. We had a minimum at $\frac{1}{3}$ where the polynomial function obtains a negative value. Thus $P_q(\kappa) \leq 0$ for all $\kappa \in (0, \eta_q]$. Hence

$$1 \leq 1 - P_q(\kappa) = q(1+\kappa)(1-\kappa^2).$$

Using the inequality $\alpha(\mathbf{r}) < -\kappa < \beta(\mathbf{r})$ we see that

$$1 \leq q(1+\kappa)(1-\kappa^2) < q(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2).$$

Division by the positive term $(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)$ proves the lemma. ☐

Lemma 2.2.4 (*cf.* [6]). *Let $q \in \mathbb{N}_0$, $\kappa \in (0,\eta_q]$ and $\mathbf{r} = (x,y) \in R_\kappa$. For each $\langle a_0, \ldots, a_{l-1}\rangle \in \mathcal{O}(C(\mathbf{r}))$ there exists a $j \in \{0, \ldots, l-1\}$ with $|a_j| < q$.*

Proof. We will show that there is one index j with

$$-\frac{1}{(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)} \leq a_j \leq \frac{1}{(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)}.$$

Then the theorem follows directly by Lemma 2.2.3. Suppose that this is not the case and for all $i \in \{0, \ldots, l-1\}$ we have

$$|a_i| > \frac{1}{(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)}.$$

Take the inequality of Lemma 2.2.1 and sum up over all indices $i \in \{0, \ldots, l-1\}$. This yields

$$\frac{l\beta(\mathbf{r})}{1-\beta(\mathbf{r})^2} \leq (1-\alpha(\mathbf{r}))\sum_{i=0}^{l-1} a_{i+1} \leq \frac{l}{1-\beta(\mathbf{r})^2}.$$

Division by $(1-\alpha(\mathbf{r})) > 0$ and l shows that neither $a_i < -\frac{1}{(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)}$ nor $a_i > \frac{1}{(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)}$ for all $i \in \{0, \ldots, l-1\}$. For the rest of the proof consider the indices of a modulo l. Suppose that there is an index $i \in \{0, \ldots, l-1\}$ with $a_i, a_{i+1} < -\frac{1}{(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)}$. Then $a_i - \alpha(\mathbf{r})a_{i+1} < -\frac{1-\alpha(\mathbf{r})}{(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)} = -\frac{l}{1-\beta(\mathbf{r})^2}$ contradicting Lemma 2.2.1. Similarly it can be shown that $a_i, a_{i+1} > \frac{1}{(1-\alpha(\mathbf{r}))(1-\beta(\mathbf{r})^2)}$ cannot hold for two consecutive indices. This induces l to be even and, w.l.o.g.,

$$a_0, a_2, a_4, \ldots, a_{l-2} > 0, \quad a_1, a_3, a_5, \ldots, a_{l-1} < 0.$$

Note that

$$0 \leq a_i x + a_{i+1} y + a_{i+2} < 1$$

holds for all $i \in \{0, \ldots, l-1\}$. Summing up yields $0 \leq \sum_{i=0}^{l} a_i$. Hence, w.l.o.g., we may suppose $a_0 \geq -a_1$. Then, by observing that $y - x < 1$ we have

$$a_2 < -a_0 x - a_1 y + 1 \leq -a_1(y-x) + 1 < -a_1 + 1$$

and since a_1 and a_2 can only obtain integer values this induces $-a_1 \geq a_2$. Now a similar estimation for a_3 gives

$$-a_3 \leq a_1x + a_2y \leq a_2(y-x) < a_2$$

and therefore $a_2 > -a_3$. Continuing in this way gives the sequence

$$a_0 \geq -a_1 \geq a_2 > -a_3 \geq a_4 > a_5 \geq \cdots.$$

Because of periodicity this is only possible for $l = 2$ and $a_0 = -a_1 = c > 0$. However, since $y - x - 1 < 0$ we have $-cx + cy - c < 0$ which contradicts $0 \leq xa_1 + ya_0 + a_1$. □

Define the set

$$A_{\kappa,q} := \left\{ (a,b) \in \mathbb{Z}^2 \middle| -q < a < q, -\frac{\kappa}{1-\kappa^2} - q + 1 < b < \frac{\kappa}{1-\kappa^2} + q - 1 \right\}.$$

Theorem 2.2.5 (*cf.* [6]). *Let $q \in \mathbb{N}_0$ and $\kappa \in (0, \eta_q]$ and Π_{R_κ} the set of graph-cycles induced by $G(A_{\kappa,q}, R_\kappa)$. If $(a_0, \dots, a_l - 1) \in \overline{C(\mathbf{r})}$ for some $\mathbf{r} \in R_\kappa$ then $(a_0, \dots, a_{l-1}) \in \Pi_{R_\kappa}$.*

Proof. By Lemma 2.2.4 we can suppose that, w.l.o.g., $-q < a_0 < q$. Due to Lemma 2.2.1 we have

$$-\frac{1}{1-\beta(\mathbf{r})^2} + \alpha(\mathbf{r})a_0 \leq a_1 \leq \frac{1}{1-\beta(\mathbf{r})^2} + \alpha(\mathbf{r})a_0.$$

The usage of of Lemma 2.2.3 immediately yields $(a_0, a_1) \in A_{\kappa,q}$. Now, by Definition 2.1.8, (a_i, a_{i+1}) is a vertex of $G(A_{\kappa,q}, R_\kappa)$ and additionally there exists an edge from (a_i, a_{i+1}) to (a_{i+1}, a_{i+2}) for all $i \in \{0, \dots, l-1\}$ (indices modulo l). Therefore $\langle a_0, \dots, a_{l-1} \rangle \in \Pi_{R_\kappa}$. □

Corollary 2.2.6. *Let $\kappa \in (0, \gamma_q]$ for some $q \in \mathbb{N}_0$, $Q \subset \mathcal{D}_2$ and Π_Q the set of all nontrivial cycles induced by $G(A_{\kappa,q}, Q)$. Then*

$$\left(Q \cap \bigcup_{0<\iota\leq\kappa} R_\iota \right) \setminus \bigcup_{\pi \in \Pi_Q} P_2(\pi) \subset \mathcal{D}_2^0.$$

Proof. Observe that $A_{\iota,q} \subseteq A_{\kappa,q}$ for $0 < \iota \leq \kappa$. Then the corollary follows immediately from Theorem 2.2.5. □

The following is due to [52]. For closed Q we can use the results of Subsection 2.1.4 to state an algorithm for determining which areas of R are contained in $\mathcal{D}_2^0$. Whenever parts of the line $y = x + 1$ are included in Q, $G(A_{\kappa,q}, Q)$ will contain a lot of cycles of the form $\langle a, -a \rangle$. These cycles correspond exactly to the line $y = x + 1$ which we already know not to be part of $\mathcal{D}_2^0$ (see Theorem 1.1.16). Based on these considerations we can adopt Algorithm 5. Since the Algorithm is based on results presented by Akiyama, Brunotte, Pethő and Thuswaldner [6] its application with parameters Q, q and p is denoted by $\mathrm{ABPT}(Q, q, p)$. In Subsection 2.1.4 we mentioned the advantage of using convex sets Q. Note that R is not convex and thus it may happen that Q is not fully contained in R. Note that from Lemma 2.2.2 we see that for growing q the sets R_{η_q} become smaller and move towards the point $(2, 1)$. The closer we come to this point the bigger $A_{\eta_q,q}$ is. This means that for big q it is difficult to obtain characterisation results for $R_{\eta_q} \cap \mathcal{D}_2^0$.

2.2.2 Computational results

In this subsection we present characterisation results concerning $\mathcal{D}_2^0$ obtained by the application of Algorithm 4 and Algorithm 5. It is a summary of several lemmas and theorems taken from [52]. Figure 2.1 shows a sketch of the subsets of $\mathcal{D}_2$ treated in this and the next subsection. The black areas are known not to belong to $\mathcal{D}_2^0$. The light grey sets are depicted in a magnified way in Figure 2.2. In [6, section 4.1] the set R was analysed for $x \leq 5/6$ without having found any cycle. Additionally the sets R_{η_q} for $q = 3, \dots, 6$ have also been shown to belong to $\mathcal{D}_2^0$ ([6, Theorem 4.8]). It is possible to continue this series.

Algorithm 5 ABPT(Q, q, p), search for all cycles within an area $Q \cap \bigcup_{0<\iota<\eta_q} R_\iota$.

Input: Q, q, p
Output: list of cycles Π_Q

calculate $A_{\eta_q,q}$
$G(A_{\eta_q,q}, Q) \leftarrow \mathrm{Gr}(A_{\eta_q,q}, Q, p)$
if $\neg$(overflow) **then**
  $\Pi_Q \leftarrow$ all cycles of $G(A_{\eta_q,q}, Q)$
  $\Pi_Q \leftarrow \Pi_Q \setminus (\{\langle 0 \rangle\} \cup \{\langle a, -a \rangle \,|\, a \in \mathbb{N}_0\})$
  $\Pi_Q \leftarrow \Pi_Q \setminus \{\pi \in \Pi_Q | P_2(\pi) = \emptyset\}$
  Return(Π_Q)
else
  Return(The corresponding graph is bigger than the given bound)
end if

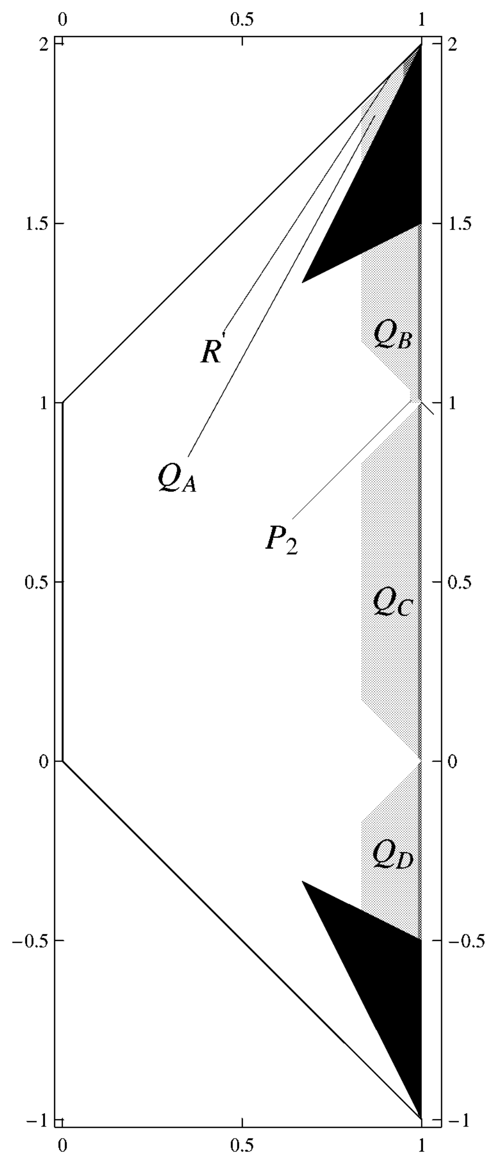


Figure 2.1: Overview of the location of several subsets of $\mathcal{D}_2$

Theorem 2.2.7. *$R_{\eta_q} \subset \mathcal{D}_2^0$ for all $q \in \{7,\dots,11\}$.*

Proof. The usage of ABPT$(\overline{R_{\eta_q}}, q, \infty)$ returns no cycles for $q \in \{7,\dots,11\}$. $\square$

Note that $R_{\eta_q} \cup R_{\eta_{q+1}} \subset \bigcup_{\eta_q \le \kappa \le \eta_{q+1}} R_\kappa$. Let

$$E_q := \left\{(x,y) \in R \,\middle|\, x \ge \eta_{q-1} y - \eta_{q-1}^2, x \ge \eta_q y - \eta_q^2 \right\}.$$

We have

$$E_q = \bigcup_{\eta_q \le \kappa \le \eta_{q+1}} R_\kappa \setminus \left(R_{\eta_q} \cup R_{\eta_{q+1}}\right).$$

Therefore, For $4 \le q \le 11$, E_q remained uninvestigated. We will analyse these sets in the next theorem.

Theorem 2.2.8. *$E_q \subset \mathcal{D}_2^0$ for all $q \in \{4,\dots,11\}$.*

Proof. Application of ABPT$(\overline{E_q}, q, \infty)$ returns no cycles for $q \in \{4,\dots,11\}$. $\square$

In Figure 2.1 we denoted union of the sets treated in Theorem 2.2.7 and Theorem 2.2.8 by R'. We cannot recognise them since they are very small but we get an idea of their location. Summing up all the results concerning R gives

Theorem 2.2.9. *$\bigcup_{0<\kappa\le\gamma_{11}} R_\kappa \supset \left\{(x,y) \in R \,\middle|\, x \le \frac{19}{20}\right\}$ is contained in $\mathcal{D}_2^0$.*

Theorem 2.2.10.

$$Q_A := \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \frac{5}{6} \le x \le \frac{19}{20}, 2x \le y, x \ge \frac{y^4}{4}\right\} \subset \mathcal{D}_2^0.$$

Proof. The application of Algorithm 4 with $c = 200$ (see the considerations after Theorem 2.1.18) returns no cycles. $\square$

Up to now the analysis of $\mathcal{D}_2^0$ seems to be easy and the number of characterising cycles accessible. This will change now. The next results are based on Algorithm 4 with $c = 20$.

Theorem 2.2.11.

$$Q_B := \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \frac{5}{6} \le x \le \frac{99}{100} \wedge -x+2 \le y \le 1 + \frac{x}{2}\right\} \cap \mathcal{D}_2^0$$

can be described by 787 *cycles.*

Theorem 2.2.12.

$$Q_C := \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \frac{5}{6} \le x \le \frac{99}{100} \wedge -x+1 \le y \le x\right\} \cap \mathcal{D}_2^0$$

can be described by 1010 *cycles.*

Theorem 2.2.13.

$$Q_D := \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \frac{5}{6} \le x \le \frac{99}{100} \wedge -\frac{x}{2} \le y \le x-1\right\} \cap \mathcal{D}_2^0$$

can be described by 402 *cycles.*

For computational processing a list of all these cycles is available as a `Mathematica`® notebook file in the internet [51]. The maximal length of these cycles is 130, the entries have modulus 74 at most. Note that it is not verified whether all of them are really necessary to characterise $\mathcal{D}_2^0$ in the particular region and which ones are possibly totally covered by others. In view of the number of the cycles of several hundreds this seems to be a difficult job. Figure 2.2 shows the three sets Q_B, Q_C and Q_D in more detail. The grey parts do not belong to the sets and mark the missing gap to the right boundary of $\mathcal{D}_2$. In Subsection 2.2.4 we will present cycles that lie within these parts but with the currently available methods it seems to be impossible to analyse them more closely. The cutout polygons that are induced to the computed cycles are depicted in black. The white areas are definitely subsets of $\mathcal{D}_2^0$.

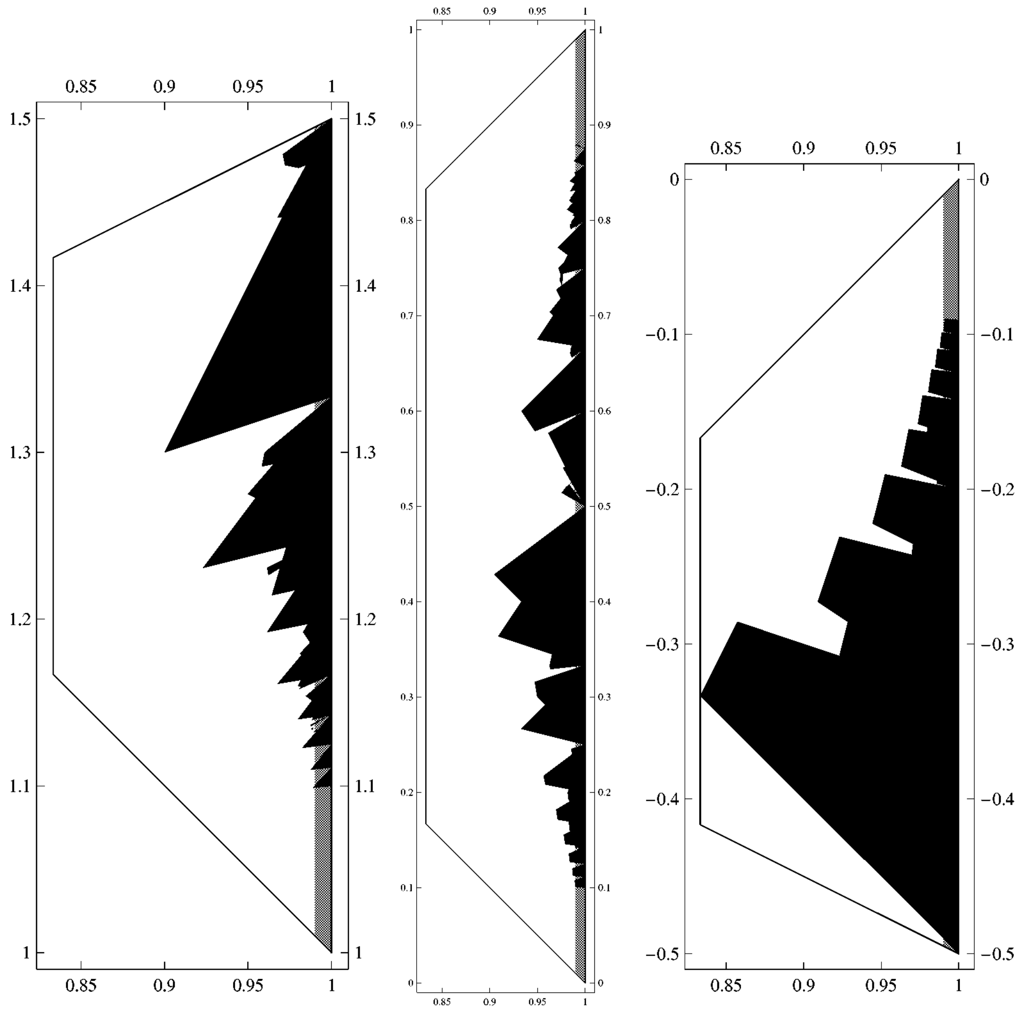


Figure 2.2: The position of the cutout polyhedra within the sets Q_B, Q_C and Q_D (from left to right)

2.2.3 An area near the right boundary of $\mathcal{D}_2$

For areas near the right boundary of $\mathcal{D}_2$ an application of the presented algorithms fails. The only possible way to prove a set Q to be a subset of $\mathcal{D}_2^0$ seems to be a direct observation of the behaviour of the orbits of the mapping $\tau_{\mathbf{r}}$ for $\mathbf{r} \in Q$. The following results are taken from [52]. Note that this result is similar to [6, Theorem 4.27].

Theorem 2.2.14.

$$P_2 := \left\{ (1-T, 1+\delta T) \,\middle|\, 0 < T < \frac{1}{30}, 0 \leq \delta \leq 1 \right\} \subset \mathcal{D}_2^0.$$

Proof. Fix $T \in (0, \frac{1}{30})$ and $\delta \in [0,1]$. Let $\mathbf{r} = (1-T, 1+\delta T) \in P_2$. Furthermore define

$$A := \left\{ (x,y) \in \mathbb{Z}^2 \,|\, x \leq 0, y < 0 \right\},$$
$$B := \left\{ (x,y) \in \mathbb{Z}^2 \,|\, x \geq 0, y > 0 \right\}.$$

These sets represent the third quadrant together with the negative y-axis and the first quadrant together with the positive y-axis, respectively. We will prove the statement by showing that $\tau_{\mathbf{r}}^p$

sends each point of $\mathbb{Z}^2$ to $\mathbf{0}$ for some $p \in \mathbb{N}$. The idea is not very complicated, but several technical lemmas are needed. These lemmas are proven afterwards. Everything is based on the fact that the application of $\tau_{\mathbf{r}}^3$ changes a point only little. Figure 2.3 shows the orbit of the point $(-200, 200)$ for $T = \frac{1}{50}$ and $\delta = 1$. It is divided into three branches. After three applications of $\tau_{\mathbf{r}}$ we return to one branch. We now look at the sequence $\{\tau_{\mathbf{r}}^n(\mathbf{z})\}_{n\in\mathbb{N}}$ of a point $\mathbf{z} \in \mathbb{Z}^2$. We will show the

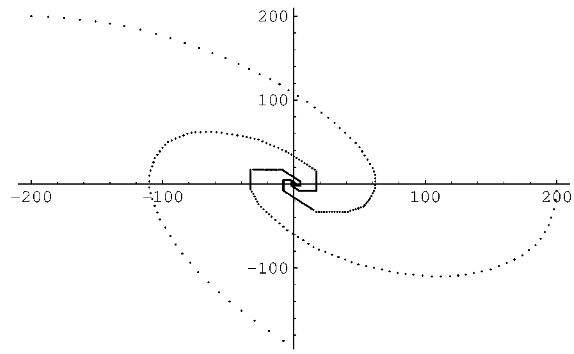


Figure 2.3: The orbit of the point $(-200, 200)$

existence of a finite subsequence $\{\mathbf{z}_0, \ldots, \mathbf{z}_{q_0}\}$ that ends up in $\mathbf{0}$. This proves the theorem. We first assert that each point in $\mathbb{Z}^2$ has an orbit that intersects with $A \cup B \cup \{\mathbf{0}\}$. This is shown in Lemma 2.2.25. Hence, without loss of generality, we can start our subsequence with $\mathbf{z}_0 \in A$ (for B the proof runs analogously). For a $\mathbf{z}_q \in A$, $q > 0$, construct $\mathbf{z}_{q+1}$ in the following way: Let $(u_0, v_0) := \mathbf{z}_q$. For an $i \geq 0$ set $(u_{i+1}, v_{i+1}) := \tau_{\mathbf{r}}^3(u_i, v_i)$. Then for $(u_i, v_i) \in A$ the following points (which are shown in the mentioned lemmas) are true:

$$u_{i+1} \leq 0 \quad \text{(Lemma 2.2.15)}, \tag{2.2.1}$$
$$u_{i+1} + v_{i+1} \geq -\left\|(u_i, v_i)\right\|_1 \quad \text{(Lemma 2.2.17)}, \tag{2.2.2}$$
$$v_{i+1} - v_i \geq 1 \quad \text{(Lemma 2.2.19)}. \tag{2.2.3}$$

Formula (2.2.3) ensures that there is no repetition possible and hence there cannot exist a cycle within the set A. By (2.2.1) and (2.2.3) can further be seen that either $(u_{i+1}, v_{i+1}) \in A$ or $v_{i+1} \geq 0$. Thus there exists a $j \in \mathbb{N}$ with $(u_i, v_i) \in A$ for $i \leq j$ and $(u_{j+1}, v_{j+1}) \notin A$ where (u_{j+1}, v_{j+1}) lies on or above the x-axis. Additionally the length of (u_i, v_i) is not growing with respect to the 1-norm. Now apply $\tau_{\mathbf{r}}$ once. Then Lemma 2.2.21 says that either $\tau_{\mathbf{r}}(u_{j+1}, v_{j+1}) = \mathbf{0}$ or $\tau_{\mathbf{r}}(u_{j+1}, v_{j+1}) \in B$. Moreover we always have $\left\|\tau_{\mathbf{r}}(u_{j+1}, v_{j+1})\right\|_1 \leq \left\|(u_j, v_j)\right\|_1$ which is shown in Lemma 2.2.23. Now, if $(x_0, y_0) = \mathbf{0}$, set $\mathbf{z}_{q+1} := \mathbf{0}$. Otherwise we proceed in an analogous manner as before for the set B. Start with $(x_0, y_0) := \tau_{\mathbf{r}}(u_{j+1}, v_{j+1})$ and define $(x_{k+1}, y_{k+1}) := \tau_{\mathbf{r}}^3(x_k, y_k), k \geq 0$. Then for each $(x_k, y_k) \in B$ we have

$$x_{k+1} \geq 0 \quad \text{(Lemma 2.2.16)}, \tag{2.2.4}$$
$$x_{k+1} + y_{k+1} \leq \left\|(x_k, y_k)\right\|_1 \quad \text{(Lemma 2.2.18)}, \tag{2.2.5}$$
$$y_{k+1} - y_k \leq -1 \quad \text{(Lemma 2.2.20)}. \tag{2.2.6}$$

Thus, again, there exists an $l \in \mathbb{N}$ with $(x_k, y_k) \in B$ for $k \leq l$ and $(x_{l+1}, y_{l+1}) \notin B$. (2.2.5) ensures that $\left\|(x_{k+1}, y_{k+1})\right\|_1 \leq \left\|(x_k, y_k)\right\|_1$ for $k < l$. We set $\mathbf{z}_{q+1} := \tau_{\mathbf{r}}(x_{l+1}, y_{l+1})$ and see that $\mathbf{z}_{q+1} \in A$ or $\mathbf{z}_{q+1} = \mathbf{0}$ (Lemma 2.2.22) and this time $\left\|\mathbf{z}_{q+1}\right\|_1 < \left\|(x_l, y_l)\right\|_1$ (Lemma 2.2.24). Hence we have

$$\begin{aligned} \left\|\mathbf{z}_q\right\|_1 &= \left\|(u_0, v_0)\right\|_1 \leq \left\|(u_1, v_1)\right\|_1 \leq \ldots \leq \left\|(u_j, v_j)\right\|_1 \\ &\leq \left\|(x_0, y_0)\right\|_1 \leq \left\|(x_1, y_1)\right\|_1 \leq \ldots \leq \left\|(x_l, y_l)\right\|_1 < \left\|\mathbf{z_{q+1}}\right\|_1. \end{aligned}$$

It is easy to see that any $\mathbf{z}_q$ is a member of our sequence $(\tau_{\mathbf{r}}^n \mathbf{z})_{n\in\mathbb{N}}$ and there exists an $q_0 > 0$ with $\left\|\mathbf{z}_0\right\|_1 < \left\|\mathbf{z}_1\right\|_1 < \ldots < \left\|\mathbf{z}_{q_0}\right\|_1 = 0$. Hence $\{\mathbf{z}_0, \ldots, \mathbf{z}_{q_0}\}$ really ends up in $\mathbf{0}$. □

We need some preparatory definitions. Let $(u,v) \in \mathbb{Z}^2$. Using the abbreviations

$$\begin{aligned}
\iota(u,v) &:= v\delta T - uT,\\
\kappa(u,v) &:= -u\delta T - v(T+\delta T) - \lfloor \iota(u,v) \rfloor \delta T,\\
\lambda(u,v) &:= u(T+\delta T) + vT + \lfloor \iota(u,v) \rfloor (T+\delta T) - \lfloor \kappa(u,v) \rfloor \delta T
\end{aligned}$$

yields

$$\begin{aligned}
\tau_{\mathbf{r}}(u,v) &= (v, -u-v-\lfloor \iota(u,v) \rfloor),\\
\tau_{\mathbf{r}}^2(u,v) &= (-u-v-\lfloor \iota(u,v) \rfloor, u+\lfloor \iota(u,v) \rfloor - \lfloor \kappa(u,v) \rfloor),\\
\tau_{\mathbf{r}}^3(u,v) &= (u+\lfloor \iota(u,v) \rfloor - \lfloor \kappa(u,v) \rfloor, v+\lfloor \kappa(u,v) \rfloor - \lfloor \lambda(u,v) \rfloor).
\end{aligned}$$

For some proofs it is better to choose another representation. By direct calculation we gain

$$\tau_{\mathbf{r}}^3((u,v)) = (u+\alpha_1 u + \alpha_2 v + \alpha_3, v + \beta_1 u + \beta_2 v + \beta_3) \tag{2.2.7}$$

with

$$\begin{aligned}
\alpha_1 &:= T(-1+\delta) - T^2\delta,\\
\alpha_2 &:= T(1+2\delta) + T^2\delta^2,\\
\beta_1 &:= T(-1-2\delta) + T^2(1+2\delta-\delta^2) + T^3\delta^2,\\
\beta_2 &:= T(-2-\delta) + T^2(-2\delta-3\delta^2) - T^3\delta^3,\\
\alpha_3 &:= (-1-\delta T)\{\iota(u,v)\} + \{\kappa(u,v)\},\\
\beta_3 &:= (T+2\delta T+\delta^2T^2)\{\iota(u,v)\} + (-1-\delta T)\{\kappa(u,v)\} + \{\lambda(u,v)\}.
\end{aligned}$$

where $\{a\}$ denotes the fractional part of a. These expressions satisfy the following inequalities:

$$\begin{aligned}
-T \leq{} & \alpha_1 < 0, & (2.2.8)\\
T \leq{} & \alpha_2 < 4T, & (2.2.9)\\
-3T <{} & \beta_1 < 0, & (2.2.10)\\
-4T <{} & \beta_2 \leq -2T, & (2.2.11)\\
-T \leq{} & \alpha_2+\beta_2 < 0. & (2.2.12)
\end{aligned}$$

The estimations are partly very crude, but easy to verify and sufficient for our aims. Because of monotonicity the extreme values of α_3 and β_3 can only occur, if $\{\iota(u,v)\}$, $\{\kappa(u,v)\}$ and $\{\lambda(u,v)\}$ take extreme values. We have $0 \leq \{\iota(u,v)\}, \{\kappa(u,v)\}, \{\lambda(u,v)\} < 1$. For our estimations of α_3 and β_3 we use 1 as an upper bound of $\{\iota(u,v)\}$, $\{\kappa(u,v)\}$ and $\{\lambda(u,v)\}$. This gives the following table:

$\{\lambda(u,v)\}$	$\{\kappa(u,v)\}$	$\{\iota(u,v)\}$	α_3	β_3
0	0	0	0	0
0	0	1	$-1-\delta T$	$T+2\delta T+\delta^2T^2$
0	1	0	1	$-1-\delta T$
0	1	1	$-\delta T$	$-1+T+\delta T+\delta^2T^2$
1	0	0	0	1
1	0	1	$-1-\delta T$	$1+T+2\delta T+\delta^2T^2$
1	1	0	1	$-\delta T$
1	1	1	$-\delta T$	$+T+\delta T+\delta^2T^2$

This table shows that

$$\begin{aligned}
-1-\delta T &< \alpha_3 < 1, & (2.2.13)\\
-1-\delta T &< \beta_3 < 1+T+2\delta T+\delta^2T^2, & (2.2.14)\\
-1+T+\delta^2T^2 &< \alpha_3+\beta_3 < 1. & (2.2.15)
\end{aligned}$$

Note that $\{\iota(u,v)\}, \{\kappa(u,v)\}$ and $\{\iota(u,v)\}$ cannot be equal to 1 hence all inequalities are strict. While proving the lemmas, we always have to keep track of the signs of the α_i and β_i as well as the possible values δ and T can obtain.

Lemma 2.2.15. *Let $(u_i, v_i) \in A$ and $(u_{i+1}, v_{i+1}) = \tau_{\mathbf{r}}^3(u_i, v_i)$. Then $u_{i+1} \leq 0$.*

Proof.

$$u_{i+1} = u_i + u_i\alpha_1 + v_i\alpha_2 + \alpha_3 = u_i(1+\alpha_1) + v_i\alpha_2 + \alpha_3$$

By the definition of A we have $u_i \leq 0$ and $v_i < 0$. Because v_i is an integer this implies $v_i \leq -1$. $(1+\alpha_1) > 0$ and $\alpha_2 > 0$ by (2.2.8) and (2.2.9). Since by (2.2.13) we have $\alpha_3 < 1$ we obtain

$$u_{i+1} < -\alpha_2 + 1 = -T(1+2\delta) - T^2\delta^2 + 1 < 1.$$

The fact that u_{i+1} is an integer allows the final conclusion $u_{i+1} \leq 0$. □

Lemma 2.2.16. *Let $(x_k, y_k) \in B$ and $(x_{k+1}, y_{k+1}) = \tau_{\mathbf{r}}^3(x_k, y_k)$. Then $x_{k+1} \geq 0$.*

Proof. Analogously to Lemma 2.2.15, by using (2.2.8), (2.2.9) and (2.2.13), we get

$$\begin{aligned}
x_{k+1} =& x_k + x_k\alpha_1 + y_k\alpha_2 + \alpha_3 \\
=& x_k(1+\alpha_1) + y_k\alpha_2 + \alpha_3 \\
>& \alpha_2 - 1 - \delta T \\
=& -1 + T(1+\delta) + T^2\delta^2 > -1
\end{aligned}$$

and therefore $x_{k+1} \geq 0$. □

Lemma 2.2.17. *Let $(u_{i+1}, v_{i+1}) = \tau_{\mathbf{r}}^3(u_i, v_i)$. Then $(u_i, v_i) \in A$ and $\|(u_i, v_i)\|_1 = m$ implies that $u_{i+1} + v_{i+1} \geq -m$.*

Proof. Since $(u_i, v_i) \in A$ we have $\|(u_i, v_i)\|_1 = -u_i - v_i$. Thus

$$\begin{aligned}
u_{i+1} + v_{i+1} =& u_i + u_i\alpha_1 + v_i\alpha_2 + \alpha_3 + v_i + u_i\beta_1 + v_i\beta_2 + \beta_3 \\
>& u_i(\alpha_1 + \beta_1) + v_i(\alpha_2 + \beta_2) - m - 1 + T + \delta^2 T^2
\end{aligned}$$

where (2.2.15) gives the lower bound for $\alpha_3 + \beta_3$. Considering (2.2.8), (2.2.9) and (2.2.12) yields $u_{i+1} + v_{i+1} > -m - 1$ and for integer values u_{i+1}, v_{i+1}, m we get $u_{i+1} + v_{i+1} \geq -m$. □

Lemma 2.2.18. *Let $(x_{k+1}, y_{k+1}) = \tau_{\mathbf{r}}^3(x_k, y_k)$. If $(x_k, y_k) \in B$ and $\|(x_k, y_k)\|_1 = m$, then $x_{k+1} + y_{k+1} \leq m$.*

Proof. Since $(x_k, y_k) \in B$ we have $\|(x_k, y_k)\|_1 = x_k + y_k$. Again (2.2.8), (2.2.9), (2.2.12) and (2.2.15) are used for the following estimation.

$$\begin{aligned}
x_{k+1} + y_{k+1} =& x_k + x_k\alpha_1 + y_k\alpha_2 + \alpha_3 + y_k + x_k\beta_1 + y_k\beta_2 + \beta_3 \\
<& x_k(\alpha_1 + \beta_1) + y_k(\alpha_2 + \beta_2) + m + 1 \\
<& m + 1
\end{aligned}$$

and thus $x_{k+1} + y_{k+1} \leq m$. □

Lemma 2.2.19. *Let $(u_{i+1}, v_{i+1}) = \tau_{\mathbf{r}}^3(u_i, v_i)$. Then $(u_i, v_i) \in A$ implies that $v_{i+1} - v_i \geq 1$.*

Proof. Since $(u_i, v_i) \in A$ we have $u_i \leq 0$ and $v_i < 0$. Thus

$$\begin{aligned}
\kappa(u_i, v_i) =& -u_i\delta T - v_i(T + \delta T) - \lfloor \iota(u_i, v_i) \rfloor \delta T \\
\geq& -u_i\delta T - v_i(T + \delta T) - \iota(u_i, v_i)\delta T \\
=& u_i(-\delta T + \delta T^2) + v_i(-T - \delta T - \delta^2 T^2) > 0 \Rightarrow \lfloor \kappa(u_i, v_i) \rfloor \geq 0 \\
\lambda(u_i, v_i) =& u_i(T + \delta T) + v_i T + \lfloor \iota(u_i, v_i) \rfloor (T + \delta T) - \lfloor \kappa(u_i, v_i) \rfloor \delta T \\
\leq& u_i(T + \delta T) + v_i T + \iota(u_i, v_i)(T + \delta T) \\
=& u_i(T + \delta T - T^2 - \delta T^2) + v_i(T + \delta T^2 + \delta^2 T^2) \\
\leq& -T - \delta T^2 - \delta^2 T^2 < 0 \Rightarrow \lfloor \lambda(u_i, v_i) \rfloor \leq -1.
\end{aligned}$$

Finally the simple computation

$$v_{i+1} - v_i = \lfloor \kappa(u_i, v_i) \rfloor - \lfloor \lambda(u_i, v_i) \rfloor \geq 1$$

shows the statement. $\square$

Lemma 2.2.20. *Let $(x_k, y_k) \in B$ and $(x_{k+1}, y_{k+1}) = \tau_{\mathbf{r}}^3(x_k, y_k)$. Then $y_{k+1} - y_k \leq -1$.*

Proof.

$$\begin{aligned}
\kappa(x_k, y_k) =& - x_k \delta T - y_k(T + \delta T) - \lfloor \iota(x_k, y_k) \rfloor \delta T \\
\leq& - x_k \delta T - y_k(T + \delta T) - (\iota(x_k, y_k) - 1)\delta T \\
=& x_k(-\delta T + \delta T^2) + y_k(-T - \delta T - \delta^2 T^2) + \delta T \\
\leq& - T - \delta^2 T^2 < 0 \Rightarrow \lfloor \kappa(x_k, y_k) \rfloor \leq -1 \\
\lambda(x_k, y_k) =& x_k(T + \delta T) + y_k T + \lfloor \iota(x_k, y_k) \rfloor (T + \delta T) - \lfloor \kappa(x_k, y_k) \rfloor \delta T \\
\geq& x_k(T + \delta T) + y_k T + (\iota(x_k, y_k) - 1)(T + \delta T) + \delta T \\
=& x_k(T + \delta T - T^2 - \delta T^2) + y_k(T + \delta T^2 + \delta^2 T^2) - T \\
\geq& \delta T^2 + \delta^2 T^2 \geq 0 \Rightarrow \lfloor \lambda(x_k, y_k) \rfloor \geq 0.
\end{aligned}$$

Hence,

$$y_{k+1} - y_k = \lfloor \kappa(x_k, y_k) \rfloor - \lfloor \lambda(x_k, y_k) \rfloor \leq -1.$$

$\square$

Lemma 2.2.21. *If $(u_j, v_j) \in A$ and $(u_{j+1}, v_{j+1}) = \tau_{\mathbf{r}}^3(u_j, v_j) \notin A$ then $\tau_{\mathbf{r}}(u_{j+1}, v_{j+1}) \in B$ or $\tau_{\mathbf{r}}(u_{j+1}, v_{j+1}) = \mathbf{0}$.*

Proof. Let $(u', v') := \tau_{\mathbf{r}}(u_{j+1}, v_{j+1})$. We will show that $u' \geq 0$ and $v' > 0$. We have $u_{j+1} \leq 0$ (according to Lemma 2.2.15) and $v_{j+1} > v_j$ (according to Lemma 2.2.19). Since $(u_{j+1}, v_{j+1}) \notin A$ we can conclude that $v_{j+1} \geq 0$ and therefore $u' = v_{j+1} \geq 0$. The proof of the other statement requires some more estimations. Set $m := -u_j - v_j$. Suppose first that $m < 2$. This is only true for $u_j + v_j = -1$ and therefore $u_i = 0$ and $v_i = -1$. Then $\lfloor \iota(u_j, v_j) \rfloor \in \{-1, 0\}$, $\lfloor \kappa(u_j, v_j) \rfloor = 0$ and $\lfloor \lambda(u_j, v_j) \rfloor = -1$. Hence either $(u_{j+1}, v_{j+1}) = \mathbf{0}$, which means that $(u', v') = \mathbf{0}$, or $(u_{j+1}, v_{j+1}) = (1, 0)$, which implies that $v' = -\lfloor -1 + T \rfloor = 1 > 0$. If $m \geq 2$ then

$$\begin{aligned}
u_{j+1} =& u_j(1 + \alpha_1) + v_j \alpha_2 + \alpha_3 \\
>& (-v_j - m)(1 + \alpha_1) + v_j \alpha_2 - 1 - \delta T \\
=& v_j(-1 - \alpha_1 + \alpha_2) - m(1 + \alpha_1) - 1 - \delta T.
\end{aligned}$$

Note that $(u_j, v_j) \in A$ and so $v_j \leq -1 < 0$. Thus

$$\begin{aligned}
u_{j+1} \geq& 1 + \alpha_1 - \alpha_2 - m - m\alpha_1 - 1 - \delta T \\
=& - m + \alpha_1(1 - m) - \alpha_2 - \delta T \\
\geq& - m - \alpha_2 - \delta T.
\end{aligned}$$

Since u_{j+1} and m are integers, the conclusion

$$u_{j+1} \geq -m \tag{2.2.16}$$

holds. Furthermore by (2.2.7) and (2.2.15)

$$\begin{aligned}
u_{j+1} + v_{j+1} =& u_j(1 + \alpha_1 + \beta_1) + v_j(1 + \alpha_2 + \beta_2) + \alpha_3 + \beta_3 \\
<& (-m - v_j)(1 + \alpha_1 + \beta_1) + v_j(1 + \alpha_2 + \beta_2) + 1 \\
=& - m + 1 - m\alpha_1 - m\beta_1 + v_j(\alpha_2 + \beta_2 - \alpha_1 - \beta_1).
\end{aligned}$$

Inserting $v_j \leq -1$ and using (2.2.8)-(2.2.11) yields

$$\begin{aligned}
u_{j+1} + v_{j+1} \leq & -m + 1 - m\alpha_1 - m\beta_1 - (\alpha_2 + \beta_2 - \alpha_1 - \beta_1) \\
=&(-m+1)(1+\alpha_1+\beta_1) - \alpha_2 - \beta_2 \\
\leq&(-m+1)(1-4T) + 3T \\
=&m(-1+4T) + 1 - T.
\end{aligned}$$

Together with (2.2.16) this implies

$$\begin{aligned}
&u_{j+1}(1-T) + v_{j+1}(1+\delta T) \\
&\leq u_{j+1}(1-T) + (m(-1+4T) + 1 - T - u_{j+1})(1+\delta T) \\
&= m(-1+4T)(1+\delta T) + (1-T)(1+\delta T) - u_{j+1}(T+\delta T) \\
&\leq m(-1+4T)(1+\delta T) + (1-T)(1+\delta T) + m(T+\delta T) \\
&= m(-1+5T+4\delta T^2) + 1 - T + \delta T - \delta T^2 \\
&\leq -2 + 10T + 8\delta T^2 + 1 - T + \delta T - \delta T^2 \\
&= -1 + 9T + \delta T + 7\delta T^2 < 0
\end{aligned}$$

and therefore $v' = -\lfloor u_{j+1}(1-T) + v_{j+1}(1+\delta T) \rfloor \geq 1 > 0$. Hence $\tau_{\mathbf{r}}(u_{j+1}, v_{j+1}) = (u', v')$ is really inside B, when it is not $\mathbf{0}$. □

Lemma 2.2.22. *$(x_l, y_l) \in B$ and $(x_{l+1}, y_{l+1}) = \tau_{\mathbf{r}}^3(x_l, y_l) \notin B$ implies that $\tau_{\mathbf{r}}(x_{l+1}, y_{l+1}) \in A$ or $\tau_{\mathbf{r}}(x_{l+1}, y_{l+1}) = \mathbf{0}$.*

Proof. Let $(x', y') := \tau_{\mathbf{r}}(x_{l+1}, y_{l+1})$. Analogously to Lemma 2.2.21 we have to show that $x' \leq 0$ and $y' < 0$. The claim $(x_{l+1}, y_{l+1}) \notin B$ together with Lemma 2.2.16 and Lemma 2.2.20 implies that $y_{l+1} \leq 0$ and therefore $x' = y_{l+1} \leq 0$. The second estimation comes from the following computations: Let $m := x_l + y_l$ and suppose that $m < 3$. There are three possibilities:

$(x_l, y_l) = (0, 1)$

$$\begin{aligned}
\lfloor \iota(0,1) \rfloor &= \lfloor \delta T \rfloor = 0 \\
\lfloor \kappa(0,1) \rfloor &= \lfloor -T - \delta T \rfloor = -1 \\
\lfloor \lambda(0,1) \rfloor &= \lfloor T + \delta T \rfloor = 0
\end{aligned}$$

and therefore $(x_{l+1}, y_{l+1}) = (0+0+1, 1-1+0) = (1, 0)$ and further $(x', y') = \tau_r(1, 0) = (0, -\lfloor (1-T) \rfloor) = \mathbf{0}$.

$(x_l, y_l) = (1, 1)$

$$\begin{aligned}
\lfloor \iota(1,1) \rfloor &= \lfloor \delta T - T \rfloor \in \{0, -1\} \\
\lfloor \kappa(1,1) \rfloor &= \lfloor -T - 2\delta T - \lfloor \iota(1,1) \rfloor \delta T \rfloor = -1 \\
\lfloor \lambda(1,1) \rfloor &= \lfloor 2T + 2\delta T + \lfloor \iota(1,1) \rfloor (T + \delta T) \rfloor = 0.
\end{aligned}$$

So either $(x_{l+1}, y_{l+1}) = (1, 0)$, which goes to $\mathbf{0}$ by the calculation above, or $(x_{l+1}, y_{l+1}) = (2, 0)$ and $y' = -\lfloor 2 - 2T \rfloor = -1 < 0$.

$(x_l, y_l) = (0, 2)$

$$\begin{aligned}
\lfloor \iota(0,2) \rfloor &= \lfloor 2\delta T \rfloor = 0 \\
\lfloor \kappa(0,2) \rfloor &= \lfloor -2T - 2\delta T \rfloor = -1 \\
\lfloor \lambda(0,2) \rfloor &= \lfloor 2T + \delta T \rfloor = 0.
\end{aligned}$$

This yields $(x_{l+1}, y_{l+1}) = (1, 1)$. This case does not fulfill the condition $(x_{l+1}, y_{l+1}) \notin B$. Thus it is irrelevant for the present lemma.

Now let $m \geq 3$. Note that always $y_l \geq 1$.

$$\begin{aligned}
x_{l+1} =& x_l(1+\alpha_1) + y_l\alpha_2 + \alpha_3 \\
<& (-y_l + m)(1+\alpha_1) + y_l\alpha_2 + 1 \\
=& y_l(-1-\alpha_1+\alpha_2) + m(1+\alpha_1) + 1 \\
\leq& -1 - \alpha_1 + \alpha_2 + m + m\alpha_1 + 1 \\
=& m + \alpha_1(m-1) + \alpha_2 \\
\leq& m + \alpha_2.
\end{aligned}$$

Again x_{l+1} and m are integers and from there follows

$$x_{l+1} \leq m. \tag{2.2.17}$$

Analogously to Lemma 2.2.21, we need a lower bound for $x_{l+1} + y_{l+1}$:

$$\begin{aligned}
x_{l+1} + y_{l+1} =& x_l(1+\alpha_1+\beta_1) + y_l(1+\alpha_2+\beta_2) + \alpha_3 + \beta_3 \\
>& (m - y_l)(1+\alpha_1+\beta_1) + y_l(1+\alpha_2+\beta_2) - 1 + T + \delta^2T^2 \\
\geq& m - 1 + m\alpha_1 + m\beta_1 + y_l(\alpha_2+\beta_2-\alpha_1-\beta_1) + T \\
\geq& m - 1 + m\alpha_1 + m\beta_1 + (\alpha_2+\beta_2-\alpha_1-\beta_1) + T \\
=& (m-1)(1+\alpha_1+\beta_1) + \alpha_2 + \beta_2 + T \\
\geq& (m-1)(1-4T) - 2T \\
=& m(1-4T) - 1 + 2T
\end{aligned}$$

where (2.2.8)-(2.2.11) yielded the last inequality. With the help of these two results we show the estimation

$$\begin{aligned}
&x_{l+1}(1-T) + y_{l+1}(1+\delta T) \\
&\geq x_{l+1}(1-T) + (m(1-4T) - 1 + 2T - x_{l+1})(1+\delta T) \\
&= m(1-4T)(1+\delta T) + (-1+2T)(1+\delta T) - x_{l+1}(T+\delta T) \\
&\geq m(1-4T)(1+\delta T) + (-1+2T)(1+\delta T) - m(T+\delta T) \\
&= m(1-5T-4\delta T^2) - 1 + 2T - \delta T + 2\delta T^2 \\
&\geq 3 - 15T - 12\delta T^2 - 1 + 2T - \delta T + 2\delta T^2 \\
&= 2 - 13T - \delta T - 10\delta T^2 > 1.
\end{aligned}$$

Therefore $y' = -\lfloor x_{l+1}(1-T) + y_{l+1}(1+\delta T)\rfloor \leq -1 < 0$. □

Lemma 2.2.23. *If $(u_i, v_i) \in A$ and $(u_{i+1}, v_{i+1}) = \tau_{\mathbf{r}}^3(u_i, v_i) \notin A$ then we have $\|\tau_{\mathbf{r}}(u_{i+1}, v_{i+1})\|_1 \leq \|(u_i, v_i)\|_1 = m$.*

Proof. Let $(u', v') := \tau_{\mathbf{r}}(u_{i+1}, v_{i+1})$. Lemma 2.2.21 says that $\tau_{\mathbf{r}}(u_{i+1}, v_{i+1})$ is an element of B and therefore $\|\tau_{\mathbf{r}}(u_{i+1}, v_{i+1})\|_1 = u' + v'$, while the inside A lying point (u_i, v_i) induces the condition $u_i + v_i = -m$. According to Lemma 2.2.17, $u_{i+1} + v_{i+1} \geq -m$ is valid, although the point is not an element of A.

$$\begin{aligned}
u' + v' =& v_{i+1} - u_{i+1} - v_{i+1} - \lfloor -u_{i+1}T + v_{i+1}\delta T \rfloor \\
=& -\lfloor u_{i+1}(1-T) + v_{i+1}\delta T \rfloor \\
\leq& -\lfloor u_{i+1}(1-T) + (-u_{i+1} - m)\delta T \rfloor \\
=& -\lfloor u_{i+1}(1-T-\delta T) - m\delta T \rfloor.
\end{aligned}$$

Using (2.2.16) yields

$$\begin{aligned}
u' + v' \leq& -\lfloor -m(1-T-\delta T) - m\delta T \rfloor \\
\leq& m - \lfloor mT \rfloor \leq m
\end{aligned}$$

and shows that $u' + v' \leq m$ holds. □

Lemma 2.2.24. *If $(x_l, y_l) \in B$ and $(x_{l+1}, y_{l+1}) = \tau_{\mathbf{r}}^3(x_l, y_l) \notin B$ then we have $\|\tau_{\mathbf{r}}(x_{l+1}, y_{l+1})\|_1 < \|(x_l, y_l)\|_1 = m$.*

Proof. Let $(x', y') = \tau_{\mathbf{r}}(x_{l+1}, y_{l+1})$. Referring to Lemma 2.2.22 we have $(x', y') \in A$ and therefore $\|(x', y')\|_1 = -x' - y'$. On the other hand $\|(x_l, y_l)\|_1 = x_l + y_l =: m$. According to Lemma 2.2.18, $x_{l+1} + y_{l+1} \leq m$ holds, although the point is not an element of B.

$$\begin{aligned} x' + y' =& y_{l+1} - x_{l+1} - y_{l+1} - \lfloor -x_{l+1}T + y_{l+1}\delta T \rfloor \\ =& - \lfloor x_{l+1}(1-T) + y_{l+1}\delta T \rfloor \\ \geq& - \lfloor x_{l+1}(1-T) + (-x_{l+1} + m)\delta T \rfloor \\ =& - \lfloor x_{l+1}(1 - T - \delta T) + m\delta T \rfloor . \end{aligned}$$

Now we use (2.2.17) to get

$$\begin{aligned} x' + y' \geq& - \lfloor m(1 - T - \delta T) + m\delta T \rfloor \\ \geq& - m - \lfloor -mT \rfloor \geq -m + 1. \end{aligned}$$

This shows the validity of $\|(x', y')\|_1 \leq m - 1 < m$. □

Lemma 2.2.25. *Let $(u, v) \in \mathbb{Z}^2$. Then there is an $i \in \mathbb{N}$ with either $\tau_{\mathbf{r}}^i(u, v) \in A \cup B$ or $\tau_{\mathbf{r}}^i(u, v) = \mathbf{0}$.*

Proof. Consider the line $x + y + \iota(x, y) = 0$. It runs through the origin and splits the second quadrant into two pieces for each possible T and δ. It allows the partition of $\mathbb{Z}^2$ into $\mathbf{0}$ and the sets

$$\begin{aligned} B :=& \left\{ (x, y) \in \mathbb{Z}^2 \,|\, x \geq 0, y > 0 \right\}, \\ A :=& \left\{ (x, y) \in \mathbb{Z}^2 \,|\, x \leq 0, y < 0 \right\}, \\ U_1 :=& \left\{ (x, y) \in \mathbb{Z}^2 \,|\, x < 0, y \geq 0, x + y + \iota(x, y) < 0 \right\}, \\ U_2 :=& \left\{ (x, y) \in \mathbb{Z}^2 \,|\, x < 0, y \geq 0, x + y + \iota(x, y) \geq 0 \right\}, \\ U_3 :=& \left\{ (x, y) \in \mathbb{Z}^2 \,|\, x > 0, y \leq 0 \right\}. \end{aligned}$$

There are the following cases:

$(u, v) \in U_1$ We have $v \geq 0$ and $-\lfloor u + v + \iota(u, v) \rfloor \geq 1 > 0$. Thus $\tau_{\mathbf{r}}(u, v) = (v, -\lfloor u + v + \iota(u, v) \rfloor) \in B$.

$(u, v) \in U_2$ (u, v) cannot be an element of the x-axis. Suppose $v = 1$ and $u \leq -2$. Then $u + v + \iota(u, v) \leq -2 + 2T + 1 + \delta T = -1 - 2T - \delta T < 0$ shows that (u, v) does not lie in U_2. If $u = -1$ then $(u, v) = (-1, 1)$. This point is an element of U_2. $\tau_{\mathbf{r}}(-1, 1) = (1, -1 + 1 - \lfloor T + \delta T \rfloor) = (1, 0)$ and $\tau_{\mathbf{r}}(1, 0) = (0, -1 - \lfloor -T \rfloor) = \mathbf{0}$ shows that this point goes to $\mathbf{0}$ after 2 applications of $\tau_{\mathbf{r}}$. For the rest of U_2 we can assume $u \leq -1$ and $v \geq 2$.

$$\begin{aligned} v + u\beta_1 + v\beta_2 + \beta_3 \geq& 2 + 2\beta_2 - \beta_1 - 1 - \delta T \\ =& 1 - T + T^2(-1 - 4\delta - 2\delta^2) + T^3(-\delta^2 - \delta^3) > 0 \end{aligned}$$

and therefore $v + u\beta_1 + v\beta_2 + \beta_3 \geq 1$. Further we have

$$\begin{aligned} \iota(u, v) =& v\delta T - uT \geq 2\delta T + T > 0 \Rightarrow \lfloor \iota(u, v) \rfloor \geq 0, \\ \kappa(u, v) =& - u\delta T - v(T + \delta T) - \lfloor \iota(u, v) \rfloor \delta T \\ =& - vT - \lfloor u + v + \iota(u, v) \rfloor \delta T \\ \leq& - vT < 0 \end{aligned}$$

which shows that $\lfloor \kappa(u, v) \rfloor \leq -1$. Furthermore $\lfloor \iota(u, v) \rfloor - \lfloor \kappa(u, v) \rfloor \geq 1$. Thus we can conclude that the point

$$\tau_{\mathbf{r}}^3(u, v) = (u + \lfloor \iota(u, v) \rfloor - \lfloor \kappa(u, v) \rfloor, v + \beta_1 u + \beta_2 v + \beta_3)$$

is above the x-axis and right of (u,v). This induces that $\tau_{\mathbf{r}}^m(u,v)$ is either $\mathbf{0}$, an element of U_1 or an element of B for an $m \in \mathbb{N}$.

$(u,v) \in U_3$ This implies that $v \le 0$. Thus $\tau_{\mathbf{r}}(u,v) = (v, -u-v-\lfloor \iota(u,v) \rfloor) \notin U_3$.

Hence $\exists i \in \mathbb{N}$ with $\tau_{\mathbf{r}}^i(u,v) \in A \cup B \cup \{\mathbf{0}\}$ for each $(u,v) \in \mathbb{Z}^2$. □

2.2.4 Two infinite families of cycles

We already noticed the existence of infinite families of pairwise disjoint cycles. The first one was presented in [3, Section 6] without an explicit analysis. In [52] a second one was found and both of these families have been investigated. For each $n \in \mathbb{N}$, $n \ge 1$, consider the cycle

$$\omega_n := \left\langle 2n+1, -2n, \bigsqcup_{i=1}^{n} (2i-1, 2n-2i+1, -2n), \bigsqcup_{i=1}^{n-1} (2n+1, -2i, -2n+2i) \right\rangle$$

where $\bigsqcup$ denotes the sequence gained by concatenation, *e.g.* $\bigsqcup_{i=1}^{n} a_i = a_1, \ldots, a_n$. For an empty set of indices the corresponding sequence is empty. Hence for each $n \ge 1$ we have a sequence of length $6n-1$.

$$\begin{aligned}
\omega_1 &= \langle 3, -2, 1, 1, -2 \rangle, \\
\omega_2 &= \langle 5, -4, 1, 3, -4, 3, 1, -4, 5, -2, -2 \rangle, \\
\omega_3 &= \langle 7, -6, 1, 5, -6, 3, 3, -6, 5, 1, -6, 7, -2, -4, 7, -4, -2 \rangle, \\
&\vdots
\end{aligned}$$

Our aim is to show that $P_2(\omega_n) \cap \mathcal{D}_2 \neq \emptyset$ for $n \in \mathbb{N}_0$, which induces ω_n to be a cycle, and $P_2(\omega_{n_1}) \cap P_2(\omega_{n_2}) = \emptyset$ for positive integers n_1, n_2 with $n_1 \neq n_2$. We already know that ω_1 induces a non-degenerated cutout polyhedra: the set E_2 defined in (1.1.6) equals $P_2(\omega_1) \cap \overline{D_2}$. In the following we will see that the case $n = 1$ behaves a little different than the other ones. For some $n > 0$ the set $P_2(\omega_n)$ consists of the points $(x,y) \in \mathbb{R}$ satisfying the following system of inequalities, deduced from (1.1.5):

$$\begin{array}{lcll}
0 \le & x - 2ny + 2n + 1 & < 1, & \qquad (2.2.18) \\
0 \le & -2nx + (2n+1)y - 2 & < 1, & \qquad (2.2.19) \\
0 \le & -2x + (2n+1)y - 2n & < 1, & \qquad (2.2.20) \\
0 \le & (2n+1)x - 2ny + 1 & < 1, & \qquad (2.2.21) \\
0 \le & -2nx + (2j+1)y + 2n - 2j - 1 & < 1 \quad (0 \le j < n), & \qquad (2.2.22) \\
0 \le & (2j+1)x + (2n-2j-1)y - 2n & < 1 \quad (0 \le j < n), & \qquad (2.2.23) \\
0 \le & (2n+1)x - 2jy - 2n + 2j & < 1 \quad (0 < j < n), & \qquad (2.2.24) \\
0 \le & -2jx + (-2n+2j)y + 2n + 1 & < 1 \quad (0 < j < n). & \qquad (2.2.25)
\end{array}$$

For $n = 1$, (2.2.19) and (2.2.20) are equal and (2.2.24) as well as (2.2.25) do not exist. For a point $\mathbf{r} \in P_2(\omega_n)$, the function $\tau_{\mathbf{r}}$ maps as follows:

$$\begin{array}{rcl}
(1, -2n) & \mapsto & (-2n, 2n+1), \\
(-2n, 2n+1) & \mapsto & (2n+1, -2), \\
(-2, 2n+1) & \mapsto & (2n+1, -2n), \\
(2n+1, -2n) & \mapsto & (-2n, 1), \\
(-2n, 2j+1) & \mapsto & (2j+1, 2n-2j-1) \quad (0 \le j < n), \\
(2j+1, 2n-2j-1) & \mapsto & (2n-2j-1, -2n) \quad (0 \le j < n), \\
(2n+1, -2j) & \mapsto & (-2j, -2n+2j) \quad (0 < j < n), \\
(-2j, -2n+2j) & \mapsto & (-2n+2j, 2n+1) \quad (0 < j < n).
\end{array}$$

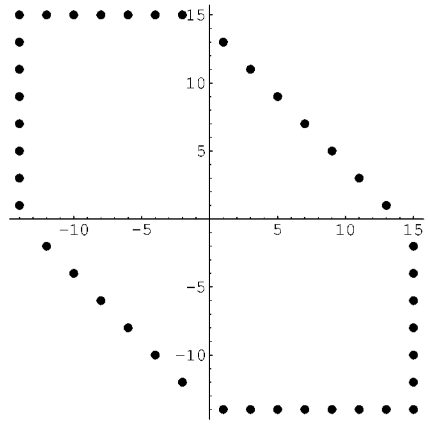


Figure 2.4: The points of the cycle ω_7

Figure 2.4 shows these points for $n = 7$. For showing that the set $P_2(\omega_n)$ of points satisfying the inequalities (2.2.18)-(2.2.25) equals a nonempty polygon for each $n \geq 1$, let

$$\begin{aligned}
\mathbf{x}_n^{(1)} &:= \left(1, \frac{2n+1}{2n}\right), \\
\mathbf{x}_n^{(2)} &:= \left(\frac{2n(2n+1)}{4n^2+2n-1}, \frac{(2n+1)^2}{4n^2+2n-1}\right), \\
\mathbf{x}_n^{(3)} &:= \left(\frac{2n(2n-1)}{4n^2-2n+1}, \frac{4n^2}{4n^2-2n+1}\right), \\
\mathbf{x}_n^{(4)} &:= \begin{cases} \left(\frac{3}{4}, \frac{3}{2}\right) & (n=1) \\ \left(1, \frac{2n}{2n-1}\right) & (\text{otherwise}) \end{cases}.
\end{aligned}$$

Denote by $\square(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ the convex hull of the points $\mathbf{a}_1, \ldots, \mathbf{a}_k$.

Theorem 2.2.26. *For any $n \geq 1$ the polygon $P_2(\omega_n)$ equals the non-empty open set $S :=$ $\operatorname{int}\left(\square(\mathbf{x}_n^{(1)}, \ldots, \mathbf{x}_n^{(4)})\right)$.*

Proof. We chose from our list (2.2.18)-(2.2.25) the four right hand (strict) inequalities (2.2.18), (2.2.21), (2.2.22) with $j = n - 1$ and (2.2.23) with $j = 0$. For $n = 1$ take (2.2.19) instead of the last one. They form a subsystem of the system (2.2.18)-(2.2.25). Each of these four inequalities describes an open half plane:

$$\begin{aligned}
U_n^{(1)} &: \{(x,y) \,|\, x - 2ny + 2n + 1 < 1\}, \\
U_n^{(2)} &: \{(x,y) \,|\, (2n+1)x - 2ny + 1 < 1\}, \\
U_n^{(3)} &: \{(x,y) \,|\, -2nx + (2n-1)y + 1 < 1\}, \\
U_n^{(4)} &: \begin{cases} \{(x,y) \,|\, -2x + 3y - 2 < 1\} & (n=1) \\ \{(x,y) \,|\, x + (2n-1)y - 2n < 1\} & (\text{otherwise}) \end{cases}.
\end{aligned}$$

Obviously we have $P_2(\omega_n) \subset \bigcap_{i=1}^4 U_n^{(i)}$. The lines

$$\begin{aligned}
g_n^{(1)} &: & x - 2ny + 2n = 0, \\
g_n^{(2)} &: & (2n+1)x - 2ny = 0, \\
g_n^{(3)} &: & -2nx + (2n-1)y = 0, \\
g_n^{(4)} &: & \begin{cases} -2x + 3y - 3 = 0 & (n=1) \\ x + (2n-1)y - 2n - 1 = 0 & (\text{otherwise}) \end{cases}
\end{aligned}$$

are the boundary lines of these half planes: $g_n^{(i)}$ bound $U_n^{(i)}$ for $i = 1, \ldots, 4$. Now it is easy to verify that

$$\begin{aligned} x_n^{(1)} &= g_n^{(1)} \wedge g_n^{(2)},\ x_n^{(1)} \in U_n^{(3)} \cap U_n^{(4)}, \\ x_n^{(2)} &= g_n^{(2)} \wedge g_n^{(4)},\ x_n^{(2)} \in U_n^{(1)} \cap U_n^{(3)}, \\ x_n^{(3)} &= g_n^{(1)} \wedge g_n^{(3)},\ x_n^{(3)} \in U_n^{(2)} \cap U_n^{(4)}, \\ x_n^{(4)} &= g_n^{(3)} \wedge g_n^{(4)},\ x_n^{(4)} \in U_n^{(1)} \cap U_n^{(2)}. \end{aligned}$$

This shows that $S = \bigcap_{i=1}^4 U_n^{(i)}$ and thus $S \supset P_2(\omega_n)$. On the other hand simple calculations show that for each $i = 1, \ldots, 4$, $\mathbf{x}_n^{(i)}$ satisfies all the other inequalities but the four chosen ones of the system (2.2.18)-(2.2.25). Hence we also have $S \subset P_2(\omega_n)$. □

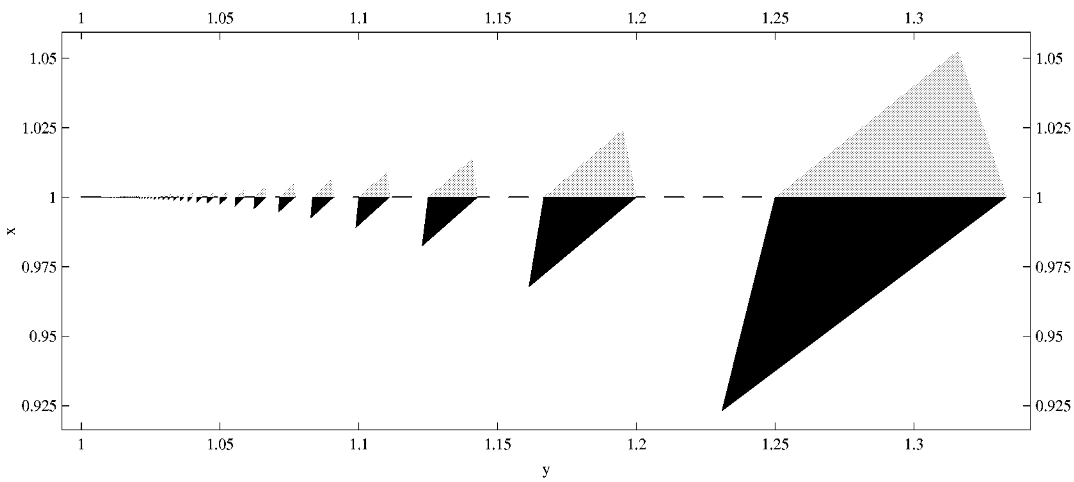


Figure 2.5: The sets $P_2(\omega_n)$

From the first coordinates of $\mathbf{x}_n^{(2)}$ and $\mathbf{x}_n^{(3)}$

$$\begin{aligned} \frac{2n(2n+1)}{4n^2+2n-1} &= 1 + \frac{1}{4n^2+2n-1}, \\ \frac{2n(2n-1)}{4n^2-2n+1} &= 1 - \frac{1}{4n^2-2n+1} \end{aligned}$$

we see that only a part of $\Pi_2(\omega_n)$ lies in $\mathcal{D}_2$. Figure 2.5 shows the cutout polygons for $n > 1$. (The axes are reversed to save space.) $P_2(\omega_1) \cap \mathcal{D}_2$ is shown in Figure 1.3 as E_2. We see the following: each of the polygons $P_2(\omega_n)$ corresponds to a quadrangle. But only $P_2(\omega_n) \cap \mathcal{D}_2$ is of real interest. And here only $P_2(\omega_1) \cap \mathcal{D}_2$ really gives a quadrangle. For $n \geq 2$ we see $P_2(\omega_n) \cap \mathcal{D}_2$ to be a triangle. These (interesting) triangles are depicted in black while the parts not intersecting with $\mathcal{D}_2$ are grey.

The second family was discovered earlier by Akiyama *et al.* [3]. A more detailed investigation of it was given in [52]. Consider the family of sequences

$$\zeta_n = \left\langle n+1, \bigsqcup_{i=1}^{n} (i, -n-1+i, -i, n+1-i) \right\rangle.$$

We have

$$
\begin{aligned}
\zeta_1 &= \langle 2, 1, -1, -1, 1 \rangle, \\
\zeta_2 &= \langle 3, 1, -2, -1, 2, 2, -1, -2, 1 \rangle, \\
\zeta_3 &= \langle 4, 1, -3, -1, 3, 2, -2, -2, 2, 3, -1, -3, 1 \rangle, \\
&\vdots
\end{aligned}
$$

These sequences also will turn out to be cycles corresponding to nonempty polygons. Contrary to the polygons $P_2(\omega_n)$, which are located in the upper half plane, we will show $y < 0$ for $P_2(\zeta_n)$. We will abbreviate the analysis of this family since it runs more or less analogously to above. $P_2(\zeta_n)$

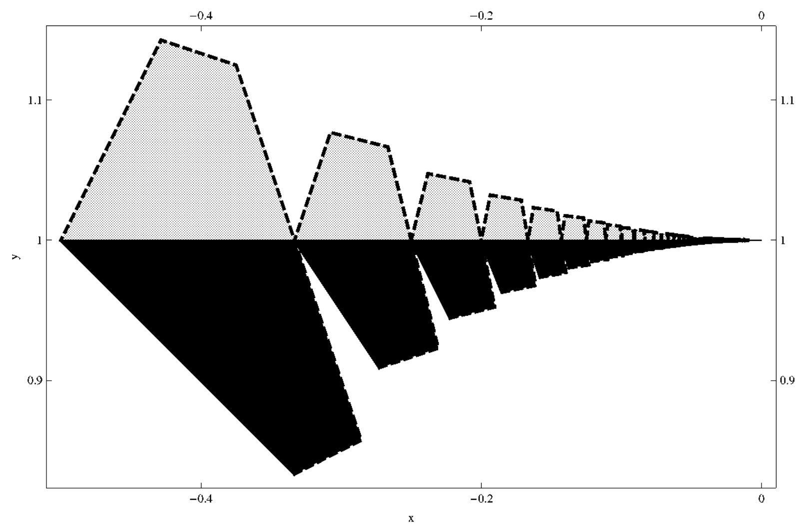


Figure 2.6: The sets $P_2(\zeta_n)$

is characterised by the pairs of inequalities

$$
\begin{aligned}
0 \leq &\quad x + (n+1)y + 1 &< 1, & & (2.2.26)\\
0 \leq &\quad jx + (j-n-1)y - j &< 1 &\quad (0 < j \leq n), & (2.2.27)\\
0 \leq &\quad (j-n-1)x - jy + n + 1 - j &< 1 &\quad (0 < j \leq n), & (2.2.28)\\
0 \leq &\quad -jx + (n+1-j)y + (j+1) &< 1 &\quad (0 < j \leq n), & (2.2.29)\\
0 \leq &\quad (n+1-j)x + (j+1)y + j - n &< 1 &\quad (0 \leq j < n). & (2.2.30)
\end{aligned}
$$

Note that the length of ζ_n is $4n+1$ and that we have exactly one inequality of each type for the case $n = 1$ which we will see to behave a little different again. We direct our attention to five lines, for $n = 1$ to three lines, respectively. They are deduced from the strict sides of the inequalities (2.2.26) and (2.2.27) with $j = 1$, and the not strict side of inequality (2.2.30) with $j = n-1$ and additionally, for $n > 1$, the strict sides of (2.2.29) with $j = n$ and (2.2.30) with $j = n-1$. These inequalities induce the lines

$$
\begin{aligned}
h_1^{(1)} &: x + 2y = 0, \\
h_1^{(2)} &: x - y = 2, \\
h_1^{(3)} &: 2x + y = 1
\end{aligned}
$$

and for $n > 1$

$$
\begin{aligned}
h_n^{(1)} &:(n+1)x + y = n+1,\\
h_n^{(2)} &:x - ny = 2,\\
h_n^{(3)} &:2x + ny = 1,\\
h_n^{(4)} &:nx - y = n,\\
h_n^{(5)} &:x + (n+1)y = 0.
\end{aligned}
$$

Furthermore define the points

$$
\begin{aligned}
\mathbf{y}_n^{(1)} &:= \left(\frac{n^2+n+2}{n^2+n+1}, -\frac{n+1}{n^2+n+1}\right),\\
\mathbf{y}_n^{(2)} &:= \left(1, -\frac{1}{n}\right),\\
\mathbf{y}_n^{(3)} &:= \left(\frac{n^2+1}{n^2+2}, -\frac{n}{n^2+2}\right),\\
\mathbf{y}_n^{(4)} &:= \left(\frac{n(n+1)}{n^2+n+1}, -\frac{n}{n^2+n+1}\right),\\
\mathbf{y}_n^{(5)} &:= \left(\frac{(n+1)^2}{n(n+2)}, -\frac{n+1}{n(n+2)}\right).
\end{aligned}
$$

The points $\mathbf{y}_1^{(1)}$ and $\mathbf{y}_1^{(5)}$ as well as $\mathbf{y}_1^{(3)}$ and $\mathbf{y}_1^{(4)}$ are identical, such that there are only the three different points $\mathbf{y}_1^{(i)}, i = 1, 2, 3$ for the case $n = 1$.

Theorem 2.2.27.

$$
\begin{aligned}
P_2(\zeta_1) &= \square(\mathbf{y}_1^{(1)}, \mathbf{y}_1^{(2)}, \mathbf{y}_1^{(3)}) \setminus (h_1^{(1)} \cup h_1^{(2)}),\\
P_2(\zeta_n) &= \square(\mathbf{y}_1^{(1)}, \dots, \mathbf{y}_1^{(5)}) \setminus (h_1^{(1)} \cup h_1^{(2)} \cup h_1^{(4)} \cup h_1^{(5)}), \quad n > 1.
\end{aligned}
$$

Proof. (sketch) For $n > 1$ we have $h_n^{(i)} \wedge h_n^{(i+1)} = \mathbf{y}_n^{(i)}$ for $i = 1, \dots, 5$ (upper indices are taken modulo 5) and $h_1^{(i)} \wedge h_1^{(i+1)} = \mathbf{y}_1^{(i)}$ for $i = 1, 2, 3$ (upper indices are taken modulo 3), respectively. Therefore the five (three, resp.) lines bound the stated area. Only $h_n^{(3)}$ is deduced from a not strict inequality, all other lines come from strict ones. Hence these lines have to be removed. Additionally all points satisfy the remaining inequalities. $\square$

ζ_1 describes a triangle, the others form pentagons. In Figure 2.6 this is shown graphically, starting with $P_2(\zeta_1)$ on the left (reversed axes). As can be verified easily, ζ_n induces the cut of a quadrangle out of $\mathcal{D}_2$ for $n > 1$ while $P_2(\zeta_1) \cap \mathcal{D}_2$ equals a triangle (black parts). Observe that we already met this triangle as E_3 (see (1.1.6)).

Summarising the results of this subsection we see that there definitely exist infinitely many cycles of $\mathcal{D}_2$ (and because of the Lifting Theorem 1.1.12 the same is true for $\mathcal{D}_d$ for any $d \geq 2$) corresponding to pairwise disjoint polyhedra. Several questions arise immediately from this observation.

Open question 1. Are there more families of cycles (there definitely exists cycles that do not belong to one of the analysed ones)?

Open question 2. Can one give a set of infinite families of cycles of $\mathcal{D}_d$ such that the set of cycles that do not belong to one of these families is finite?

Figure 2.7 shows an approximation of $\mathcal{D}_2^0$ that involves all the results we found up to now. Of course, we cannot depict all cutout polyhedra that correspond to cycles. But $P_2(\omega_n)$, $P_2(\zeta_n)$ for large n are that small that they cannot be recognised any more. Only for the light grey areas we do not known whether they belong to $\mathcal{D}_2^0$ or not.

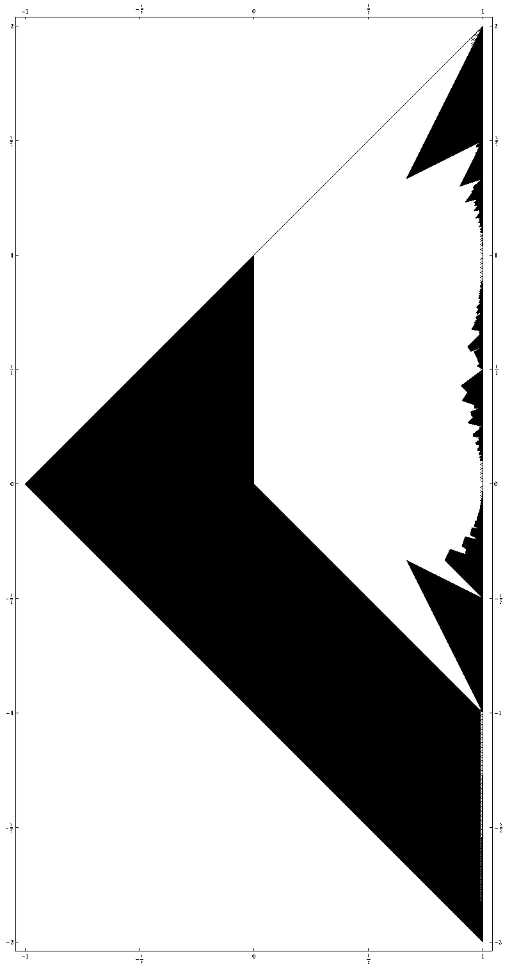


Figure 2.7: An approximation of D_2^0

2.2.5 Topological observations

We already noted that the point $K_d^{(1)} := (0, \ldots, 0, 1, 0)$ is a critical point for $d \geq 2$ due to [3]. By using the results of the previous subsection we will be able prove that $K_d^{(2)} := (0, \ldots, 0, 1, 1) \in \overline{\mathcal{D}_d}$ is also a critical point for $d \geq 2$ (*cf* [52]). Afterwards we will show the existence of a cutpoint of $\mathcal{D}_2^0$. the last subsection. We start with a simple lemma.

Lemma 2.2.28. *Let π be any cycle. Then* $\operatorname{int} P_d(\pi) \cap \mathbb{Z}^d = \emptyset$.

Proof. $P_d(\pi)$ is described be several inequalities of the form

$$0 \leq a_1 r_1 + \ldots + a_d r_d + a_{d+1} < 1$$

with integers $a_1, \ldots, a_d$. To ensure that a point is an inner point of $P_d(\pi)$, also the left hand side of the inequality have to be strict. For a point of $\mathbb{Z}^d$ this is impossible to fulfil. $\square$

Theorem 2.2.29. *The point $K_d^{(2)}$ is critical.*

Proof. Because of the lifting Theorem 1.1.12 it suffices to show the assertion only for $K_2^{(2)} = (1, 1)$. From Theorem 2.2.26 we know that we can construct a sequence of points $(x_k, y_k)_{k \in \mathbb{N}}$ converging to $K_2^{(2)}$ with $x_k < 1$, $y_k > 1$ and $(x_k, y_k) \notin \mathcal{D}_2^0$ for all $k \in \mathbb{N}$. Suppose $K_2^{(2)}$ were not a critical point. Then there must exist a cycle $\pi = \langle a_0, a_1, a_2, \ldots, a_{l-1}\rangle$ such that $P_2(\pi)$ includes all but finitely many elements of the sequence. For this cycle we can deduce the following properties (for the rest of the proof, take the indices of a modulo l):

1. $P_2(\pi) \supset Q := (1-\delta, 1)\times(1, 1+\varepsilon)$ for some $\delta, \varepsilon > 0$. The set $P_2(\pi)$ is described by inequalities
$$0 \le a_{i-1}x + a_i y + a_{i+1} < 1$$
with $i \in \{0, \ldots, l-1\}$. The points of Q have to suffice each of these inequalities. Together with Lemma 2.2.28 we obtain
$$\begin{array}{lcl} a_i < 0 & \Rightarrow & a_{i-1} + a_i + a_{i+1} = 1, \\ a_i > 0 & \Rightarrow & a_{i-1} + a_i + a_{i+1} = 0, \\ a_i = 0, a_{i-1} < 0 & \Rightarrow & a_{i-1} + a_i + a_{i+1} = 0, \\ a_i = 0, a_{i-1} > 1 & \Rightarrow & a_{i-1} + a_i + a_{i+1} = 1. \end{array}$$
Especially we have
$$\begin{array}{rcl} a_i \le 0, a_{i+1} < 0 & \Rightarrow & a_{i+2} = 1 - a_i - a_{i+1} > 0 \quad \Rightarrow \quad a_{i+3} = a_i - 1 < 0, \\ & \Rightarrow & a_{i+4} = a_{i+1} + 1, \end{array} , \tag{2.2.31}$$
$$\begin{array}{rcl} a_i \ge 0, a_{i+1} > 0 & \Rightarrow & a_{i+2} = -a_i - a_{i+1} < 0 \quad \Rightarrow \quad a_{i+3} = a_i + 1 > 0, \\ & \Rightarrow & a_{i+4} = a_{i+1} - 1. \end{array} . \tag{2.2.32}$$

2. π has to include zeros. To see this, we first note that π consists of positive and negative numbers. Summing up over all triples $a_{i-1} + a_i + a_{i+1}$ and observing the rules from (1) yields
$$3\sum_{i=0}^{l-1} a_i = |\{i \mid a_i < 0 \vee a_i = 0, a_{i-1} > 1\}\,| > 0.$$
For any $i \in \{0, \ldots, l-1\}$ we have
$$\begin{array}{cccccl} a_i & +a_{i+1} & +a_{i+2} & & \in & \{0,1\} \\ & -a_{i+1} & -a_{i+2} & -a_{i+3} & \in & \{0,-1\} \\ \hline a_i & & & -a_{i+3} & \in & \{0,1,-1\}. \end{array}$$
Therefore the element a_{i+3} differs from the element a_i by one at most. If $l \not\equiv 0(3)$, the elements of π can be rearranged as
$$a_0, a_3, \ldots, a_{3j+3(\mathrm{mod}\, l)}, \ldots, \; (0 \le j \le l-1)$$
Observing that this list includes positive numbers as well as negative ones and that neighboured elements differ by one at most shows that necessarily it has to include at least one zero. In the case $n \equiv 0(\mathrm{mod}\, 3)$ we can make such a rearrangement for each of the sets $A_k := \{a_i | i \equiv k(\mathrm{mod}\, 3)\}$, $k \in \{0, 1, 2\}$ separately. Suppose all of these sets consist of equal signed integers only. Then there must be at least one such set including only positive numbers, say A_1. The calculation
$$0 < \sum_{i=0}^{n-1} a_i = \sum_{j=0}^{n/3-1} (a_{3j} + a_{3j+1} + a_{3j+2}) = 0,$$
where $a_{3j+1} \in A_1$ for all j, shows the impossibility of this. Hence A_1 has to contained both positive and negative numbers and therefore zeros too.

We are now in the position to prove that there cannot exist a cycle π having the desired properties. We do this by constructing a sequence $(b_i)_{i\ge 0}$ of integers sufficing items 1. and 2. and showing that this sequence ends up in a series of zeros. Without loss of generality we may start with $b_0 = 0$. Suppose $b_1 = k > 0$. Successive application of (2.2.32) shows $b_{3k} = k > 0$ and $b_{3k+1} = 0$. Hence $b_{3k+2} = -k + 1$. Now we apply (2.2.31) sufficiently often to see that $b_{6k-2} = -k + 1, b_{6k-1} = 0$. This implies $b_{6k} = k - 1$. Repeating this procedure yields $b_{3k^2+2k-1} = 0$ and $b_{3k^2+2k} = 0$. Similar calculations for the case $k < 0$ shows the same result finishing the proof. □

When we look in Figure 2.2 more closely at Q_C we see that there obviously exists cycles that lie not within some neighbourhood of the boundary of $\mathcal{D}_2$. We enlarge the interesting area, *i.e.*, the quadrangle $[0.95, 1] \times [0.725, 0.755]$ (see Figure 2.8 left). There seems to exist a point on which it

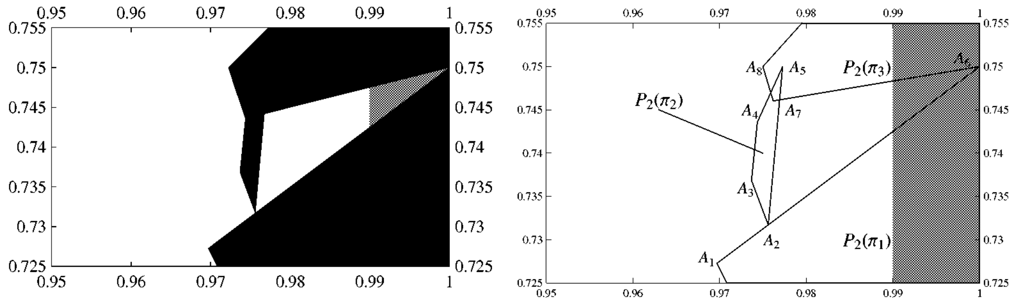


Figure 2.8: Magnification of a small subset of Q_C (left) and the position of three interesting cutout polygons (right).

depends whether $\mathcal{D}_2^0$ is possibly disconnected. To become more concrete consider the three cycles

$$\begin{aligned}
\pi_1 &:= \langle -8, 1, 8, -6, -3, 9, -3, -6, 8, 1, -8, 5, 5 \rangle, \\
\pi_2 &:= \langle -10, 1, 10, -8, -3, 11, -5, -7, 11, -1, -9, 8, 3, -10, 5, 7 \rangle, \\
\pi_3 &:= \langle -12, 3, 10, -10, -2, 12, -7, -6, 12, -3, -9, 10, 2, -11, 7, 6, -11, 3, 9, -9, -2, 11, -6, \\
&\quad -6, 11, -2, -9, 9, 3, -11, 6, 7, -11, 2, 10, -9, -3, 12, -6, -7, 12, -2, -10, 10, 3, -12, 7, 7 \rangle.
\end{aligned}$$

The corresponding polygons are depicted on the right hand side of Figure 2.8 and suffice to investigate the situation more closely. Obviously there are more cycles necessary in order to characterise $\mathcal{D}_2^0 \cap ([0.95, 1] \times [0.725, 0.755])$ as can be seen easily by comparing the two images in Figure 2.8. Define the points

$$\begin{aligned}
&A_0 := \left(1, \tfrac{2}{3}\right), && A_1 := \left(\tfrac{32}{33}, \tfrac{8}{11}\right), && A_2 := \left(\tfrac{40}{41}, \tfrac{30}{41}\right), && A_3 := \left(\tfrac{37}{38}, \tfrac{14}{19}\right), && A_4 := \left(\tfrac{38}{39}, \tfrac{29}{39}\right), \\
&A_5 := \left(\tfrac{43}{44}, \tfrac{3}{4}\right), && A_6 := \left(1, \tfrac{3}{4}\right), && A_7 := \left(\tfrac{41}{42}, \tfrac{47}{63}\right), && A_8 := \left(\tfrac{39}{40}, \tfrac{3}{4}\right), && A_9 := \left(1, \tfrac{7}{9}\right).
\end{aligned}$$

One can show that $P_2(\pi_1) \cap \mathcal{E}_2 \subset \operatorname{int}(\square(A_0, A_1, A_6))$, $P_2(\pi_2) \cap \mathcal{E}_2 \subset \operatorname{int}(\square(A_1, A_2, A_3, A_4))$ and $P_2(\pi_3) \cap \mathcal{E}_2 \subset \operatorname{int}(\square(A_6, A_7, A_8, A_9))$. We will not give a full characterisation of these sets. Just note that some of the bounding lines are included in the corresponding polygon and some are not. It is easy to see that the previously mentioned point is A_2. Thus we will restrict to the 3 involved lines. We have

$$\begin{aligned}
\overline{A_1A_6} &\in \{(x, y) \in \mathbb{R}^2 | -3x + 4y + 1 = 1\}, \\
\overline{A_2A_3} &\in \{(x, y) \in \mathbb{R}^2 | -8x - 3y + 11 = 1\}, \\
\overline{A_2A_5} &\in \{(x, y) \in \mathbb{R}^2 | \, 11x - y - 9 = 1\}.
\end{aligned}$$

Note that $A_2 \in \overline{A_1A_6}$. The first line is induced by the strict inequality which corresponds to the subsequence $(-3, 4, 1)$ of π_1, the second one by the strict inequality which corresponds to the subsequence $(-8, -3, 11)$ of π_2 and the last one is induced by the strict inequality corresponding to the subsequence $(11, -1, -9)$ of π_3. Hence, A_2 is not included in any of these cutout polygons and an application of Algorithm 3 of A_2 confirms that $A_2 \in \mathcal{D}_2^0$. Summing up the previous discussion gives

Theorem 2.2.30. $\left(\frac{40}{41}, \frac{30}{41}\right)$ *is a cutpoint of* $\mathcal{D}_2^0$.

We could not verify $\mathcal{D}_2^0$ to be disconnected. When we look carefully at Q_C in Figure 2.2 we can see that the topmost cutouts could possibly separate a subset of $\mathcal{D}_2^0$ from the rest of $\mathcal{D}_2^0$. For verifying this we would need better characterisation results for this questionable area since it lies inside the grey area that is not completely analysed yet.

Chapter 3

Numeration systems and tilings

In the present Chapter we start in Section 3.1 with the observation that the mapping $\tau_{\mathbf{r}}$ itself can be used to define a representation of integer vectors. We will also define a new type of tiles using the SRS notations and call these tiles SRS-tiles. In Section 3.2 we will deal with canonical number systems. We start with a generalisation of CNS to non-monic polynomials and call them generalise canonical number systems (GCNS for short). It will turn out that these GCNS can be analysed in an analogue way as the classical CNS. In Theorem 3.2.12 we will show that GCNS and SRS are strongly related. In Subsection 3.2.3 we will also see that tiles associated to expanding polynomials are also closely related to SRS-tiles. In Section 3.3 we will do a similar investigation for the β-expansion. We already mentioned in Theorem 1.3.5 the connection between the finiteness property (F) and SRS. We will investigate this relation more closely. In Subsection 3.3.3 we are going to investigate how new β-tiles and SRS-tiles are connected. In Theorem 3.3.22 we will show that they are connected by a linear map.

3.1 SRS representation and SRS tiles

The definitions and results of this section are taken from [17].

3.1.1 SRS representation

We can define a representation for d-dimensional integer vectors that is based on the mapping $\tau_{\mathbf{r}}$ for $\mathbf{r} \in \mathbb{R}^d$.

Definition 3.1.1. For $\mathbf{r} \in \mathbb{R}^d$ and $\mathbf{z} \in \mathbb{Z}^d$ the sequence

$$X_{\mathbf{r}}(\mathbf{z}) := (v_1, v_2, v_3, \ldots),$$

with v_i being the last coordinate of the vector

$$\tau_{\mathbf{r}}^i(\mathbf{z}) - R(\mathbf{r})(\tau_{\mathbf{r}}^{i-1}(\mathbf{z})),$$

is called the *SRS representation* of $\mathbf{z}$ with respect to $\mathbf{r}$ (no matter whether $\tau_{\mathbf{r}}$ is an SRS or not). For notating SRS representations we can adapt Notation 1.2.2.

For the SRS representation we have the following easy properties.

Lemma 3.1.2. *Let* $\mathbf{r} \in \mathbb{R}^d$ *and*

$$X_{\mathbf{r}}(\mathbf{z}) = (v_i)_{i \in \mathbb{N}^*}$$

the SRS representation of $\mathbf{z} \in \mathbb{Z}^d$ *with respect to* $\mathbf{r}$. *Then for the reals* $v_1, v_2, v_3, \ldots$ *the following assertions hold.*

1. $-1 < v_i \leq 0$,

2. $\tau_{\mathbf{r}}^k(\mathbf{z})$ *has the SRS representation* $X_{\mathbf{r}}(\tau_{\mathbf{r}}^k(\mathbf{z})) = (v_{i+k})_{i\in\mathbb{N}^*}$ *for all* $k \in \mathbb{N}$,

3.
$$\mathbf{z} = R(\mathbf{r})^{-k}\tau_{\mathbf{r}}^k(\mathbf{z}) + \sum_{j=1}^{k} R(\mathbf{r})^{-j}(0,\ldots,0,v_j)^T$$

for all $k \in \mathbb{N}$.

Proof. Observe that for each $\mathbf{x} \in \mathbb{Z}^d$ the first $d-1$ components of $\tau_{\mathbf{r}}(\mathbf{x}) - R(\mathbf{r})\tau_{\mathbf{r}}^{i-1}(\mathbf{x})$ are 0 and the last entry equals
$$-\lfloor \mathbf{r}\mathbf{x} \rfloor + \mathbf{r}\mathbf{x} \in (-1,0].$$
The rest can easily be proven by using the definition. □

The SRS representation is unique in the following sense.

Theorem 3.1.3. *Let* $\mathbf{r} \in \mathbb{R}^d$, $\mathbf{z}_0 \in \mathbb{Z}^d$, $X_{\mathbf{r}}(\mathbf{z}) = (w_i)_{i\in\mathbb{N}^*}$ *the SRS representation of* $\mathbf{x}$ *and* $v_1, v_2, \ldots, v_n \in (-1,0]$ *such that for all* k *with* $0 \le k \le n$ *the vector*
$$\mathbf{z}_k := R(\mathbf{r})^{-k}\mathbf{z}_0 + \sum_{j=1}^{k} R(\mathbf{r})^{-j}(0,\ldots,0,v_j)^T$$
is an integer vector. Then $\tau_{\mathbf{r}}^k(\mathbf{z}_k) = \mathbf{z}_0$ *and* $(v_k,\ldots,v_1,w_1,w_2,w_3,\ldots)$ *is the SRS representation of*$\mathbf{z}_k$.

Proof. The assertion is obviously true for $k = 0$. Now continue by induction on k. Suppose we already knew that $\mathbf{z}_k$ has the SRS representation $(v_k,\ldots,v_1,w_1,w_2,\ldots)$. By definition we have
$$\mathbf{z}_{k+1} = R(\mathbf{r})^{-k-1}\mathbf{z}_0 + \sum_{j=1}^{k+1} R(\mathbf{r})^{-j}(0,\ldots,0,v_j)^T \in \mathbb{Z}^d.$$
By the assumption on the induction this gives
$$\mathbf{z}_k = R(\mathbf{r})\mathbf{z}_{k+1} - (0,\ldots,0,v_{k+1})^T \in \mathbb{Z}^d.$$
By the definition of $R(\mathbf{r})$ this induces that the first $d-1$ entries of $\mathbf{z}_{k+1}$ equal the last $d-1$ entries of $\mathbf{z}_k$. Let α be the the last entry of $\mathbf{z}_k$. Then, by the conditions made on v_{k+1}, we see
$$0 \le \alpha + \mathbf{r}\mathbf{z}_{k+1} < 1.$$
Since α is an integer, we conclude that $\alpha = -\lfloor \mathbf{r}\mathbf{z}_{k+1} \rfloor$ and therefore $\mathbf{z}_k = \tau_{\mathbf{r}}(\mathbf{z}_{k+1})$. Thus $\mathbf{z}_{k+1}$ has the SRS representation $(v_{k+1},v_k,\ldots,v_1,,w_1,w_2,w_3,\ldots)$ by item 2. of Lemma 3.1.2. □

Lemma 3.1.4. *If* $\mathbf{r} \in \mathcal{D}_d$ *then for each* $\mathbf{z} \in \mathbb{Z}^d$ *the SRS representation is periodic. If* $\mathbf{r} \in \mathcal{D}_d^0$ *then, for each* $\mathbf{z} \in \mathbb{Z}^d$, *the SRS representation is finite.*

Proof. This follows directly by the definitions of $\mathcal{D}_d$ and $\mathcal{D}_d^0$. □

3.1.2 SRS-Tiles

We now define a new type of tiles based on the mapping $\tau_{\mathbf{r}}$. Denote by $\gamma(\cdot,\cdot)_{A,\delta}$ the Hausdorff metric induced by $\|\cdot\|_{A,\delta}$.

Definition 3.1.5. Let $\mathbf{r} \in \mathrm{int}(\mathcal{D}_d)$ and set
$$T_{\mathbf{r},n}(\mathbf{x}) := \{\mathbf{z} \in \mathbb{Z}^d \mid \tau_{\mathbf{r}}^n(\mathbf{z}) = \mathbf{x}\}$$
for $\mathbf{x} \in \mathbb{Z}^d$. The set
$$T_{\mathbf{r}}(\mathbf{x}) = \overline{\lim_{n\to\infty} R(\mathbf{r})^n T_{\mathbf{r},n}(\mathbf{x})}$$
is called an *SRS tile* associated to $\mathbf{r}$. The limit is taken with respect to $\gamma(\cdot,\cdot)_{R(\mathbf{r}),\delta}$ for some δ with $\varrho(R(\mathbf{r})) < \delta < 1$. The tile $T_{\mathbf{r}}(\mathbf{0})$ is called the *central SRS tile.*

Proposition 3.1.6.
$$\bigcup_{\mathbf{x}\in\mathbb{Z}^d} \mathcal{T}_{\mathbf{r}}(\mathbf{x}) = \mathbb{R}^d.$$

Proof. This can easily be seen by observing that the set $\lim_{n\to\infty} R(\mathbf{r})^n\mathbb{Z}^d$ is dense in $\mathbb{R}^d$. $\square$

Proposition 3.1.7. *$\mathcal{T}_{\mathbf{r},n}(\mathbf{x})$ consists exactly of those points $\mathbf{z} \in \mathbb{Z}^d$ that have an SRS representation of the shape $(v_1,\ldots,v_n,w_1,w_2,w_3,\ldots)$ where $(w_1,w_2,w_3,\ldots)$ is the SRS representation of $\mathbf{x}$.*

Proposition 3.1.8.
$$\mathcal{T}_{\mathbf{r}}(\mathbf{y}) = \bigcup_{\mathbf{x}\in\mathcal{T}_{\mathbf{r},1}(\mathbf{y})} R(\mathbf{r})\mathcal{T}_{\mathbf{r}}(\mathbf{x})$$

Proof. It is easy to see that we have
$$\mathcal{T}_{\mathbf{r},n}(\mathbf{y}) = \bigcup_{\mathbf{x}\in\mathcal{T}_{\mathbf{r},1}(\mathbf{y})} \mathcal{T}_{\mathbf{r},n-1}(\mathbf{x}).$$

Multiplying $R(\mathbf{r})^n$ and taking the limit on both sides yields
$$\begin{aligned}\mathcal{T}_{\mathbf{r}}(\mathbf{y}) &= \lim_{n\to\infty} R(\mathbf{r})^n\mathcal{T}_{\mathbf{r},n}(\mathbf{y}) = \lim_{n\to\infty} R(\mathbf{r})^n \bigcup_{\mathbf{x}\in\mathcal{T}_{\mathbf{r},1}(\mathbf{y})} \mathcal{T}_{\mathbf{r},n-1}(\mathbf{x}) \\ &= \bigcup_{\mathbf{x}\in\mathcal{T}_{\mathbf{r},1}(\mathbf{y})} R(\mathbf{r}) \lim_{n\to\infty} R(\mathbf{r})^{n-1}\mathcal{T}_{\mathbf{r},n-1}(\mathbf{x}) = \bigcup_{\mathbf{x}\in\mathcal{T}_{\mathbf{r},1}(\mathbf{y})} R(\mathbf{r})\mathcal{T}_{\mathbf{r}}(\mathbf{x}).\end{aligned}$$

$\square$

Example 3.1.9. For $\mathbf{r} = (\frac{3}{4}, 1)$ the tiles $\mathcal{T}_{\mathbf{r}}(\mathbf{x})$ with $\|\mathbf{x}\|_\infty \le 2$ are shown in Figure 3.1.

Let us study how many different tiles can share one point. This investigation immediately implies the boundedness of SRS tiles.

Lemma 3.1.10. *Let $\mathbf{r} \in \operatorname{int}\mathcal{D}_d$. For any $\mathbf{s} \in \mathbb{R}^d$ the set*
$$\left\{\mathbf{x} \in \mathbb{Z}^d \,|\, \mathbf{s} \in \mathcal{T}_{\mathbf{r}}(\mathbf{x})\right\}$$
is finite.

Proof. Since $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$ we can find a $\delta \in \mathbb{R}$ with $\varrho(R(\mathbf{r})) < \delta < 1$. From Proposition 3.1.7 we know that each $\mathbf{z} \in \mathcal{T}_{\mathbf{r},n}(\mathbf{x})$ can be represented as
$$\mathbf{z} = \sum_{i=1}^{n} R(\mathbf{r})^{-i}\mathbf{v}_i + R(\mathbf{r})^{-n}\mathbf{x}$$
with $\|\mathbf{v}_i\|_{R(\mathbf{r}),\delta} < \|(0,\ldots,0,1)^T\|_{R(\mathbf{r}),\delta}$ using the SRS representation. Hence
$$\|R(\mathbf{r})^n\mathbf{z} - \mathbf{x}\|_{R(\mathbf{r}),\delta} \le \sum_{i=0}^{n-1} \|R(\mathbf{r})^i\mathbf{v}_i\|_{R(\mathbf{r}),\delta}.$$

Set $M := \frac{\|(0,\ldots,0,1)^T\|_{R(\mathbf{r}),\delta}}{1-\delta}$. Then
$$\gamma(R(\mathbf{r})^n\mathbf{z}, \{\mathbf{x}\})_{R(\mathbf{r}),\delta} < M.$$

Thus $\gamma(R(\mathbf{r})^n\mathcal{T}_{\mathbf{r},n}(\mathbf{x}), \{\mathbf{x}\})_{R(\mathbf{r}),\delta} < M$ and therefore
$$\gamma(\mathcal{T}_{\mathbf{r}}(\mathbf{x}), \{\mathbf{x}\}) \le M. \tag{3.1.1}$$

Since M is independent of the choice of $\mathbf{x}$ we have
$$\left\{\mathbf{x} \in \mathbb{Z}^d \,|\, \mathbf{s} \in \mathcal{T}_{\mathbf{r}}(\mathbf{x})\right\} \subseteq \left\{\mathbf{x} \in \mathbb{Z}^d \,\middle|\, \|\mathbf{s}-\mathbf{x}\|_{R(\mathbf{r}),\delta} \le M\right\}$$
where the set on the right hand side is obviously finite. $\square$

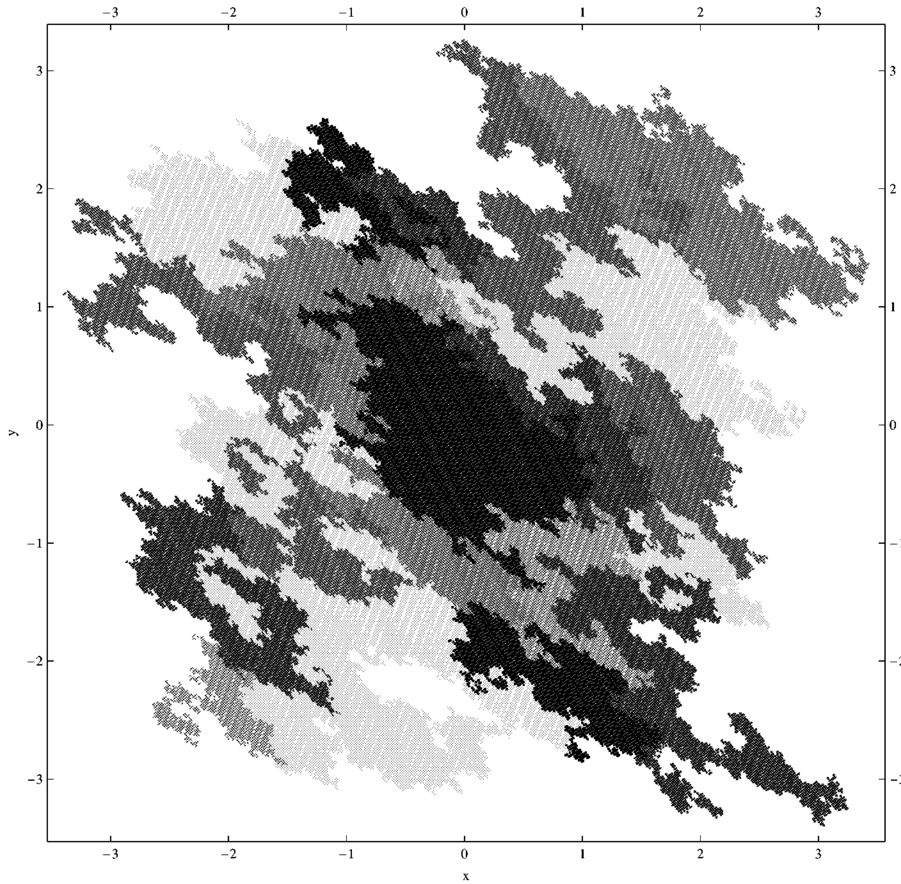


Figure 3.1: SRS tiles associated to $\mathbf{r} = (\frac{3}{4}, 1)$

Proposition 3.1.11. *For $\mathbf{r} \in \operatorname{int} \mathcal{D}_d$ and $\mathbf{x} \in \mathbb{Z}^d$, the tile $\mathcal{T}_\mathbf{r}(\mathbf{x})$ is compact.*

Proof. By definition SRS tiles are closed and (3.1.1) shows that $\mathcal{T}_\mathbf{r}(\mathbf{x})$ is bounded. □

Next we will answer the question which tiles contain the origin. Obviously $\mathbf{0}$ is always an element of the central tile. But it can be shared by finitely many other tiles.

Theorem 3.1.12. *Let $\mathbf{r} \in \operatorname{int} \mathcal{D}_d$. $\mathbf{0} \in \mathcal{T}_\mathbf{r}(\mathbf{x})$ if and only if $\mathbf{x} \in \mathbb{Z}^d$ is purely periodic with respect to $\tau_\mathbf{r}$.*

Proof. We first show that $\mathbf{0} \in \mathcal{T}_\mathbf{r}(\mathbf{x})$ if $\mathbf{x}$ is purely periodic with a period of length l. We have $\tau_\mathbf{r}^l(\mathbf{x}) = \mathbf{x}$. Therefore, for all $k \in \mathbb{N}$, $\mathbf{x} \in \mathcal{T}_{\mathbf{r},kl}(\mathbf{x})$ and since $R(\mathbf{r})$ is contractive we have

$$\mathbf{0} = \lim_{k \to 0} R(\mathbf{r})^{kl} \mathbf{x} \in \mathcal{T}_\mathbf{r}(\mathbf{x}).$$

On the other hand suppose there is some non-purely periodic $\mathbf{x}_0 \in \mathbb{Z}^d$ with $\mathbf{0} \in \mathcal{T}_\mathbf{r}(\mathbf{x}_0)$. We will derive a contradiction by showing that this implies the existence of a sequence $(\mathbf{x}_i)_{i \in \mathbb{N}}$ with the following properties for all $i \in \mathbb{N}$:

1. $\mathbf{x}_i$ is not purely periodic,
2. $\mathbf{0} \in \mathcal{T}_\mathbf{r}(\mathbf{x}_i)$,
3. $\tau_\mathbf{r}^i(\mathbf{x}_i) = \mathbf{x}_0$,
4. $\mathbf{x}_i \neq \mathbf{x}_j$ for $j < i$.

We do this by induction on i. $\mathbf{x}_0$ satisfies items 1.-4. by definition. Now suppose that we already have points $\mathbf{x}_0, \dots, \mathbf{x}_i$ with the desired properties. By Proposition 3.1.8 we have

$$T_{\mathbf{r}}(\mathbf{x}_i) = \bigcup_{\mathbf{z} \in T_{\mathbf{r},\mathbf{1}}(\mathbf{x}_i)} R(\mathbf{r}) T_{\mathbf{r}}(\mathbf{z}).$$

Since $\mathbf{0} \in T_{\mathbf{r}}(\mathbf{x}_i)$ we can find a $\mathbf{z} \in T_{\mathbf{r},\mathbf{1}}(\mathbf{x}_i)$ with $\mathbf{0} \in R(\mathbf{r})T_{\mathbf{r}}(\mathbf{z})$ and therefore $\mathbf{0} \in T_{\mathbf{r}}(\mathbf{z})$. Set $\mathbf{x}_{i+1} := \mathbf{z}$. Then $\mathbf{x}_{i+1}$ satisfies item 2. Since $\tau_{\mathbf{r}}(\mathbf{x}_{i+1}) = \mathbf{x}_i$ and $\tau_{\mathbf{r}}^i(\mathbf{x}_i) = \mathbf{x}_0$ we also have $\tau_{\mathbf{r}}^{i+1}(\mathbf{x}_{i+1}) = \mathbf{x}_0$, proving item 3. To show item 1. assume that $\mathbf{x}_{i+1}$ were purely periodic. Then $\tau_{\mathbf{r}}(\mathbf{x}_{i+1}) = \mathbf{x}_i$ would be purely periodic, too, which is not the case by the assumption on the induction. Finally $\mathbf{x}_{i+1} = \mathbf{x}_j = \tau_{\mathbf{r}}^{i+1-j}(\mathbf{x}_{i+1})$ for a $j < i$ would contradict the the fact that $\mathbf{x}_{i+1}$ is not purely periodic, proving item 4. Thus we have shown the existence of our sequence. But the existence of such a sequence contradicts Lemma 3.1.10. □

Corollary 3.1.13. *For $\mathbf{r} \in \operatorname{int} \mathcal{D}_d$ we have $\mathbf{r} \in \mathcal{D}_d^0$ if and only if $\mathbf{0}$ is an exclusive inner point of $T_{\mathbf{r}}(\mathbf{0})$, i.e., $\mathbf{0}$ is an inner point of $T_{\mathbf{r}}(\mathbf{0})$ and $\mathbf{0} \notin T_{\mathbf{r}}(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{0}$.*

It immediately follows that for $\mathbf{r} \in \mathcal{D}_d^0 \cap \mathcal{E}_d$ the central tile $T_{\mathbf{r}}(\mathbf{0})$ has a nonempty interior. One may ask if $T_{\mathbf{r}}(\mathbf{x})$ has non-empty interior for each $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$, $\mathbf{x} \in \mathbb{Z}^d$. The answer is no, as the following example shows.

Example 3.1.14. Set $\mathbf{r} = (\frac{9}{10}, -\frac{11}{20})$. Consider the points

$$\mathbf{z}_1 = (-1,-1), \mathbf{z}_2 = (-1,1), \mathbf{z}_3 = (1,2), \mathbf{z}_4 = (2,1), \mathbf{z}_5 = (1,-1).$$

It can easily be verified that

$$\tau_{\mathbf{r}} : \mathbf{z}_1 \mapsto \mathbf{z}_2 \mapsto \mathbf{z}_3 \mapsto \mathbf{z}_4 \mapsto \mathbf{z}_5 \mapsto \mathbf{z}_1.$$

Thus, each of these points is purely periodic. In fact, we have $\langle -1,-1,1,2,1 \rangle \in \mathcal{O}(C(\mathbf{r}))$. Now calculate $T_{\mathbf{r},1}(\mathbf{z}_1) = \tau_{\mathbf{r}}^{-1}(\mathbf{z}_1)$:

$$\begin{aligned} T_{\mathbf{r},1}(\mathbf{z}_1) &= \left\{ (x,-1)^T \,\middle|\, x \in \mathbb{Z}, 0 \le \frac{9}{10}x + \frac{11}{20} - 1 < 1 \right\} \\ &= \left\{ (x,-1)^T \,\middle|\, x \in \mathbb{Z}, \frac{1}{2} \le x < \frac{29}{18} \right\} = \{(1,-1)^T\} = \{\mathbf{z}_5\}. \end{aligned}$$

Similarly it can be shown that $T_{\mathbf{r},1}(\mathbf{z}_i) = \{\mathbf{z}_{i-1}\}$ for $i \in \{2,3,4,5\}$. Hence $T_{\mathbf{r}}(\mathbf{z}_i) = \mathbf{0}$ for $i \in \{1,2,3,4,5\}$. Figure 3.2 shows the tiles $T_{\mathbf{r}}(\mathbf{x})$ for all $\mathbf{x}$ with $\|\mathbf{x}\|_\infty \le 5$.

We have seen that for $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$ the union of all tiles $T_{\mathbf{r}}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{Z}^d$ cover the d-dimensional Euclidean space. We now ask whether the tiles provide a tiling, *i.e.*, $\operatorname{int}(T_{\mathbf{r}}(\mathbf{x})) \cap \operatorname{int}(T_{\mathbf{r}}(\mathbf{y})) = \emptyset$ for $\mathbf{x} \neq \mathbf{y}$.

Conjecture 3.1.15. The family $(T_{\mathbf{r}}(\mathbf{x}))_{\mathbf{x} \in \mathbb{Z}^d}$ tiles the d-dimensional Euclidean space for all $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$.

In the general case this question is up to now unsolved. Various computer experiments confirm the conjecture. We will give a necessary and sufficient condition for the tiles to provide a tiling.

Lemma 3.1.16. *Let $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$, $\boldsymbol{x} \in \mathbb{Z}^d$ and $\mathbf{s} \in \operatorname{int}(T_{\mathbf{r}}(\mathbf{x}))$. For each open ball $K(\mathbf{s}, \varepsilon)$ around $\mathbf{s}$ with radius $\varepsilon > 0$ such that $K(\mathbf{s}, \varepsilon) \subset T_{\mathbf{r}}(\mathbf{x})$ there exist $n \in \mathbb{N}$, $\mathbf{z} \in T_{\mathbf{r},n}(\mathbf{x})$ such that $T_{\mathbf{r}}(\mathbf{z})$ has nonempty interior and $R(\mathbf{r})^n T_{\mathbf{r}}(\mathbf{z}) \subset K(\mathbf{s}, \varepsilon)$.*

Proof. This is easy to see by the boundedness of SRS tiles (Formula (3.1.1)) and the set equation in Proposition 3.1.8. □

Theorem 3.1.17. *Let $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$. The SRS-tiles $(T_{\mathbf{r}}(\mathbf{x}))_{\mathbf{x} \in \mathbb{Z}^d}$ tile the d-dimensional real space if and only if each tile $T_{\mathbf{r}}(\mathbf{x})$ with nonempty interior has at least one exclusive inner point.*

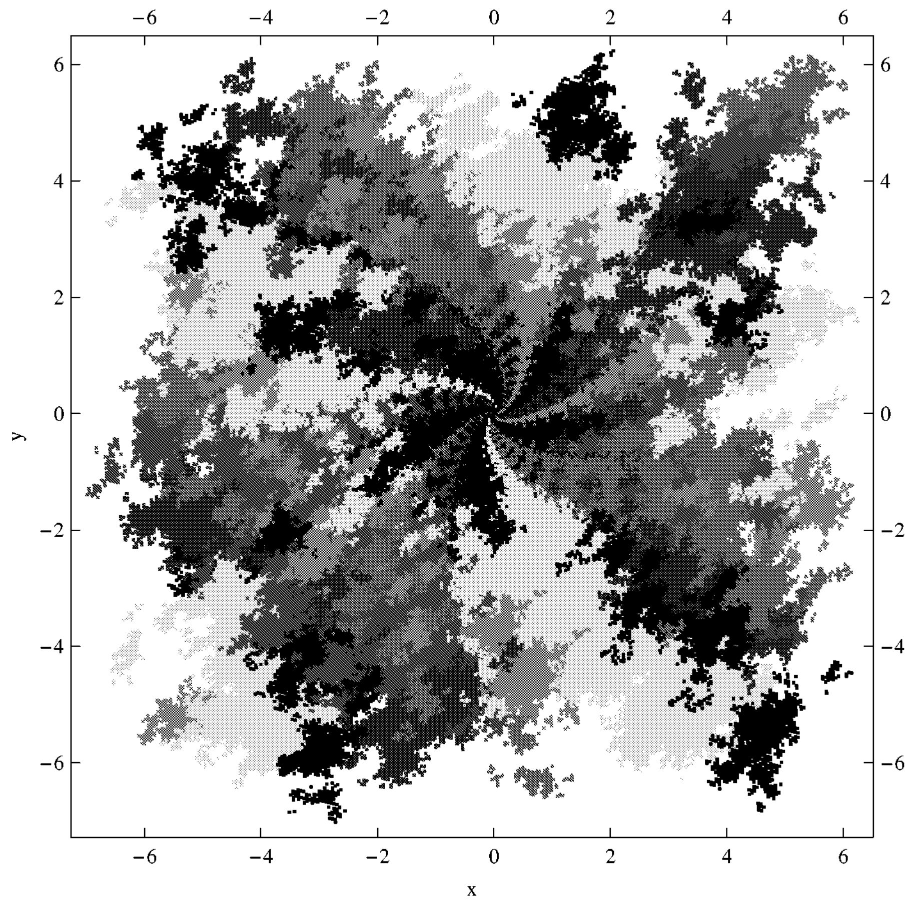


Figure 3.2: SRS tiles associated to $(\frac{9}{10}, \frac{11}{20})$

Proof. Since for a tiling each interior point is exclusive this direction is trivial. To show the opposite direction suppose that there are two points $\mathbf{y}, \mathbf{y}'$ such that the corresponding tiles have an inner point $\mathbf{s}$ in common. We will show that under this circumstances not all tiles with nonempty interior can have an exclusive inner point.

We can find an $\varepsilon > 0$ with $T_{\mathbf{r}}(\mathbf{y}) \cap T_{\mathbf{r}}(\mathbf{y}') \supset K := K(\mathbf{s}, \varepsilon)$. Then, by Lemma 3.1.16, there exists $n \in \mathbb{N}$, $\mathbf{z} \in T_{\mathbf{r},n}(\mathbf{y})$, such that $R(\mathbf{r})^n T_{\mathbf{r}}(\mathbf{z}) \subset K$ and $\operatorname{int}(T_{\mathbf{r}}(\mathbf{z})) \neq \emptyset$. Thus, since $T_{\mathbf{r}}(\mathbf{y}') = \bigcup_{\mathbf{z}' \in T_{\mathbf{r},n}(\mathbf{y}')} R(\mathbf{r})^n T_{\mathbf{r}}(\mathbf{z}')$, we have

$$R(\mathbf{r})^n T_{\mathbf{r}}(\mathbf{z}) \subset K \subset T_{\mathbf{r}}(\mathbf{y}') = \bigcup_{\mathbf{z}' \in T_{\mathbf{r},n}(\mathbf{y}')} R(\mathbf{r})^n T_{\mathbf{r}}(\mathbf{z}')$$

and therefore

$$T_{\mathbf{r}}(\mathbf{z}) \subset \bigcup_{z' \in T_{\mathbf{r},n}(\mathbf{y}')} T_{\mathbf{r}}(\mathbf{z}').$$

But this is a contradiction to the existence of an exclusive inner point of $T_{\mathbf{r}}(\mathbf{z})$. $\square$

3.2 Canonical number systems

3.2.1 Generalised Canonical number systems

After our survey concerning canonical number systems (CNS) in Section 1.2 we now show how we can generalise CNS. In a recent paper Scheicher, Surer, Thuswaldner and van de Woestijne [46] treat the non-monic case and call it generalised canonical number system. In fact, the authors present results in a much more general way.

Fix a commutative ring $\mathcal{E}$ with identity.

Definition 3.2.1. Let $P = p_d x^d + \cdots + p_1 x + p_0$ such that p_d is no zero divisor, p_0 is not zero divisors and no unit and $E/(p_0)$ is finite. Set $\mathcal{R} = \mathcal{E}[x]/(P)$ and denote by X is the image of x under the canonical epimorphism $\mathcal{E}[x] \to \mathcal{E}[x]/(P)$. Denote by $\mathcal{N} \subset \mathcal{E}$ a system of representatives of $\mathcal{R}/(X)$. We call the triple $(\mathcal{R}, X, \mathcal{N})$ a *digit system* in $\mathcal{R}$. Define maps

$$\begin{aligned} m_{\mathcal{N}} :& \mathcal{R} \to \mathcal{N} : m_{\mathcal{N}}(A) = e, \text{ the unique } e \in \mathcal{N} \text{ with } A \equiv e \pmod{X}, \\ T_P :& \mathcal{R} \to \mathcal{R} : T(A) = \frac{A - m_{\mathcal{N}}(A)}{X}. \end{aligned}$$

For an $A \in \mathcal{R}$ we call $X_P(A) = \left(m_{\mathcal{N}}(T_P^n(A))\right)_{n\in\mathbb{N}}$ the X-ary representation of A. If there exists an $h \in \mathbb{N}$ such that

$$A = \sum_{i=0}^{h} m_{\mathcal{N}}(T_P^i(A))X^i$$

this sum is called a finite X-ary expansion of A. The sum is a minimal expansion of A if h is minimal. We say that $(\mathcal{R}, X, \mathcal{N})$ has the periodic representation property (PRP) if the representations $X_P(A)$ are eventually periodic for all $A \in \mathcal{R}$ and that it has the finite expansion property (FEP) if every $A \in \mathcal{R}$ has a finite X-ary expansion with digits in $\mathcal{N}$.

One notes that $\mathcal{R}/(X) \cong \mathcal{E}/(p_0)$, so that $|\mathcal{N}| = |\mathcal{E}/(p_0)|$. The X-ary representation of $A \in \mathcal{R}$ clearly exists and is unique, because $\mathcal{N}$ is a system of representatives of $\mathcal{R}$ modulo X. If $\mathcal{N}$ were larger, we would have non-uniqueness and redundance in the X-ary expansion of A.

Lemma 3.2.2. *$A \in \mathcal{R}$ has a finite X-ary expansion if and only if $T_P^n(A) = 0$ for some $n \in \mathbb{N}$.*

Proof. Let $A = \sum_{i=0}^{n-1} e_i X^i$ with $e_i \in \mathcal{N}$. We obviously have $m_{\mathcal{N}}(A) = e_0$ and $T_P(A) = \sum_{i=0}^{n-2} e_{i+1} X^i$. Thus $T_P^n(A) = 0$. Now suppose there exists an $n \in \mathbb{N}$ such that $T_P^n(A) = 0$. Let $(a_n)_{n\in\mathbb{N}}$ the X-ary representation of A. Then it is easy to see that $A = \sum_{i=0}^{n-1} a_i X^i$. □

Lemma 3.2.3. *The finite expansion property implies the periodic representation property.*

Proof. Assume we had a digit system $(\mathcal{R}, X, \mathcal{N})$ that has the finite expansion property but not the periodic representation property. Then we can find an $A \in \mathcal{R}$ having finite X-ary expansion and no periodic X-ary representation. By Lemma 3.2.2 there exists an $n \in \mathbb{N}$ with $T_P^n(A) = 0$. Since A is not periodic we have $T_P^k(A) \neq 0$ for all $k > n$. This means that $T_P^{n+1}(A) = T_P(0) \neq 0$ cannot have a finite expansion by Lemma 3.2.2. That is a contradiction to our assumption. □

Denote by $C(\mathcal{R}, X, \mathcal{N})$ the set of all elements A of $\mathcal{R}$ with $T_P^n(A) = A$ for some $n \geq 1$ and write $C(\mathcal{R}, X, \mathcal{N})/T_P$ for the set of orbits.

Lemma 3.2.4. *Let $(\mathcal{R}, X, \mathcal{N})$ have the periodic representation property. $(\mathcal{R}, X, \mathcal{N})$ has the finite expansion property if and only if $0 \in C(\mathcal{R}, X, \mathcal{N})$ and $|C(\mathcal{R}, X, \mathcal{N})/T_P| = 1$.*

Proof. Since $(\mathcal{R}, X, \mathcal{N})$ has the periodic representation property we have that the X-ary representation of each element of $\mathcal{R}$ ends up periodically. $C(\mathcal{R}, X, \mathcal{N})/T_P$ is the set of all possible periods. Thus $|C(\mathcal{R}, X, \mathcal{N})/T_P| = 1$ and $0 \in C(\mathcal{R}, X, \mathcal{N})$ implies that for each $A \in \mathcal{R}$ there exists an $n \in \mathbb{N}$ with $T^n(A) = 0$. Then Lemma 3.2.2 shows the necessity of our statement.

We now show the opposite. Since $(\mathcal{R}, X, \mathcal{N})$ has the periodic representation property, for each $A \in \mathcal{R}$ there must exist an $n \in \mathbb{N}$ and a $B \in C(\mathcal{R}, X, \mathcal{N})/T_P$ such that $T_P^n(A) = B$. Moreover, as $|C(\mathcal{R}, X, \mathcal{N})/T_P| = 1$ and $0 \in C(\mathcal{R}, X, \mathcal{N})$, there must exist an $k \in \mathbb{N}$ with $T_P^k(B) = 0$. This immediately implies $T_P^{n+k}(A) = 0$ and therefore each $A \in \mathcal{R}$ has a finite X-ary representation by Lemma 3.2.2. □

As $\mathcal{R}$ is a quotient of the polynomial ring $\mathcal{E}[x]$, the usual generating set taken for $\mathcal{R}$ is $1, X, X^2, \ldots$. For CNS (see for example [22, 48]) a special type of generating set of $\mathcal{R}$ has proved to be very useful in order to simplify the backward division algorithm. This so-called Brunotte basis can be used in our more general case as well.

$$\begin{aligned}
w_0 &:= p_d, \\
w_1 &:= w_0 X + p_{d-1}, \\
&\vdots \\
w_k &:= w_{k-1} X + p_{d-k}, \\
&\vdots \\
w_{d-1} &:= w_{d-2} X + p_1 = p_d X^{d-1} + p_{d-1} X^{d-2} + \cdots + p_1.
\end{aligned} \tag{3.2.1}$$

Note that modulo P we have $X w_{d-1} + p_0 = 0$.

Definition 3.2.5. The $\mathcal{E}$-submodule of $\mathcal{R}$ generated by the w_i will be called the P-lattice of $\mathcal{R}$ and written $\Lambda_P(\mathcal{R})$.

Up to now in the literature this construction was used in the monic case, because then $\mathcal{R}$ is an $\mathcal{E}$-module with $w_0, \ldots, w_{d-1}$ acting as basis and each element of $\mathcal{R}$ has a unique representation of the shape $\sum_{i=0}^{d-1} a_i w_i$ with $a_i \in \mathcal{E}$. For non-monic P this is obviously not the case; in fact, $\Lambda_P(\mathcal{R})$ is a sublattice of index p_d^d of the lattice generated by $1, X, \ldots, X^{d-1}$. However, we have the following modified representation.

Lemma 3.2.6. *Let $M \subset \mathcal{E}$ a set of representatives of $\mathcal{E}/p_d$ including 0. For each $A \in \mathcal{R}$ there exists unique $A' \in \Lambda_P(\mathcal{R})$ and $(r_0, \ldots, r_{k-1}) \in M^k$ for some $k \in \mathbb{N}$ such that*

$$A = A' + r$$

with $r = \sum_{i=0}^{k-1} r_i X^i$ and $r_{d-1} \neq 0$.

Proof. Let $A \in \mathcal{R}$ be represented by $f = \sum_{i=0}^{l} b_i X^i \in \mathcal{E}[X]$. There is an $r_l \in M$ such that $b_l = r_l + q_l p_d$. If $l \geq d$ replace f by $f - q_l P X^{l-d}$, then we may assume $b_l \in M$. Then, for $i = l-1, l-2, \ldots, d$ let $r_i \in M$ such that $b_i = r_i + q_i p_d$ and replace f by $f - q_i P X^{i-d}$. Thus we may assume that $f = \sum_{i=d}^{l} r_i X^i + \sum_{i=0}^{d-1} b_i X^i$ with $r_i \in M$ and $b_i \in \mathcal{E}$ and f still represents A. $\Lambda_P(\mathcal{R})$ is a sublattice of the lattice $\{\sum_{i=0}^{d-1} c_i X^i | \, c_i \in \mathcal{E}\}$ with index p_d^d and thus there must exist unique $r_0, \ldots r_{d-1} \in M$, $a_1, \ldots, a_{d-1} \in \mathcal{E}$ such that

$$\sum_{i=0}^{d-1} b_i X^i = \sum_{i=0}^{d-1} a_i w_i + \sum_{i=0}^{d-1} r_i X^i.$$

The assertion follows immediately by setting $k = \max_{i \in \{0, \ldots, l\}} \{r_i \neq 0\}$ and $A' = \sum_{i=0}^{d-1} a_i w_i$. □

Definition 3.2.7. Let $M \subset \mathcal{E}$ be a set of representatives of $\mathcal{E}/p_d$. For an element $A \in \mathcal{R}$ the representation

$$A = A' + r$$

from the above Lemma is called the *standard representation* of A with respect to M. We say that r is the *residue polynomial* of A.

Note that in the monic case $\Lambda_P(\mathcal{R}) = \mathcal{R}$. The P-lattice itself does not depend on the choice of M and has the nice property that the behaviour of the X-ary representation is completely characterised by it.

Theorem 3.2.8. *Suppose $\mathcal{N} \subset \mathcal{E}$. $(\mathcal{R}, X, \mathcal{N})$ has the finite expansion property (periodic representation property) if and only if each element of $\Lambda_P(\mathcal{R})$ has a finite X-ary expansion (periodic X-ary representation).*

Proof. We will prove the theorem for the finite expansion property. The condition is obviously necessary for $(\mathcal{R}, X, \mathcal{N})$ to have the finite expansion property since $\Lambda_P(\mathcal{R}) \subset \mathcal{R}$. Now suppose each element of $\Lambda_P(\mathcal{R})$ has a finite X-ary expansion. Let $M \subset \mathcal{E}$ a set of representatives of $\mathcal{E}/p_d$, $0 \in M$, and $A = A^{(0)} \in \mathcal{R}$ with standard representation

$$A = \sum_{i=0}^{d-1} a_i w_i + r$$

with $r = r^{(0)} = r_k X^k + \cdots + r_1 X + r_0$, $r_i \in M$. We will investigate the action of T_P on A in terms of the standard representation. The constant term of $A^{(0)}$ equals $\sum_{i=0}^{d-1} a_i p_{d-i} + r_0 = qp_0 + e_0$ for some $q \in \mathcal{E}$, $e_0 = m_{\mathcal{N}}(A) \in \mathcal{N}$. Thus

$$A^{(0)} = e_0 + qp_0 + \sum_{i=0}^{d-1} a_i(w_i - p_{d-i}) + Xr^{(1)} \text{ in } \mathcal{E}[X]$$

with $r^{(1)} = r_k X^{k-1} + \cdots + r_2 X + r_1$. Observe that $w_0 - p_d = 0$, $w_i - p_{d-i} = Xw_{i-1}$ for $i \in \{1, \ldots, d-1\}$. Now switch to $\mathcal{R}$ and note that $qp_0 = -qXw_{d-1}$. Set $a_d := -q$. Then $A^{(0)} = e_0 + XA^{(1)}$ with $A^{(1)} = T_P(A^{(0)})$ having standard representation

$$A^{(1)} = \sum_{i=0}^{d-1} a_{i+1} w_i + r^{(1)}.$$

In other words: after one application of T_P the degree of the residue polynomial r decreases by one and the first k coefficients do not change. Hence, after $k+1$ applications, we have

$$T_P^{k+1}(A) = A^{(k+1)} \in \Lambda_P(\mathcal{R}).$$

Since each element of $\Lambda_P(\mathcal{R})$ has a finite X-ary expansion by assumption, $A^{(k+1)}$ and therefore A has a finite expansion. The proof for the periodic representation property runs analogously. □

Theorem 3.2.9. *Let $\mathcal{N} \subset \mathcal{E}$ and $\mathcal{E}$ Euclidean with value function $g : \mathcal{E} \mapsto \mathbb{R}^+ \cup \{0, -\infty\}$ where $g(0) = -\infty$. Further suppose $g(e) < g(p_0)$ for all $e \in \mathcal{N}$. If $g(p_d) \geq g(p_0)$ no element of $\Lambda_P(\mathcal{R})$ but 0 has a finite X-ary expansion.*

Proof. First note that the assumption on $\mathcal{N}$ implies $0 \in \mathcal{N}$. Let $\pi : \mathcal{E}[x] \to \mathcal{R}$ the canonical epimorphism and $A \in \mathcal{E}[x]$. Because the leading coefficient of w_k is p_d for each $k \in \{0, \ldots, d-1\}$ it is easy to see that $\pi(A) \in \Lambda_P(\mathcal{R})$ implies that the leading coefficient of A is a multiple of p_d. Now suppose that there is a $B \in \Lambda_P(\mathcal{R})$, $B \neq 0$ with finite X-ary expansion

$$B = \sum_{i=0}^{h} e_i X^i, \quad e_i \in \mathcal{N}, e_h \neq 0.$$

Thus $B = \pi(\sum_{i=0}^{h} e_i x^i) \in \Lambda_P(\mathcal{R})$. By assumption we have that $g(e) < g(p_0)$ for $e \in \mathcal{N}$ and therefore $g(e_h) < g(p_0) \leq g(p_d)$. As observed above we also must have that $e_h = qp_d$ for some nonzero $q \in \mathcal{E}$. But $\mathcal{E}$ is Euclidean and $q, p_d \neq 0$ implies $g(e_h) = g(qp_d) \geq g(p_d)$ which is a contradiction. □

Corollary 3.2.10. *With the above assumptions on $\mathcal{E}$ and $\mathcal{N}$, $g(p_d) < g(p_0)$ is necessary for $(\mathcal{R}, X, \mathcal{N})$ to have the finite expansion property.*

Now assume the ring $\mathcal{E} = \mathbb{Z}$ and $|e| < |p_0|$ for all $e \in \mathcal{N}$. Here Corollary 3.2.10 gives us a first characterisation. $(\mathcal{R}, X, \mathcal{N})$ can only be a digit system if $|p_0| > |p_n|$. Let $\mathcal{N} = \{0, \ldots, |p_0| - 1\}$. If P is monic and $(\mathcal{R}, X, \mathcal{N})$ has the finite expansion property the pair $(P, \mathcal{N})$ is a canonical number systems. We will generalise this in the following

Definition 3.2.11. Let $P(x) = p_d x^d + \cdots + p_1 x + p_0 \in \mathbb{Z}[x]$, $\mathcal{R} = \mathbb{Z}[x]/(P)$ and $\mathcal{N} = \{0, \ldots, |p_0| - 1\}$. If $(\mathcal{R}, X, \mathcal{N})$ has the finite expansion property we call $(P, \mathcal{N})$ a Generalised Canonical Number System (GCNS).

We now can reformulate Theorem 1.2.6 with respect to GCNS.

Theorem 3.2.12. *Let $P(x) = p_d x^d + p_{d-1} x^{d-1} + \cdots + p_1 x + p_0 \in \mathbb{Z}[x]$ and $\mathcal{N} = \{0, \ldots, |p_0| - 1\}$. $(P, \mathcal{N})$ is a GCNS if and only if $\mathbf{r} = \left(\frac{p_d}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right) \in \mathcal{D}_d^0$. Each element of $\mathcal{R}$ has an ultimately periodic X-ary expansion if and only if $\mathbf{r} = \left(\frac{p_d}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right) \in \mathcal{D}_d$.*

Proof. For $k \in \{0, \ldots, d-1\}$ define w_k as in (3.2.1), thus

$$w_k = \sum_{i=0}^{k} p_{d-i} X^{k-i}.$$

By Theorem 3.2.8 $(P, \mathcal{N})$ is a GCNS if and only if each elements A of the shape

$$A = A^{(0)} = \sum_{i=0}^{d-1} w_i a_i, \quad a_i \in \mathbb{Z}$$

has a finite X-ary expansion. In the proof of Theorem 3.2.8 we showed that an application of T_P yields $A^{(0)} = e_0 + XA^{(1)}$ where $e_0 = m_{\mathcal{N}}(A^{(0)}) \in \mathcal{N}$ and

$$A^{(1)} = T_P(A^{(0)}) = \sum_{i=0}^{d-1} a_{i+1} w_i$$

with $a_d = -\frac{\sum_{i=0}^{d-1} a_i p_{d-i} + e_0}{p_0} \in \mathbb{Z}$. Thus, by the choice of $\mathcal{N}$, we have $0 \le \sum_{i=0}^{d-1} a_i \frac{p_{d-i}}{p_0} + a_d = \frac{e_0}{p_0} < 1$ which shows that

$$a_d = -\lfloor \mathbf{r}(a_0, \ldots, a_{d-1}) \rfloor .$$

Thus $A^{(1)} = (w_0, \ldots, w_{d-1}) \tau_{\mathbf{r}}(a_0, \ldots, a_{d-1})$. $(P, \mathcal{N})$ is a GCNS exactly if successive application of T_P on each $A \in \Lambda_P(\mathcal{R})$ ends up in 0 and we see that this is equivalent for $\tau_{\mathbf{r}}$ to be an SRS. The second part can be shown analogously. $\square$

3.2.2 X-ary representation and the SRS-representation

We are going to analyse the connection between the X-ary representation and the SRS representation more closely. Let $P(x) = p_d x^d + \ldots + p_1 x + p_0 \in \mathbb{Z}[x]$, $\mathcal{R} = \mathbb{Z}[x]/(P)$, X the image of x under this canonical epimorphism and $\mathcal{N} = \{0, \ldots, |p_0| - 1\}$. Additionally, since $(P) = (-P)$ we can assume $p_0 \ge 2$. Set $\mathbf{r} = \left(\frac{p_d}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right)$.

First suppose that $p_d \in \{-1, 1\}$. Then each element of $A \in \mathcal{R}$ can be represented uniquely as

$$A = \sum_{i=0}^{d-1} a_i X^i.$$

Consider the embedding

$$\begin{aligned} \Phi_C : \mathcal{R} &\rightarrow \mathbb{Z}^d, \\ A &\mapsto (a_0, \ldots, a_{d-1}). \end{aligned}$$

By the definition of T_P the mapping

$$\begin{aligned} \tilde{T}_P : \qquad \mathbb{Z}^d &\rightarrow \mathbb{Z}^d, \\ (a_0, \ldots, a_{d-1})^T &\mapsto (a_1 - qp_1, \ldots, a_{d-1} - qp_{d-1}, -qp_d) \quad \left(q = -\left\lfloor \frac{a_0}{p_0} \right\rfloor\right) \end{aligned}$$

has the property $\tilde{T}_P \circ \Phi_C = \Phi_C \circ T_P$ (*cf.* [3]) and as

$$(w_0, \dots, w_{d-1}) = (1, X, \dots X^{d-1})V,$$

with $w_0, \dots, w_{d-1}$ as in (3.2.1) and

$$V = \begin{pmatrix} p_d & p_{d-1} & \cdots & \cdots & p_1 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & p_{d-1} \\ 0 & \cdots & \cdots & 0 & p_d \end{pmatrix},$$

we further have $V\tau_{\mathbf{r}}(\mathbf{z}) = \tilde{T}_P(V\mathbf{z})$ for all $\mathbf{z} \in \mathbb{Z}^d$ by Theorem 1.2.6.

We redefine the matrix B from Subsection 1.2.2 as

$$B := \begin{pmatrix} 0 & \cdots & \cdots & 0 & -\frac{p_0}{p_d} \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -\frac{p_{d-1}}{p_d} \end{pmatrix}.$$

It is easy to see that for an element $P(X) \in \mathcal{R}$ the multiplication with X commutes with the embedding Φ_C as

$$\Phi_2(XA) = B\Phi_2(A).$$

Finally observe the following lemma taken from [20]:

Lemma 3.2.13. *The companion matrix*

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{d-2} & -c_{n-1} \end{pmatrix} \in \mathbb{R}^{n \times n}$$

is regular if and only if $c_0 \neq 0$. *Then*

$$C^{-1} = \begin{pmatrix} -\frac{c_{n-1}}{c_0} & -\frac{c_{n-2}}{c_0} & \cdots & -\frac{c_1}{c_0} & -\frac{1}{c_0} \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Hence, as $R(\mathbf{r})$ is such a regular matrix, its inverse given by

$$R(\mathbf{r})^{-1} = \begin{pmatrix} -\frac{p_{d-1}}{p_d} & -\frac{p_{d-2}}{p_d} & \cdots & -\frac{p_1}{p_d} & -\frac{p_0}{p_d} \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

which is an integer matrix since $p_d \in \{-1, 1\}$. Then easy matrix multiplication shows $BV = VR(\mathbf{r})^{-1}$.

Summing up all these things yields the diagram

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "\Phi_C"] \arrow[d, "\cdot X"] & \mathbb{Z}^d \arrow[d, "B"] & \mathbb{Z}^d \arrow[l, "V"'] \arrow[d, "R(\mathbf{r})^{-1}"] \\
\mathcal{R} \arrow[r, "\Phi_C"] \arrow[d, "T_P"] & \mathbb{Z}^d \arrow[d, "\tilde{T}_P"] & \mathbb{Z}^d \arrow[l, "V"'] \arrow[d, "\tau_{\mathbf{r}}"] \\
\mathcal{R} \arrow[r, "\Phi_C"] & \mathbb{Z}^d & \mathbb{Z}^d \arrow[l, "V"']
\end{tikzcd} \tag{3.2.2}$$

In the present form the diagram does not hold in general for non-monic P. Φ_C is only defined for $\mathcal{R}'$, the submodule of $\mathcal{R}$ generated by $1, X, \ldots, X^{d-1}$. The upper left part is only true for $A \in \mathcal{R}'$ such that $XA \in \mathcal{R}'$. Note that $XA \in \mathcal{R}'$ if and only if $B\Phi_C(A)$ is an integer vector. The upper right part of the diagram is just a matrix identity and holds for all rational (and even real) vectors. Lets consider the restriction of Diagram 3.2.2 to $\Lambda_P(\mathcal{R})$. Note that then Φ_C as well as V are not surjective any more. Actually $\Phi_C(\Lambda_P(\mathcal{R})) = V\mathbb{Z}^d \subset \mathbb{Z}^d$, where the inclusion is proper, and $V^{-1} \circ \Phi_C$ maps $\Lambda_P(\mathcal{R})$ onto $\mathbb{Z}^d$ bijectively. Since $\Lambda_P(\mathcal{R})$ is closed under application of T_P the lower part of the diagram holds for all $A \in \Lambda_P(\mathcal{R})$.

We are now in the position to make some assertions. At first it is easy to see that T_P, $\tilde{T}_P$ and $\tau_{\mathbf{r}}$ are conjugate mappings. We also have a simple formula to obtain the X-ary representation of some $A \in \mathcal{R}$ using $\tau_{\mathbf{r}}$.

Theorem 3.2.14. *Let $P(x) = p_d x^d + \ldots + p_1 x + p_0 \in \mathbb{Z}[x]$, $\mathcal{R} = \mathbb{Z}[x]/(P)$, $\mathcal{N} = \{0, \ldots, |p_0| - 1\}$ and $\mathbf{r} = \left(\frac{p_d}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right)$. For an $A \in \Lambda_P(\mathcal{R})$ denote by $X_P(A) = (a_i)_{i \in \mathbb{N}}$ the X-ary representation of A. Then we have*

$$X_P(A) = -p_0 (v_i)_{i \in \mathbb{N}^*}$$

where $(v_i)_{i \in \mathbb{N}^*}$ is the SRS representation of $\mathbf{z} := V^{-1}\Phi_C(A)$ with respect to $\mathbf{r}$.

Proof. From the definition of T_P and Φ_C we know that

$$\Phi_C\left(m_{\mathcal{N}}(T_P^i(A))\right) = \Phi_C\left(T_P^i(A) - XT_P^{i+1}(A)\right) = (a_i, 0, \ldots, 0). \tag{3.2.3}$$

Note that $T_P^i(A) \in \Lambda_P(\mathcal{R}) \subset \mathcal{R}'$. We claim that $XT_P^{i+1}(A) \in \mathcal{R}'$ for $i \geq 0$. By the discussion from above this is equivalent to $B\Phi_C\left(T_P^{i+1}(A)\right) \in \mathbb{Z}^d$. But $\Phi_C\left(T_P^{i+1}(A)\right) = \tilde{T}_P^{i+1}(\Phi_C(A))$, as the lower part of Diagram 3.2.2 holds for the elements of $\Lambda_P(\mathcal{R})$, and by the definition of $\tilde{T}_P$ we see that the last entry of $\tilde{T}_P^{i+1}(\Phi_C(A))$ is a multiple of p_d. Hence $B\Phi_C\left(T_P^{i+1}(A)\right) \in \mathbb{Z}^d$ and $XT_P^{i+1}(A) \in \mathcal{R}'$.

Now Diagram 3.2.2 shows that (3.2.3) can be written as

$$(a_i, 0, \ldots, 0) = V\left(\tau_{\mathbf{r}}^i(\mathbf{z}) - R(\mathbf{r})^{-1}\tau_{\mathbf{r}}^{i+1}(\mathbf{z})\right).$$

As

$$\begin{aligned}
\tau_{\mathbf{r}}^{i+1}(\mathbf{z}) =& \tau_{\mathbf{r}}(\tau_{\mathbf{r}}^i(\mathbf{z})) = R(\mathbf{r})\tau_{\mathbf{r}}^i(\mathbf{z}) + (0, \ldots, 0, \mathbf{r}\tau_{\mathbf{r}}^i(\mathbf{z}) - \lfloor \mathbf{r}\tau_{\mathbf{r}}^i(\mathbf{z}) \rfloor) \\
=& R(\mathbf{r})\tau_{\mathbf{r}}^i(\mathbf{z}) + (0, \ldots, 0, v_i)
\end{aligned}$$

we easily get

$$\tau_{\mathbf{r}}^i(\mathbf{z}) - R(\mathbf{r})^{-1}\tau_{\mathbf{r}}^{i+1}(\mathbf{z}) = R(\mathbf{r})^{-1}(0, \ldots, 0, v_i) = \left(-\frac{p_0}{p_d}v_i, 0, \ldots, 0\right).$$

As a_i is the first entry of

$$V\left(-\frac{p_0}{p_d}v_i, 0, \ldots, 0\right)$$

we see $a_i = -p_0 v_i$ as it was claimed. □

3.2.3 The relation between tiles associated to expanding polynomials and SRS tiles

In Subsection 1.2.2 we defined tiles associated to (monic) expanding polynomials. By the previously shown relations between X-ary representation and the SRS representation one expects that the corresponding tiles also are connected. In the following suppose $P(x) = x^d + \ldots + p_1 x + p_0 \in \mathbb{Z}[x]$, $p_0 \geq 2$, an expanding monic polynomial, $\mathcal{R} = \mathbb{Z}[x]/(P)$ and $\mathcal{N} = \{0, \ldots, |p_0| - 1\}$. Set

$$E_h := \left\{ A \in \mathcal{R} \;\middle|\; A = \sum_{i=0}^{h} c_i X^i, c_i \in \mathcal{N} \right\}.$$

Then it is easy to see that

$$\mathcal{F} = \overline{\bigcup_{h \geq 1} B^{-h-1} \Phi_C(E_h)}.$$

Theorem 3.2.15. *Let $\mathcal{F}$ be the fundamental domain associated to P. Moreover, let*

$$\mathbf{r} = (\frac{1}{p_0}, \frac{1}{p_0}, \ldots, \frac{a_1}{p_0})$$

be the associated element of $\mathrm{int}(\mathcal{D}_d)$. *Then*

$$\mathcal{F} = V \mathcal{T}_{\mathbf{r}}(\mathbf{0}).$$

Proof. The element $\mathbf{r}$ is easily seen to be contained in $\mathrm{int}(\mathcal{D}_d)$ because P is an expanding polynomial. Indeed, the reciprocal polynomial of P is the minimal polynomial of $R(\mathbf{r})$.

Since $E_h \subset E_{h+1}$ and $B^{-h-1} E_h \subset B^{-h-2} E_{h+1}$ we have

$$\mathcal{F} = \lim_{h \to \infty} B^{-h-1} \Phi_C(E_h),$$

the limit with respect to $\gamma(\cdot, \cdot)_{R(\mathbf{r}), \delta}$.

By our Diagram (3.2.2) we immediately see that

$$E_h = \{ A \in \mathcal{R} \mid T_P^{h+1}(A) = 0 \}$$

and $\Phi_C(\mathcal{E}_h) = V \mathcal{T}_{\mathbf{r},h+1}(\mathbf{0})$. Furthermore, since $B^{-h-1} = V R(\mathbf{r})^{h+1} V^{-1}$, we have $B^{-h-1} \Phi_C(E_h) = V R(\mathbf{r})^{h+1} \mathcal{T}_{\mathbf{r},h+1}(\mathbf{0})$. Thus

$$\mathcal{F} = \lim_{h \to \infty} B^{-h-1} \Phi_C(E_h) = \lim_{h \to \infty} V R(\mathbf{r})^{h+1} \mathcal{T}_{\mathbf{r},h+1}(\mathbf{0}) = V \mathcal{T}_{\mathbf{r}}(\mathbf{0}).$$

□

Corollary 3.2.16. *With the definitions of Theorem 3.2.15 we have*

$$\mathcal{T}_{\mathbf{r}}(\mathbf{z}) = V^{-1} \mathcal{F} + \mathbf{z}.$$

Proof. Since

$$\{ A \in \mathcal{R} \mid T_P^{h+1}(A) = \Phi_C^{-1}(V\mathbf{z}) \} = E_h + X^{h+1} \Phi_C^{-1}(V\mathbf{z})$$

we see that $B^{-h-1} \Phi_C(E_h + X^{h+1} \Phi_C^{-1}(V\mathbf{z})) = V R(\mathbf{r})^{h+1} (\mathcal{T}_{\mathbf{r},h+1}(\mathbf{0})) + \mathbf{z}$. The rest can be proven in the same way as Theorem 3.2.15. □

Example 3.2.17. Consider the expanding polynomial $P(x) = x^2 - x + 2$. Since $\mathbf{r} = \left(\frac{1}{2}, -\frac{1}{2}\right) \in \mathcal{D}_2^0$ (as it can easily be seen from Figure 1.3) we have that $(P, \mathcal{N})$ with $\mathcal{N} = \{0, 1\}$ is a CNS. The central SRS tile $\mathcal{T}_{\mathbf{r}}(0)$ associated to $\mathbf{r}$ is shown in Figure 3.3 left. In order to obtain the CNS tile $\mathcal{F}$ associated to $(P, \mathcal{N})$ we have to multiply $\mathcal{T}_{\mathbf{r}}(\mathbf{0})$ with the Matrix

$$V = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

$\mathcal{F}$ is shown in Figure 3.3 right.

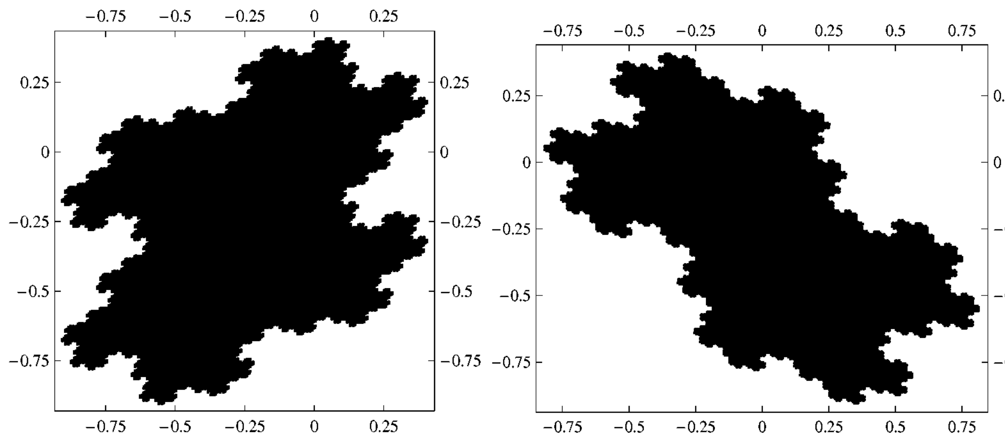


Figure 3.3: The central SRS tile $T_{\mathbf{r}}(\mathbf{0})$ for $\mathbf{r} = \left(\frac{1}{2}, -\frac{1}{2}\right)$ (left) and the CNS tile $\mathcal{F}$ associated to $(P, \mathcal{N})$ (right).

Consider a non-monic expanding polynomial $P(x) = p_d x^d + \cdots + p_1 x + p_0$ and set $\mathbf{r} := \left(\frac{p_d}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right)$. We have $\mathbf{r} \in \text{int}\,(\mathcal{D}_d)$. In Theorem 3.2.12 we showed the relation between T_P and the mapping $\tau_{\mathbf{r}}$. Thus we may ask for the generalisation of CNS-tiles to our non-monic P such that these tiles are related to the SRS-tiles associated to $\mathbf{r}$ in a similar way as in Theorem 3.2.15. For $A \in \Lambda_P$ define

$$\tilde{E}_h(A) = \left\{B \in \Lambda_P \mid T_P^{h+1}(B) = A\right\}$$

and

$$\tilde{\mathcal{F}}(A) = \lim_{h \to \infty} B^{-h-1} \Phi_C(\tilde{E}_h(A)).$$

In the same way as in Theorem 3.2.15 one easily proves the identity

$$\tilde{\mathcal{F}}(A) = V T_{\mathbf{r}}(V^{-1} \Phi_C(A))$$

for any $A \in \Lambda_P$ by observing Diagram (3.2.2) and the remarks afterwards. This shows the tiles $\tilde{\mathcal{F}}(A)$ to cover the d-dimensional Euclidean space. Since obviously neither (1.2.4) nor Corollary 3.2.16 hold we cannot expect two different tiles to differ only by translation. This expectation is confirmed by the next example.

Example 3.2.18. Let $P(x) = 2x^2 + x + 8$. P is an expanding polynomial and the corresponding $\mathbf{r} := \left(\frac{1}{4}, \frac{1}{8}\right)$ induces an SRS (as it can easily seen in Figure 1.3). Thus $(P, \mathcal{N})$ with $\mathcal{N} = \{0, \ldots, 7\}$ is a CNS. The left picture of Figure 3.4 shows the SRS-tiles $T_{\mathbf{r}}(\mathbf{x})$ associated to $\mathbf{r}$ for $\|\mathbf{x}\|_\infty \leq \mathbf{1}$. On the right the corresponding tiles $\tilde{\mathcal{F}}(A)$ are depicted. One can see that the tiles partly look very similar but they all have different shapes.

Up to now these generalised CNS-tiles have not been investigated. But it seems to be difficult to analyse them. Note that the SRS-tiles presented in Example 3.1.14 corresponds to tiles associated to the non-monic polynomial $20x^2 - 11x + 18$. Thus not even all of these tiles have a nonempty interior in general.

3.3 β-expansions

3.3.1 The relation between SRS representation and β-expansion

In this subsection let $\beta > 1$ be an algebraic integer with minimal polynomial $P(x) = x^{d+1} + p_d x^d + \cdots + p_1 x + p_0$. We will carefully analyse how SRS and β-expansions are related. A first connection

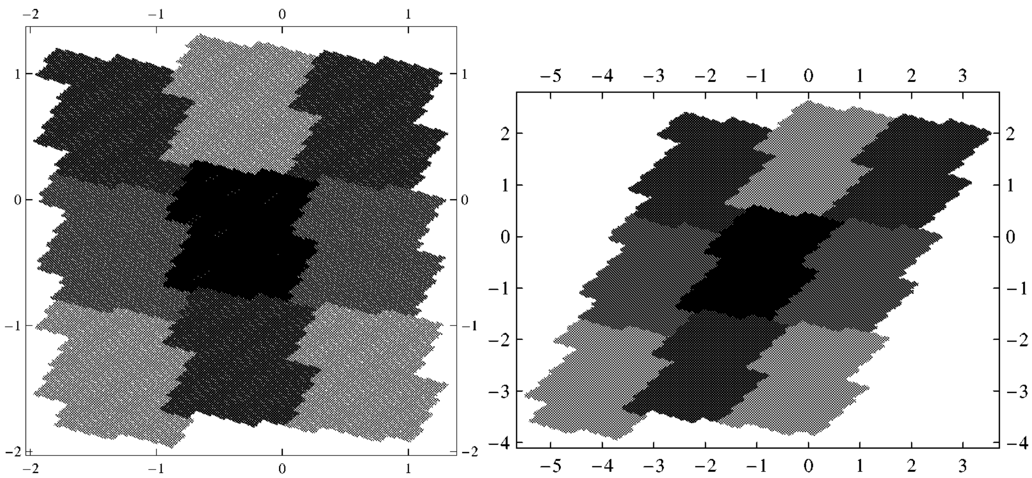


Figure 3.4: SRS-tiles for $\left(\frac{1}{4}, \frac{1}{8}\right)$ (left) and the corresponding tiles associated to the non-monic polynomial P (right).

was already given in Theorem 1.3.5. For a better understanding we will state this theorem again here and give the proof of it. Set

$$\mathbf{r} = (r_0, \dots, r_{d-1}) \quad \text{with} \quad \begin{array}{ll} r_d & = 1, \\ r_j & = a_{j+1} + \beta r_{j+1} \qquad (0 \le j \le d-1). \end{array} \tag{3.3.1}$$

Theorem 3.3.1 (*cf.* [3, Theorem 2.1] (*cf.* also [29])). *For an algebraic number $\beta > 1$ and $\mathbf{r}$ defined as in* (3.3.1) *we have $\mathbf{r} \in \mathcal{D}_d^0$ if and only if β has property (F).*

Proof. Let $\gamma \in \mathbb{Z}\left[\beta^{-1}\right] \cap [0,1)$. By the definition of T_β we have $T_\beta^k(\gamma) = \beta^k\gamma + \sum_{i=0}^{k-1} a_i\beta^i$ with integers a_i. We can choose $k \in \mathbb{N}$ such that $\beta^k\gamma \in \mathbb{Z}[\beta]$. Then $T_\beta^k(\gamma) = \beta^k\gamma + \sum_{i=0}^{k-1} a_i\beta^i \in \mathbb{Z}[\beta] \cap [0,1)$. Therefore we can restrict to $\gamma \in \mathbb{Z}[\beta] \cap [0,1)$ and it suffices to show that $\mathbf{r} \in \mathcal{D}_d^0$ if and only if $d_\beta(\gamma)$ is finite for each $\gamma \in \mathbb{Z}[\beta] \cap [0,1)$.

Now let $\gamma \in \mathbb{Z}[\beta] \cap [0,1)$. Consider the reals $r_0, \dots, r_d$ defined in (3.3.1). We see that γ can be written as

$$\gamma = \sum_{i=0}^{d} z_i r_i \text{ with } z_i \in \mathbb{Z}$$

($\mathbb{Z}[\beta]$ is a $\mathbb{Z}$-module and $r_0, \dots, r_d$ is a basis of it). An application of T_β to γ yields

$$\begin{aligned} T_\beta(\gamma) = \beta\gamma - \lfloor \beta\gamma \rfloor &= \beta \sum_{i=0}^{d} z_i r_i - \left\lfloor \beta \sum_{i=0}^{d} z_i r_i \right\rfloor \\ &= \sum_{i=0}^{d-1} z_{i+1} r_i - \sum_{i=0}^{d} a_i - \left\lfloor \sum_{i=0}^{d-1} z_{i+1} r_i - \sum_{i=0}^{d} a_i \right\rfloor \\ &= \sum_{i=0}^{d} z_{i+1} r_i \end{aligned}$$

by observing (3.3.1) and setting $z_{d+1} := -\left\lfloor \sum_{i=0}^{d-1} z_{i+1} r_i \right\rfloor$. We immediately see that $(z_2, \dots, z_{d+1}) = \tau_{\mathbf{r}}(z_1, \dots, z_d)$. The assertion now follows directly from the fact that $d_\beta(\gamma)$ is finite if and only if $T_\beta^n(\gamma) = 0$ for some $n \in \mathbb{N}$. $\square$

Let $\mathbb{Z}[\beta]/1$ be the factorisation modulo 1 of $\mathbb{Z}[\beta]$ with $\pi : \mathbb{Z}[\beta] \to \mathbb{Z}[\beta]/1$ the projection. Note that the projection restricted to $\mathbb{Z}[\beta] \cap [0,1)$ is bijective. Define $\Psi : \mathbb{Z}[\beta]/1 \to \mathbb{Z}^d$ to be the unique representation of an element $\gamma' \in \mathbb{Z}[\beta]/1$ in the basis $\{r_0, \ldots, r_{d-1}\}$ such that $\gamma' = \mathbf{r}\Psi(\gamma')$. Due to Theorem 3.3.1 the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z}[\beta] \cap [0,1) \arrow[r, "\pi"'] \arrow[d, "T_\beta"] & \mathbb{Z}[\beta]/1 \arrow[r, "\Psi"'] & \mathbb{Z}^d \arrow[d, "\tau_{\mathbf{r}}"] \\
\mathbb{Z}[\beta] \cap [0,1) \arrow[r, "\pi"'] & \mathbb{Z}[\beta]/1 \arrow[r, "\Psi"'] & \mathbb{Z}^d
\end{tikzcd}$$

Thus the dynamical systems T_β and $\tau_{\mathbf{r}}$ are conjugate. Let

$$\begin{aligned} f : \mathbb{Z}^d &\to \mathbb{Z}[\beta] \cap [0,1), \\ \mathbf{x} &\mapsto \mathbf{r}\mathbf{x} - \lfloor \mathbf{r}\mathbf{x} \rfloor . \end{aligned}$$

Proposition 3.3.2. *f is bijective.*

Proof. **surjectivity** Choose an arbitrary $\gamma \in \mathbb{Z}[\beta] \cap [0,1)$. γ can be represented in the basis $\{r_0, \ldots, r_{d-1}, 1\}$ as

$$\gamma = x_0 r_0 + \cdots + r_{d-1} x_{d-1} + x_d$$

with $x_0, \ldots, x_{d-1}, x_d \in \mathbb{Z}$. Since $\gamma \in [0,1)$ we have

$$x_d = -\lfloor x_0 r_0 + \cdots + r_{d-1} x_{d-1} \rfloor$$

and therefore

$$\gamma = f((x_0, \ldots, x_{d-1})).$$

injectivity Suppose that $f(\mathbf{x}) = f(\mathbf{y})$, *i.e.*,

$$f(\mathbf{x}) = x_0 r_0 + \cdots + x_{d-1} r_{d-1} - \lfloor \mathbf{r}\mathbf{x} \rfloor = y_0 r_0 + \cdots + y_{d-1} r_{d-1} - \lfloor \mathbf{r}\mathbf{y} \rfloor = f(\mathbf{y}).$$

The fact that $\{r_0, \ldots, r_{d-1}, 1\}$ forms a basis of $\mathbb{Z}[\beta]$ implies $x_0 = y_0, \ldots, x_{d-1} = y_{d-1}$. □

This shows that f is the inverse of $\Psi \circ \pi$ and we obtain the following commutative diagram.

$$\begin{tikzcd}
\mathbb{Z}[\beta] \cap [0,1) \arrow[r, "T_\beta"] \arrow[d, "\Psi\circ\pi"'] & \mathbb{Z}[\beta] \cap [0,1) \\
\mathbb{Z}^d \arrow[r, "\tau_{\mathbf{r}}"] & \mathbb{Z}^d \arrow[u, "f"']
\end{tikzcd} \tag{3.3.2}$$

Hence, in the following lemma we are able to present a method to calculate the β-expansion of some element of $\mathbb{Z}[\beta] \cap [0,1)$ by using $\tau_{\mathbf{r}}$.

Lemma 3.3.3. *Let $\beta > 1$ be an algebraic integer with minimal polynomial $A(x) = x^{d+1} + p_d x^d + \cdots + p_1 x + p_0$ and let $\gamma \in \mathbb{Z}[\beta] \cap [0,1)$ having*

$$\gamma = \sum_{i=1}^{\infty} b_i \beta^{-i}$$

as β-expansion. Then, for $\mathbf{z} := \Psi \circ \pi(\gamma)$ and $\mathbf{r}$ defined as in (3.3.1), we have

$$b_i = \beta(\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{z}) - \lfloor \mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{z}) \rfloor) - \mathbf{r}\tau_{\mathbf{r}}^{i}(\mathbf{z}) + \lfloor \mathbf{r}\tau_{\mathbf{r}}^{i}(\mathbf{z}) \rfloor = \beta f(\tau_{\mathbf{r}}^{i-1}(\mathbf{z})) - f(\tau_{\mathbf{r}}^{i}(\mathbf{z})).$$

Proof. We have $b_i = \left\lfloor \beta T_\beta^{i-1}(\gamma) \right\rfloor$ by (1.3.2). Thus the diagram in (3.3.2) immediately yields the desired result. □

If β is a unit (which implies that $\mathbb{Z}[\beta] = \mathbb{Z}\left[\beta^{-1}\right]$) then, starting from the SRS, we get the β-expansion of each $\gamma \in \mathbb{Z}[\beta] = \mathbb{Z}\left[\beta^{-1}\right]$.

Now we compare the SRS-representation of some $\mathbf{z}$ with the β-expansion of $f(\mathbf{z})$.

Theorem 3.3.4. *Let $\beta > 0$ be an algebraic integer with minimal polynomial $A(x) = x^{d+1} + a_d x^d + \cdots + a_1 x + a_0$ and $\mathbf{r}$ defined as in (3.3.1). Furthermore, let*

$$(v_n, v_{n+1}, \ldots)$$

the SRS representation of $\mathbf{z}' \in \mathbb{Z}^d$,

$$(v_1, v_2, \ldots, v_n, v_{n+1}, \ldots)$$

the SRS representation of $\mathbf{z} \in \mathbb{Z}^d$ and

$$\gamma = \sum_{i=1}^{\infty} b_i \beta^{-i}$$

the β expansion of $\gamma := f(\mathbf{z})$. Then

1. $\gamma = -v_1$
2. $b_i = -\beta v_i + v_{i+1}$ *for* $1 \le i \le n$ *with* $v_{n+1} = f(\mathbf{z}')$
3. $f(\mathbf{z}') = \sum_{i=n}^{\infty} b_i \beta^{-i+n-1}$

Proof. By Definition 3.1.1 we see that $(0, \ldots, 0, v_i) = R(\mathbf{r})\tau_{\mathbf{r}}^{i-1}(\mathbf{z}) - \tau_{\mathbf{r}}^{i}(\mathbf{z})$ and therefore $v_i = f(\tau_{\mathbf{r}}^{i-1}(\mathbf{z}))$. Together with Lemma 3.3.3 this immediately shows items 1. and 2. To show item 3. observe that by item 2. of Lemma 3.1.2 we have $\mathbf{z}' = \tau_{\mathbf{r}}^{n}(\mathbf{z})$. Then, by the diagram in (3.3.2), we have $f(\mathbf{z}') = T_\beta(\gamma) = \sum_{i=n}^{\infty} b_i \beta^{-i+n-1}$. □

Item 2. of this theorem shows that $(b_n)_{n\in\mathbb{N}} = Y_{-\beta}\left((v_n)_{n\in\mathbb{N}}\right)$ with Y defined as in (2.1.1). Thus the identity $Y_\beta(X_{\mathbf{r}}(\mathbf{x})) = d_\beta(f(\mathbf{x}))$ holds. Item 3. induces the greedy condition since $0 \le f(\mathbf{z}') < 1$.

Corollary 3.3.5. *Let β and $\mathbf{r}$ as in Theorem 3.3.4 and $(v_1, v_2, v_3, \ldots)$ the SRS representation of $(0, \ldots, 0, 1)$. Then $d_\beta(1) = V_{-\beta}\left((-1, v_1, v_2, v_3, \ldots)\right)$.*

Proof. We have $T_\beta(1) = \beta - \lfloor\beta\rfloor$ and by the definition of r_{d-1} we get $f((0, \ldots, 1)) = a_d + \beta - \lfloor a_d + \beta\rfloor = \beta - \lfloor\beta\rfloor$. Thus, by Theorem 3.3.4, $(t_2, t_3, \ldots) := d_\beta(\beta - \lfloor\beta\rfloor) = Y_{-\beta}\left((v_1, v_2, v_3, \ldots)\right)$. We obviously have $d_\beta(1) = (\lfloor\beta\rfloor, t_2, t_3, \ldots)$. Since $v_1 = -\beta + \lfloor\beta\rfloor$ by Theorem 3.3.4 we obtain $-\beta(-1) + v1 = \lfloor\beta\rfloor$ and thus $d_\beta(1) = Y_{-\beta}\left((-1, v_1, v_2, v_3, \ldots)\right)$. □

3.3.2 Parry numbers

Motivated by the close relation between the β-expansions and SRS and by the observation that the finiteness of $d_\beta(1)$ for an algebraic number β is equivalent to the finiteness of $X_{\mathbf{r}}\big((0, \ldots, 0, 1)\big)$ for $\mathbf{r}$ as in (3.3.1) we will try to give an analogue to Theorem 1.3.8 (*cf.* [14]) for shift radix systems. For $d > 0$ define the set

$$B_d := \{\mathbf{r} \in \mathcal{D}_d \,|\, \exists k \in \mathbb{N} : \tau_{\mathbf{r}}^{k}((0, \ldots, 0, 1)) = \mathbf{0}\}.$$

Of course, $\mathcal{D}_d^0 \subset B_d \subset \mathcal{D}_d$. It is easy to see that $B_1 = [0, 1)$. The characterisation of B_2 is less obvious. We partition $\mathcal{E}_2$ into several subsets and treat them separately. An overview of the locations of these subsets is depicted on the left hand side of Figure 3.6.

Lemma 3.3.6.

$$A_1 := \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x < 1, 0 \le y < 1\right\} \subset B_2.$$

Proof. For an $\mathbf{r} \in A_1$ it is easy to see that $\tau_{\mathbf{r}}((0, 1)) = (1, 0)$ and $\tau_{\mathbf{r}}((1, 0)) = (0, 0)$. □

Lemma 3.3.7.
$$A_2 := \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1, 1 \leq y < x+1\} \subset B_2.$$

Proof. Let $\mathbf{r} = (x,y) \in A_2$. Since $y < x+1 < 2$ we have $\tau_{\mathbf{r}}((0,1)) = (1,-1)$. From $-1 < x-y < 0$ and $0 < -x+y < 1$ we obtain $\tau_{\mathbf{r}}((1,-1)) = (-1, -\lfloor x-y \rfloor) = (-1,1)$, $\tau_{\mathbf{r}}((-1,1)) = (1, -\lfloor -x+y \rfloor) = (1,0)$ and $\tau_{\mathbf{r}}(1,0) = (0,0)$. □

Lemma 3.3.8.
$$A_3 := \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1, -x \leq y < 0\} \subset B_2.$$

Proof. For an $\mathbf{r} \in A_3$ we have $\tau_{\mathbf{r}}((0,1)) = (1,1)$, $\tau_{\mathbf{r}}((1,1)) = (1,0)$ and $\tau_{\mathbf{r}}((1,0)) = (0,0)$. □

Lemma 3.3.9. *Let*
$$A_4 := \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1, -1 \leq y < -x\}.$$
Then $A_4 \cap B_2 = \emptyset$.

Proof. Let $\mathbf{r} \in A_4$. Then $\tau_{\mathbf{r}}((0,1)) = (1,1)$, $\tau_{\mathbf{r}}((1,1)) = (1,1)$ and thus we end up periodically. □

Lemma 3.3.10. *Let*
$$A_5 := \{(x,y) \in \mathbb{R}^2 \,|\, -1 \leq x < 0, -x-1 \leq y < 0\}.$$
Then $A_5 \cap B_2 = \emptyset$.

Proof. For an $\mathbf{r} \in A_5$ we have $\tau_{\mathbf{r}}((0,1)) = (1,1)$ and $\tau_{\mathbf{r}}((1,1)) = (1,1)$. □

Lemma 3.3.11. *Let*
$$A_6 := \{(x,y) \in \mathbb{R}^2 \,|\, -1 \leq x < 0, 0 \leq y \leq x+1\}.$$
Then $A_6 \cap B_2 = \emptyset$.

Proof. $\tau_{\mathbf{r}}((0,1)) = (1,0)$ and $\tau_{\mathbf{r}}((1,0)) = (0,1)$ for all $\mathbf{r} \in A_6$. □

Before we turn to the last big subset we investigate the reminding parts of the boundary, *i.e.*, the lines L_1, L_4 and L_5 (see (1.1.4)).

Lemma 3.3.12. $(L_1 \cup L_4 \cup L_5) \cap B_2 = \emptyset$.
$$(\{(x,x+1) \in \mathbb{R}^2 \,|\, 0 \leq x < 1\} \cup \{(1,y) \,|\, -2 \leq x \leq 2\}) \cap B_2 = \emptyset$$

Proof. For an $\mathbf{r} = (x,y) \in L_1$ we have $\tau_{\mathbf{r}}(0,1) = (1,-1), \tau_{\mathbf{r}}(1,-1) = (-1,1)$ and $\tau_{\mathbf{r}}(1,-1) = (1,-1)$. Let $\mathbf{r} = (1,y) \in L_4$. Suppose that there exists a $y \in (-2,2)$ such that $(1,y) \in B_2$. There must exist an $n \in \mathbb{N}$ such that $\tau^n_{(1,y)}((0,1)) = (0,0)$. Assume that n is minimal in that sense that $\tau^{n-1}_{(1,y)}((0,1)) \neq (0,0)$. By the definition of the function $\tau_{\mathbf{r}}$ we must have $\tau^{n-1}_{(1,y)}((0,1)) = (a,0)$ for some integer $a \neq 0$. But it is easy to see that the relation $\tau_{(1,y)}((a,0)) = (0,0)$ can impossibly hold. For $\mathbf{r} \in L_5$ it can easily be shown that $\left(\tau^n_{\mathbf{r}}((0,1))\right)_{n \in \mathbb{N}}$ is not even periodic. □

The last subset
$$U := \{(x,y) \in \mathbb{R}^2 | \, 0 < x < 1, -x-1 < y < -1\} \tag{3.3.3}$$
is a little more difficult to analyse. For $k \geq 1$ define
$$U_k = \{(x,y) \in \mathbb{R}^2 | \, x(k-1) + yk < -k, xk + y(k+1) \geq -k-1, 0 < x < 1\}.$$

Lemma 3.3.13. *For the sets U_k the following properties hold:*

1. $U_i \cap U_j = \emptyset$ *for* $i \neq j$.

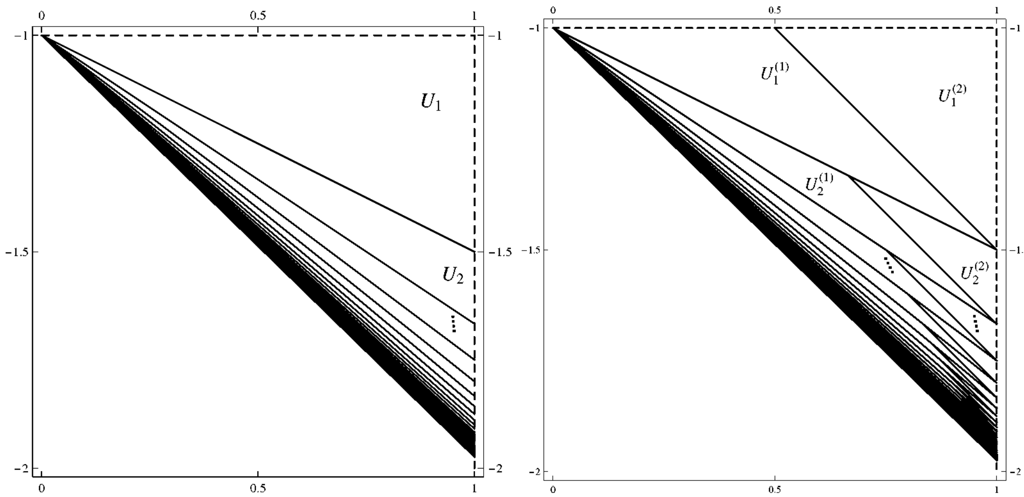


Figure 3.5: The partition of the set U

2. $\bigcup_{i\geq 1} U_i = U$.
3. *Let* $k \geq 1$. *For all points* $(x,y) \in \bigcup_{i\geq k} U_i$ *we have* $x(k-1)+yk < -k$.

Proof. The proof is trivial by observing the left hand side of Figure 3.5. $\square$

Lemma 3.3.14. $\tau_{(x,y)}\left((k-1,k)\right) = (k,k+1)$ *for* $(x,y) \in \bigcup_{i\geq k} U_i$.

Proof. Estimating the expression $(k-1)x+ky$ yields

$$(k-1)x+ky > -k-x > -k-1$$

for all $(x,y) \in U$ by (3.3.3) and

$$(k-1)x+ky < -k$$

by item 3. of Lemma 3.3.13 for all $(x,y) \in \bigcup_{i\geq k} U_i$. Thus $\lfloor (k-1)x+ky \rfloor = -k-1$ and $\tau_{(x,y)}\left((k-1,k)\right) = (k,k+1)$ for all $(x,y) \in \bigcup_{i\geq k} U_i$. $\square$

Now divide each U_k into two disjoint sets $U_k^{(1)}$ and $U_k^{(2)}$ with

$$\begin{aligned} U_k^{(1)} :=& \{(x,y) \in U_k \mid (k+1)x+(k+1)y < -k\}, \\ U_k^{(2)} :=& \{(x,y) \in U_k \mid (k+1)x+(k+1)y \geq -k\}. \end{aligned}$$

The situation is depicted in Figure 3.5 right.

Lemma 3.3.15. $(x,y) \in U_k^{(2)}$ *satisfy* $2x+y>0$ *for all* $k \geq 1$.

Proof. A point $(x,y) \in U_k^{(2)}$ satisfies $(k+1)x+(k+1)y \geq -k$ as well as $x(k-1)+yk < -k$. Multiplying the second inequality with -1 and adding the result to the first inequality gives $2x+y>0$. $\square$

Lemma 3.3.16. $\tau_{(x,y)}\left((k+1,k)\right) = (k,k-1)$ *for* $(x,y) \in \bigcup_{i\geq k} U_i^{(2)}$.

Proof. Similar as in Lemma 3.3.14 we have

$$(k+1)x+ky > (k-1)x+(k-1)y > -k+1.$$

for all $(x,y) \in \bigcup_{i\geq 1} U_i^{(2)}$ by Lemma 3.3.15 and (3.3.3). An upper estimation can be obtained by using (3) of Lemma 3.3.13 and (3.3.3):

$$(k+1)x + ky \leq -k + 2x < -k + 2 \quad \forall (x,y) \in \bigcup_{i\geq k} U_i.$$

Hence $\lfloor (k+1)x + ky \rfloor = -k+1$ and $\tau_{(x,y)}((k+1,k)) = (k,k-1)$ for all $(x,y) \in \bigcup_{i\geq k} U_i^{(2)}$. □

Lemma 3.3.17. *For all $k > 0$ we have $U_k^{(1)} \cap B_2 = \emptyset$ and $U_k^{(2)} \subset B_2$.*

Proof. Let $(x,y) \in U_k$. By Lemma 3.3.14 we have $\tau_{(x,y)}^k((0,1)) = (k,k+1)$. Another application of $\tau_{(x,y)}$ yields $\tau_{(x,y)}^k((0,1)) = (k+1,k+1)$ by the estimate $-k-1 \leq kx + (k+1)y < -k+x+y < -k$ obtained by the definition of U_k. If $(x,y) \in U_k^{(1)}$ we have

$$-k-1 < -k-1+x \geq x(k+1) + y(k+1) < -k$$

by the definitions of U_k, $U_k^{(1)}$ and U and thus $\tau_{(x,y)}^{k+1}((0,1)) = \tau_{(x,y)}^{k+2}((0,1)) = (k+1,k+1)$ shows the periodicity. If $(x,y) \in U_k^{(2)}$ we have

$$-k \leq x(k+1) + y(k+1) < -k + 2x + y < -k + 1$$

by the Definitions of U_k, $U_k^{(2)}$ and U and hence $\tau_{(x,y)}^{k+2}((0,1)) = (k+1,k)$. An application of Lemma 3.3.16 shows $\tau_{(x,y)}^{2k+2}((0,1)) = (1,0)$ and since $0 < x < 1$ we have $\tau_{(x,y)}^{2k+3}((0,1)) = (0,0)$. □

Hence Lemmas 3.3.6 to 3.3.12 together with Lemma 3.3.17 provide a full characterisation of B_2. We summarise this in a Theorem.

Theorem 3.3.18.

$$B_2 = A_1 \cup A_2 \cup A_3 \cup \bigcup_{j\geq 1} U_k^{(2)}.$$

B_2 is depicted as grey area in Figure 3.6 right (the full lines of its boundary belong to the set, the dotted lines do not).

Let $\mathbf{r} \in \mathbb{R}^d$ and set

$$N(\mathbf{r}) = \left| \{ \tau_{\mathbf{r}}^n((0,\ldots,0,1)) | \, n \in \mathbb{N} \} \right|.$$

Proposition 3.3.19. *For a Parry number β and $\mathbf{r}$ as in 3.3.1 the Automaton 1.3.5 has $N(\mathbf{r})$ stages if β is simple, otherwise it has $N(\mathbf{r})+1$ stages.*

Proof. This is an immediate consequence of Corollary 3.3.5. □

Corollary 3.3.20. *Let β a Pisot number of degree 3. Then the Automaton 1.3.5 has 3, 5 or $2k$, $k \in \mathbb{N}_0$ stages if β is a simple Parry number. Otherwise it has k stages for some $k \geq 3$. All cases really occur.*

Proof. Let $\mathbf{r} \in \mathcal{D}_2$ as in 3.3.1. From Lemma 3.3.6 - Lemma 3.3.11 and Lemma 3.3.17 we easily obtain the length of the orbit $N(\mathbf{r})$. Then the first statement is a consequence of Proposition 3.3.19. Since the Pisot numbers are dense in $\mathcal{D}_d$, *i.e.*, for any open subset of $\mathcal{D}_d$ there exists a Pisot number β such that the corresponding $\mathbf{r}$ as in (3.3.1) lies in it, all the cases really occur. □

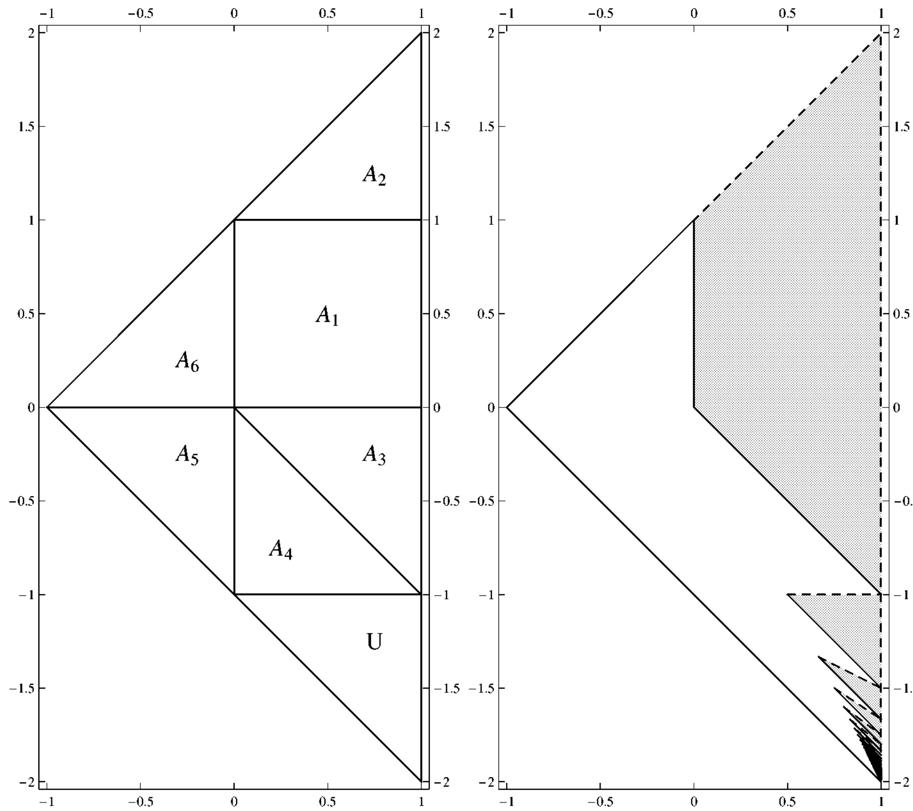


Figure 3.6: A partition of $\mathcal{D}_2$ (left) and the position of B_2 inside $\mathcal{D}_2$ (right)

3.3.3 The relation between new β-Tiles and SRS-Tiles

In the following assume β to be a Pisot number of degree $d+1$. We are going to investigate the relation between SRS tiles (see Definition 3.1.5) and new β-tiles (see Definition 1.3.12) generated by Pisot numbers. For convenience we state the notations again here.

Let $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$. For an $\mathbf{x} \in \mathbb{Z}^d$ the set

$$T_{\mathbf{r}}(\mathbf{x}) = \overline{\lim_{n\to\infty} R(\mathbf{r})^n T_{\mathbf{r},n}(\mathbf{x})},$$

where

$$T_{\mathbf{r},n}(\mathbf{x}) := \{\mathbf{z} \in \mathbb{Z}^d \mid \tau_{\mathbf{r}}^n(\mathbf{z}) = \mathbf{x}\},$$

is the SRS tile associated to $\mathbf{r}$. The limit is taken with respect to $\gamma(\cdot,\cdot)_{R(\mathbf{r}),\delta}$ for some δ with $\varrho(R(\mathbf{r})) < \delta < 1$. The tile $T_{\mathbf{r}}(\mathbf{0})$ is called the *central SRS tile.*

For the Pisot number β and $\omega \in \mathbb{Z}[\beta] \cap [0,1)$ let

$$\tilde{S}_{\beta,n}(\omega) := \{\gamma \in \mathbb{Z}[\beta] \cap [0,1) \mid T_\beta^n(\gamma) = \omega\}.$$

Then the set

$$\tilde{S}_\beta(\omega) := \overline{\lim_{n\to\infty} \Phi(\beta^n \tilde{S}_{\beta,n}(\omega))},$$

(again with respect to the Hausdorff metric) is called a *new β-tile.* The tile $S_\beta(0)$ will be called *central new β-tile.*

For preparation we use a lemma concerning companion matrices, which is a special case of a result which was presented in [15].

Lemma 3.3.21. *For a companion matrix*

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{d-2} & -c_{n-1} \end{pmatrix} \in \mathbb{R}^{n \times n}$$

with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, the left eigenvector $\mathbf{l}_j$ associated to the eigenvalue λ_j equals

$$\mathbf{l}_j = (1, \lambda_j, \ldots, \lambda_j^{d-1})H$$

with

$$H = \begin{pmatrix} c_1 & c_2 & \cdots & c_{n-1} & 1 \\ c_2 & & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ c_{n-1} & \iddots & \iddots & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Proof. See [15, Lemma 1] and observe that the transposed companion matrix is considered there. Moreover, we only need the special case where all eigenvalues have multiplicity 1. □

Set

$$Q_j(x) = \prod_{i=1, i\neq j}^{d} (x - \beta_i) = x^{d-1} + q_{d-2}^{(j)} x^{d-2} + \cdots + q_1^{(j)} x + q_0^{(j)}$$

and

$$U = \begin{pmatrix} q_0^{(1)} & q_1^{(1)} & \cdots & q_{d-2}^{(1)} & 1 \\ q_0^{(2)} & q_1^{(2)} & \cdots & q_{d-2}^{(2)} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ q_0^{(r)} & q_1^{(r)} & \cdots & q_{d-2}^{(r)} & 1 \\ \Re(q_0^{(r+1)}) & \Re(q_1^{(r+1)}) & \cdots & \Re(q_{d-2}^{(r+1)}) & 1 \\ \Im(q_0^{(r+1)}) & \Im(q_1^{(r+1)}) & \cdots & \Im(q_{d-2}^{(r+1)}) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \Re(q_0^{(r+s)}) & \Re(q_1^{(r+s)}) & \cdots & \Re(q_{d-2}^{(r+s)}) & 1 \\ \Im(q_0^{(r+s)}) & \Im(q_1^{(r+s)}) & \cdots & \Im(q_{d-2}^{(r+s)}) & 0 \end{pmatrix} \in \mathbb{R}^{d \times d}.$$

Theorem 3.3.22. *Let β be a Pisot number with minimal polynomial $x^{d+1} + p_d x^d + \cdots + p_1 x + p_0$ and $\mathbf{r} := (r_0, \ldots, r_{d-1}) \in \operatorname{int}(\mathcal{D}_d)$ defined as in (3.3.1). Then $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$ and for each $\mathbf{x} \in \mathbb{Z}^d$ and $\omega = \mathbf{x}\mathbf{r} - \lfloor \mathbf{x}\mathbf{r} \rfloor$ we have*

$$\tilde{S}_\beta(\omega) = U(R(\mathbf{r}) - \beta I_d) T_{\mathbf{r}}(\mathbf{x})$$

where I_d is the d-dimensional identity matrix.

Proof. For convenience set $r_d := 1$. First note that all Galois conjugates of β are less than 1, *i.e.*, the characteristic polynomial $(x^d + r_{d-1}x^{d-1} + \cdots + r_0)$ of $R(\mathbf{r})$ has all roots inside the unit circle. Thus $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$.

Consider the function f and Diagram 3.3.2 from Subsection 3.3.1. From there it is easy to see that

$$\tilde{S}_{\beta,n}(\omega) = f(T_{\mathbf{r},n}(\mathbf{x})).$$

Set

$$\begin{aligned} g: \mathbb{Z}^d &\to \mathbb{R}^d, \\ \mathbf{z} &\mapsto -\beta\mathbf{z} + \tau_{\mathbf{r}}(\mathbf{z}). \end{aligned}$$

and

$$\Lambda_\beta = \mathrm{diag}\left(\beta_1, \ldots, \beta_r, \begin{pmatrix} \Re(\beta_{r+1}) & \Im(\beta_{r+1}) \\ -\Im(\beta_{r+1}) & \Re(\beta_{r+1}) \end{pmatrix}, \ldots, \begin{pmatrix} \Re(\beta_{r+s}) & \Im(\beta_{r+s}) \\ -\Im(\beta_{r+s}) & \Re(\beta_{r+s}) \end{pmatrix}\right),$$

We will now prove the validity of the following diagram:

$$\begin{tikzcd} \mathbb{Z}[\beta] \arrow[r, "\Phi"] & \mathbb{R}^d & \mathbb{R}^d \arrow[l, "U"'] \\ \mathbb{Z}[\beta] \arrow[u, "\beta"] \arrow[r, "\Phi"] & \mathbb{R}^d \arrow[u, "\Lambda_\beta"] & \mathbb{R}^d \arrow[l, "U"'] \arrow[u, "R(\mathbf{r})"'] \\ & \mathbb{Z}^d \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd} \tag{3.3.4}$$

(i) We start with the upper left square: observe that $\forall \gamma \in \mathbb{Z}[\beta] : \Phi(\beta\gamma) = \Lambda_\beta \Phi(\gamma)$.

(ii) Next we will prove the commutativity of the lower triangle, *i.e.* that $\forall \mathbf{z} \in \mathbb{Z}^d : \Phi \circ f(\mathbf{z}) = Ug(\mathbf{z})$. Suppose $\mathbf{z} = (z_0, \ldots, z_{d-1})^T \in \mathbb{Z}^d$. Then, by denoting $z_d = -\lfloor \mathbf{zr} \rfloor$, we have $f(\mathbf{z}) = \sum_{i=0}^d r_i z_i$ and thus

$$\Phi \circ f(\mathbf{z}) = \sum_{i=0}^d \Phi(r_i) z_i. \tag{3.3.5}$$

Set $\Phi(r_i) = (r_i^{(1)}, \ldots, r_i^{(d)})$. By the definition of the r_i we have

$$r_d x^d + r_{d-1}x^{d-1} + \cdots + r_0 = (x - \beta_1)(x - \beta_2) \cdots (x - \beta_d).$$

Therefore, taking conjugates yields

$$r_d^{(j)} x^d + r_{d-1}^{(j)} x^{d-1} + \cdots + r_0^{(j)} = Q_j(x)(x - \beta) \quad (1 \le j \le r)$$

and

$$\begin{aligned} r_d^{(r+2j-1)} x^d + r_{d-1}^{(r+2j-1)} x^{d-1} + \cdots + r_0^{(r+2j-1)} &= \Re(Q_{r+j}(x))(x - \beta), \\ r_d^{(r+2j)} x^d + r_{d-1}^{(r+2j)} x^{d-1} + \cdots + r_0^{(r+2j)} &= \Im(Q_{r+j}(x))(x - \beta) \qquad (1 \le j \le s). \end{aligned}$$

From this we easily obtain for $1 \le j \le r$ the formulas

$$\begin{aligned} r_0^{(j)} &= -\beta q_0^{(j)}, \\ r_i^{(j)} &= q_{i-1}^{(j)} - \beta q_i^{(j)}, \quad (1 \le i < d), \\ r_d^{(j)} &= q_{d-1}^{(j)} \; (= 1), \end{aligned} \tag{3.3.6}$$

while for $1 \le j \le s$ we have

$$\begin{aligned} r_0^{(r+2j-1)} &= -\beta\Re(q_0^{(j)}), & r_0^{(r+2j)} &= -\beta\Im(q_0^{(j)}), \\ r_i^{(r+2j-1)} &= \Re(q_{i-1}^{(j)}) - \beta\Re(q_i^{(j)}), & r_i^{(r+2j)} &= \Im(q_{i-1}^{(j)}) - \beta\Im(q_i^{(j)}), \quad (1 \le i < d), \\ r_d^{(r+2j-1)} &= \Re(q_{d-1}^{(j)}) \; (= 1), & r_d^{(r+2j)} &= \Im(q_{d-1}^{(j)}) \; (= 0). \end{aligned} \tag{3.3.7}$$

Now let $(\alpha_1, \ldots, \alpha_d)^T := Ug(\mathbf{z})$. Then we have for $1 \le j \le r$

$$\begin{aligned}\alpha_j &= \sum_{i=0}^{d-1} q_i^{(j)}(-\beta z_i + z_{i+1}) \\ &= -\beta q_0^{(j)} z_0 + \sum_{i=1}^{d-1}(q_{i-1}^{(j)} - \beta q_i^{(j)}) z_i + q_{d-1}^{(j)} z_d \\ &= \sum_{i=0}^{d} r_i^{(j)} z_i\end{aligned}$$

where we used (3.3.6) to obtain the last line. Analogously from (3.3.7) we get

$$\begin{aligned}\alpha_{r+2j-1} &= \sum_{i=0}^{d} \Re(r_i^{(r+j)}) z_i, \\ \alpha_{r+2j} &= \sum_{i=0}^{d} \Im(r_i^{(r+j)}) z_i\end{aligned}$$

for $1 \le j \le s$. Hence, together with (3.3.5) this yields

$$Ug(\mathbf{z}) = (\alpha_1, \ldots, \alpha_d)^T = \sum_{i=0}^{d} \Phi(r_i) z_i = \Phi \circ f(\mathbf{z}).$$

(iii) Finally we turn to the upper right square and show that $\Lambda_\beta U = UR(\mathbf{r})$. Observe that due to Lemma 3.3.21 the left eigenvectors of $R(\mathbf{r})$ are exactly the row vectors $(q_0^{(j)}, \ldots, q_{d-2}^{(j)}, q_{d-1}^{(j)})$ and thus $U^{-1}\Lambda_\beta U = R(\mathbf{r})$ is the real Jordan decomposition of $R(\mathbf{r})$.

The diagram in (3.3.4) now easily yields the identities

$$\Phi(\beta^n \tilde{S}_{\beta,n}(\omega)) = UR(\mathbf{r})^n g(T_{\mathbf{r},n}(\mathbf{x}))$$

and

$$\tilde{S}_\beta(\omega) = \lim_{n\to\infty} \Phi(\beta^n \tilde{S}_{\beta,n}(\omega)) = U \lim_{n\to\infty} R(\mathbf{r})^n g(T_{\mathbf{r},n}(\mathbf{x})).$$

Finally we define the linear function

$$\tilde{g} : \mathbb{Z}^d \to \mathbb{R}^d : \mathbf{z} \mapsto (-\beta I_d + R(\mathbf{r}))\mathbf{z}.$$

For g and $\tilde{g}$ we have the relation $\tilde{g}(\mathbf{z}) = g(\mathbf{z}) + (0, \ldots, 0, e(\mathbf{z}))^T$ where $e(\mathbf{z}) \in (-1, 0]$ holds for all $\mathbf{z} \in \mathbb{Z}^d$. Then $\gamma(\tilde{g}(T_{\mathbf{r},n}(\mathbf{x})), g(T_{\mathbf{r},n}(\mathbf{x})))_{R(\mathbf{r}),\delta}$ is bounded for each $n \in \mathbb{N}$ and, hence, as $R(\mathbf{r})$ is contractive, $\gamma(R(\mathbf{r})^n \tilde{g}(T_{\mathbf{r},n}(\mathbf{x})), R(\mathbf{r})^n g(T_{\mathbf{r}}(\mathbf{x})^{(n)}))_{R(\mathbf{r}),\delta}$ tends to zero as $n \to \infty$. Thus

$$\lim_{n\to\infty} UR(\mathbf{r})^n \tilde{g}(T_{\mathbf{r},n}(\mathbf{x})) = \lim_{n\to\infty} UR(\mathbf{r})^n g(T_{\mathbf{r},n}(\mathbf{x})) = \lim_{n\to\infty} \Phi(\beta^n \tilde{S}_{\beta,n}(\omega)) = \tilde{S}_\beta(\omega).$$

Now observe that $R(\mathbf{z})^n$ commutates with $\tilde{g}$. Therefore $R(\mathbf{z})^n \tilde{g}(T_{\mathbf{r},n}(\mathbf{x})) = \tilde{g}(R(\mathbf{r})^n T_{\mathbf{r},n}(\mathbf{x}))$ and

$$\tilde{S}_\beta(\omega) = \lim_{n\to\infty} UR(\mathbf{r})^n \tilde{g}(T_{\mathbf{r},n}(\mathbf{x})) = U(R(\mathbf{r}) - \beta I_d) \lim_{n\to\infty} R(\mathbf{r})^n (T_{\mathbf{r},n}(\mathbf{x})) = U(R(\mathbf{r}) - \beta I_d) T_{\mathbf{r}}(\mathbf{x}).$$

□

Corollary 3.3.23. *For a Pisot number β and $\omega \in \mathbb{Z}[\beta]$ we have*

$$\tilde{S}_\beta(\omega) = \bigcup_{\gamma \in \tilde{S}_{\beta,1}(\omega)} \Lambda_\beta \tilde{S}_\beta(\gamma).$$

Proof. The equation can be derived from Proposition 3.1.8, Theorem 3.3.22 and the bijectivity of f (Proposition 3.3.2). □

Corollary 3.3.24. *The new β-tiles induced by a Pisot unit of degree $d+1$ cover the d-dimensional real space.*

Proof. This is an immediate consequence of Proposition 3.1.6 and Theorem 3.3.22. □

Corollary 3.3.25. *Let β a Pisot unit and $\mathbf{r}$ as in (3.3.1). The number of different SRS-tiles up to translation associated to $\mathbf{r}$ is $N(\mathbf{r})$ if β is a simple Parry number, otherwise it equals $N(\mathbf{r})+1$.*

Proof. This can easily be seen by Corollary 3.3.20 and Theorem 3.3.22. □

Example 3.3.26. Consider the Polynomial $x^3 - 3x^2 + 1$. Its greatest root in modulus is $\beta = 2.87938524157\ldots$, a Pisot unit. Set $\mathbf{r} = (r_0, r_1)$ with

$$r_0 = -\frac{1}{\beta}, r_1 = -\frac{1}{\beta^2}.$$

The 25 SRS-tiles $\mathcal{T}_{\mathbf{r}}(\mathbf{x}) \subset \mathbb{R}^2$ with $\|\mathbf{x}\|_\infty \leq 2$ are shown on the left hand side of Figure 3.7. Let β_1, β_2 be the Galois conjugates of β. β_1 and β_2 are real and thus the corresponding β-tiles

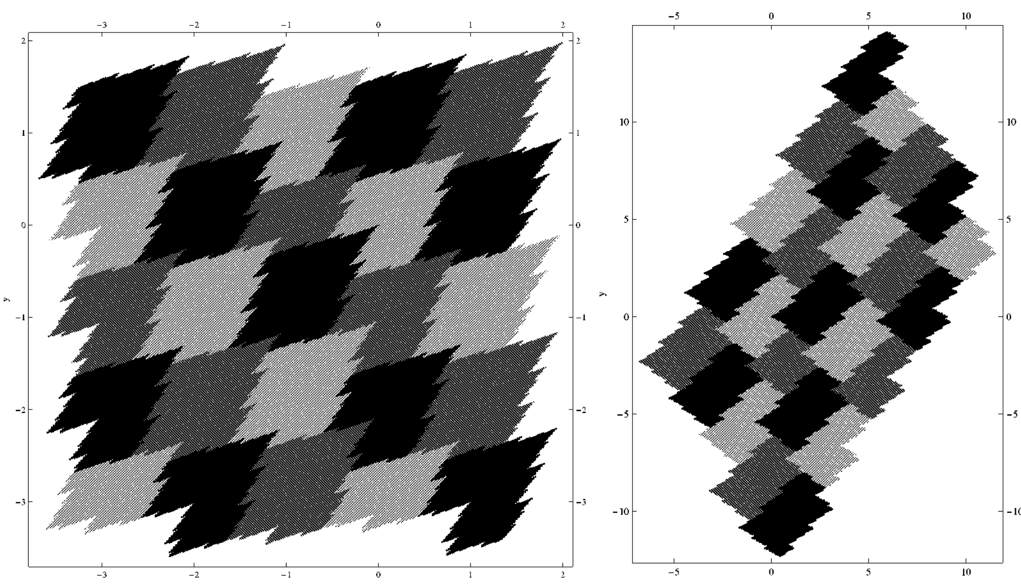


Figure 3.7: SRS tiles associated to $\mathbf{r} = \left(-\beta^{-1}, -\beta^{-2}\right)$ (left) and the corresponding new β-tiles generated by β (right).

generated by β are obtained by multiplying $\mathcal{T}_{\mathbf{r}}(\mathbf{x})$ by the matrix

$$U(\beta I_2 + R(\mathbf{r})) = \begin{pmatrix} -\beta_2 & 1 \\ -\beta_1 & 1 \end{pmatrix} \left(\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -r_0 & -r_1 \end{pmatrix} \right).$$

They are shown on the right hand side of Figure 3.7.

Chapter 4

Variations of Shift Radix Systems

In the last chapter we will deal with ε-shift radix systems. In the fist part we analyse notations and results concerning SRS for a possible generalisation to ε-SRS. We also show the relation of ε-SRS and ε-CNS to ε-β-expansions. Afterwards we concentrate on the two-dimensional case and show in Theorem 4.2.2 and Theorem 4.2.11 that $\mathcal{D}^0_{2,\varepsilon}$ can be completely described for $\varepsilon \neq 0$. We will explicitly characterise this set for two exemplary values of ε. Finally, in Section 4.3, we turn to the three-dimensional symmetric case and present a complete characterisation of $\mathcal{D}^0_{3,\frac{1}{2}} = \tilde{\mathcal{D}}^0_3$.

4.1 ε-shift radix systems

The results of this section are, if not other cited, due to [53].

4.1.1 Basic properties

We already defined ε-SRS and the sets $\mathcal{D}_{d,\varepsilon}$ and $\mathcal{D}^0_{d,\varepsilon}$ in Definition 1.1.20. Lots of basic properties and notations concerning $\mathcal{D}^0_{d,\varepsilon}$ and $\mathcal{D}_{d,\varepsilon}$ can be directly adopted from the case $\varepsilon = 0$. Since $\mathcal{D}^0_{d,\varepsilon} \subseteq \mathcal{D}_{d,\varepsilon}$ we first study the structure of $\mathcal{D}_{d,\varepsilon}$.

Theorem 4.1.1. *Let $d \in \mathbb{N}$. Then we have*

$$\mathcal{E}_d \subseteq \mathcal{D}_{d,\varepsilon} \subseteq \overline{\mathcal{E}_d},$$

$$\partial\mathcal{D}_{d,\varepsilon} = \{\mathbf{r} \in \mathbb{R}^d |\, \varrho(R(\mathbf{r})) = 1\}.$$

Proof. For 0-SRS this has been proven in [3] (see also Theorem 1.1.4) and for $\frac{1}{2}$-SRS in [12]. The proofs can be transferred to other values of ε without difficulties. □

We can easily see that the interior of $\mathcal{D}_{d,\varepsilon}$ equals $\mathcal{E}_d$ and does not depend on ε. We do not expect $\mathcal{D}_{d,\varepsilon} \setminus \mathcal{E}_d$ to be independent of ε, however, these boundaries seem to be very hard to describe (see Subsection 1.1.2).

We already defined ε-CNS. Let $P(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_0$ and $\mathcal{N} = [-\varepsilon\,|p_0|\,, (1-\varepsilon)\,|p_0|) \cap \mathbb{Z}$. In the same way as for CNS we can define our backward division algorithm. Let $A = A_0 \in \mathcal{R} = \mathbb{Z}[x]/(P)$. A_0 has a unique representation of the shape $A_0 = \sum_{i=0}^{d-1} a_i^{(0)} X^i$. For $k \geq 0$ we inductively calculate $A_{k+1} := \sum_{i=0}^{d-1} a_i^{(k+1)} X^i$ by $A_{k+1} = \sum_{i=0}^{d-1} (a_{i+1}^{(k)} + q_k p_{i+1}) X^i$ with q_k such that $a_0^{(k)} = e_k + q_k p_0$ and $e_k \in \mathcal{N}$. For each $l \in \mathbb{N}$ we then have

$$A = \sum_{i=0}^{l-1} e_i X^i + A_l X^l.$$

Thus, $(P, \mathcal{N})$ is an ε-CNS if and only if the backward division algorithm ends up in 0 for each $A \in \mathcal{R}$.

Theorem 4.1.2. *$(P,\mathcal{N})$ is an ε-CNS if and only if $(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}) \in \mathcal{D}^0_{d,\varepsilon}$.*

Proof. The proof runs analogously to the proof of [3, Theorem 3.1] (*cf.* also Theorem 1.2.6 and Theorem 3.2.12). □

Remark 4.1.3. Actually the above theorem can easily be generalised to non-monic polynomials. Suppose $P(x) = p_d x^d + p_{d-1}x^{d-1} + \cdots + p_0$, $\mathcal{R} = \mathbb{Z}[x]/(P)$, X the image of x under the canonical epimorphism and $\mathcal{N} = [-\varepsilon\,|p_0|\,, (1-\varepsilon)\,|p_0|) \cap \mathbb{Z}$. Analogously to Theorem 3.2.12 it can be shown that the digit system $(\mathcal{R}, X, \mathcal{N})$ (see Definition 3.2.1) has the periodic representation property if and only if $(\frac{p_d}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}) \in \mathcal{D}_{d,\varepsilon}$ and that $(\mathcal{R}, X, \mathcal{N})$ has the finite expansion property if and only if $(\frac{p_d}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}) \in \mathcal{D}^0_{d,\varepsilon}$.

We now turn briefly to ε-β-expansions. For $\gamma \in [-\varepsilon, 1-\varepsilon)$ the ε-β-expansion can be obtained by applying the ε-β-shift

$$T_{\beta,\varepsilon} : [-\varepsilon, 1-\varepsilon) \to [-\varepsilon, 1-\varepsilon), \gamma \mapsto \beta\gamma - \lfloor \beta\gamma + \varepsilon \rfloor .$$

Each $\gamma \in \mathbb{R}$ has exactly one ε-β-expansion except for the case $\varepsilon = 0$ where we have to restrict to $\gamma \in \mathbb{R} \cap [0, \infty)$. If $\gamma \notin [\varepsilon, 1-\varepsilon)$ there exists a $k \in \mathbb{N}$ with $\beta^{-k}\gamma \in [\varepsilon, 1-\varepsilon)$. Then the ε-β-shift can be applied to obtain the ε-β-expansion of $\beta^{-k}\gamma$. Multiplication with β^k then yields the ε-β-expansion of γ. For $\varepsilon = 0$ the ε-β-shift corresponds to the β-shift and for $\varepsilon = \frac{1}{2}$ the analogue mapping for the symmetric case (see Subsection 1.3.3).

Definition 4.1.4. Let $\varepsilon \in [0,1)$. An algebraic integer $\beta > 1$ is said to have property (ε-F) if each $\gamma \in \mathbb{Z}[\beta^{-1}] \cap [-\varepsilon, 1-\varepsilon)$ has a finite ε-β-expansion.

Property (0-F) is obviously equal to the well known property (F) while ($\frac{1}{2}$-F) corresponds to (SF). For algebraic integers β the ε-β-expansion is closely related to ε-SRS.

Theorem 4.1.5. *Let $\varepsilon \in [0,1)$ and β a positive algebraic integer with minimal polynomial $P(x) = x^{d+1} + p_d x^d + \cdots + p_1 x + p_0$ and $\mathbf{r}$ defined as in (3.3.1). Then β has property (ε-F) if and only if $\mathbf{r} \in \mathcal{D}^0_{d,\varepsilon}$.*

Proof. For $\varepsilon = 0$ this has already been shown in [3, Theorem 2.1] (*cf.* Theorem 3.3.1). One can easily generalise this proof to all values of ε by showing that the dynamical systems $T_{\beta,\varepsilon}$ and $\tau_{\mathbf{r},\varepsilon}$ are conjugated. □

This immediately shows that (ε-F) is only possible for Pisot numbers.

4.1.2 About $\mathcal{D}^0_{d,\varepsilon}$

In order to analyse the structure of $\mathcal{D}^0_{d,\varepsilon}$ we proceed similarly as we have done in Section 1.2. The following definitions and lemmas are directly adopted from there and do not require further comments or proofs.

Definition 4.1.6. Let $\mathbf{r} \in \mathcal{D}_{d,\varepsilon}$. A point $\mathbf{x} \in \mathbb{Z}^d$ is called *purely periodic* (with respect to $\tau_{\mathbf{r},\varepsilon}$) if $\tau^l_{\mathbf{r},\varepsilon}(\mathbf{x}) = \mathbf{x}$ for some $l \in \mathbb{N}$. We denote by $C_\varepsilon(\mathbf{r})$ the set of all purely periodic points with respect to $\tau_{\mathbf{r},\varepsilon}$.

Lemma 4.1.7. *Let $\mathbf{r} \in \operatorname{int}(\mathcal{D}_{d,\varepsilon})$. Then $C_\varepsilon(\mathbf{r})$ is a finite set.*

Set $\mathcal{O}(C_\varepsilon(\mathbf{r})) := C_\varepsilon(\mathbf{r})/\tau_{\mathbf{r},\varepsilon}$. Again each element of $\mathcal{O}(C_\varepsilon(\mathbf{r}))$ is completely determined by a finite sequence of integers.

Definition 4.1.8. Let $\mathbf{r} \in \mathcal{D}_{d,\varepsilon}$. We call an orbit $\pi \in \mathcal{O}(C_\varepsilon(\mathbf{r}))$ with $|\pi| = l$ a *cycle of period l of* $\tau_{\mathbf{r},\varepsilon}$. By the same discussion as after Definition 1.1.2 a cycle of period l is uniquely determined by l integers $x_0, \dots, x_{l-1}$ and will be denoted by $\langle x_0, \dots, x_{l-1} \rangle$. We refer to an element of

$$\Pi_{d,\varepsilon} := \bigcup_{\mathbf{r} \in \mathcal{D}_{d,\varepsilon}} \mathcal{O}(C_\varepsilon(\mathbf{r}))$$

more generally as a *cycle of* $\mathcal{D}_{d,\varepsilon}$.

Lemma 1.1.3 also holds for the elements of $\mathcal{O}(C_\varepsilon(\mathbf{r}))$.

Again we ask: For which $\mathbf{r} = (r_0, \ldots, r_{d-1})$ is a given $\langle z_0, \ldots, z_{l-1}\rangle$, $l \in \mathbb{N}$, $z_i \in \mathbb{Z}$, a cycle of $\tau_{\mathbf{r},\varepsilon}$? Again these points are described by a system of inequalities analogously to (1.1.5)

$$0 \le r_0 z_i + \cdots + r_{d-1} z_{i+d-1} + z_{i+d} + \varepsilon < 1, \quad \forall i \in (0, \ldots, l-1) \tag{4.1.1}$$

(indices of z modulo l). Hence, $\langle z_0, \ldots, z_{l-1}\rangle$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for those $\mathbf{r}$ that satisfy the system of inequalities (4.1.1). Define

$$P_{d,\varepsilon}(\langle z_0, \ldots, z_{l-1}\rangle) := \left\{(r_0, \ldots, r_{d-1}) \in \mathbb{R}^d \,|\, (r_0, \ldots, r_{d-1}) \text{ satisfies (4.1.1)} \right\}.$$

$P_{d,\varepsilon}(\langle z_0, \ldots, z_{l-1}\rangle)$ is a d-dimensional polyhedron which can degenerate to a lower dimension or even to the empty set. $\langle z_0, \ldots, z_{l-1}\rangle$ is a cycle of some $\tau_{\mathbf{r},\varepsilon}$ if and only if $P_{d,\varepsilon}(\langle z_0, \ldots, z_{l-1}\rangle) \cap \mathcal{D}_{d,\varepsilon} \neq \emptyset$. As in Section 1.2 we call the cycle $\langle z_0, \ldots, z_{l-1}\rangle$ degenerated, if $\dim P_{d,\varepsilon}(\langle z_0, \ldots, z_{l-1}\rangle) < d$. $\langle z_0, \ldots, z_{l-1}\rangle$ is no cycle if $P_{d,\varepsilon}(\langle z_0, \ldots, z_{l-1}\rangle) = \emptyset$. Note that, if π is a non-degenerated cycle of $\mathcal{D}_{d,\varepsilon}$, analogously to the Lifting Theorem 1.1.12, π is also a non-degenerated cycle of $\mathcal{D}_{d',\varepsilon}$ for all $d' > d$. Again we have the identity

$$\mathcal{D}^0_{d,\varepsilon} = \mathcal{D}_{d,\varepsilon} \setminus \bigcup_{\pi \in \Pi_{d,\varepsilon} \setminus \{\langle 0 \rangle\}} P_{d,\varepsilon}(\pi).$$

Before we turn to the problem of characterising $\mathcal{D}^0_{d,\varepsilon}$ we observe a nice symmetry concerning $\mathcal{D}_{d,\varepsilon}$ and $\mathcal{D}_{d,1-\varepsilon}$.

Lemma 4.1.9. *For all $\varepsilon \in (0,1)$ we have*

$$\operatorname{int}(P_{d,\varepsilon}(\langle z_0, \ldots, z_{l-1}\rangle)) = \operatorname{int}(P_{d,1-\varepsilon}(\langle -z_0, \ldots, -z_{l-1}\rangle)).$$

Proof. $\mathbf{r} = (r_0, \ldots, r_{d-1})$ is in $\operatorname{int}(P_{d,\varepsilon}(\langle z_0, \ldots, z_{l-1}\rangle))$ if and only if

$$0 < r_0 z_i + \cdots + r_{d-1} z_{i+d-1} + z_{i+d} + \varepsilon < 1 \quad \forall i \in (0, \ldots, l-1)$$

(indices of z modulo l). Thus

$$-1 < r_0 z_i + \cdots + r_{d-1} z_{i+d-1} + z_{i+d} - 1 + \varepsilon < 0 \quad \forall i \in (0, \ldots, l-1).$$

Multiplication with -1 yields

$$0 < r_0(-z_i) + \cdots + r_{d-1}(-z_{i+d-1}) - z_{i+d} + 1 - \varepsilon < 1 \quad \forall i \in (0, \ldots, l-1)$$

which is equivalent to $\mathbf{r} \in \operatorname{int}(P_{d,1-\varepsilon}(\langle -z_0, \ldots, -z_{l-1}\rangle))$. □

Corollary 4.1.10. *Let $\varepsilon \in (0,1)$ and denote the d-dimensional Lebesgue measure by μ_d. Then*

$$\mu_d(\mathcal{D}^0_{d,\varepsilon} \triangle \mathcal{D}^0_{d,1-\varepsilon}) = 0$$

holds.

In Definition 2.1.5 we defined the set of witnesses for some point $\mathbf{r} \in \mathcal{D}_d$. It is remarkable that we can give an analogue to Theorem 2.1.6 for ε-SRS without changing the definition of the set of witnesses.

Theorem 4.1.11. *Let $\varepsilon \in [0,1)$ and $\mathbf{r} \in \mathcal{D}_{d,\varepsilon}$. $\mathbf{r} \in \mathcal{D}^0_{d,\varepsilon}$ if and only if a set of witnesses $\mathcal{V}$ does not contain purely periodic points with respect to $\tau_{\mathbf{r},\varepsilon}$.*

Proof. If $\mathcal{V}$ has a periodic element then $\mathbf{r} \notin \mathcal{D}^0_{d,\varepsilon}$ by the definition of $\mathcal{D}^0_{d,\varepsilon}$. For showing the other direction we observe that the behaviour of the floor function (see proof of Theorem 2.1.6) implies that for any $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ we have

$$\tau_{\mathbf{r},\varepsilon}(\mathbf{a}+\mathbf{b}) \in \left\{\tau_{\mathbf{r},\varepsilon}(\mathbf{a}) + \tau_{\mathbf{r},0}(\mathbf{b}), \tau_{\mathbf{r},\varepsilon}(\mathbf{a}) + (-\tau_{\mathbf{r},0}(-\mathbf{b}))\right\}.$$

The rest of the proof runs analogously to that of Theorem 2.1.6. □

We therefore can use Definition 2.1.7 and the set $V(Q)$ and all the assertions concerning the finiteness in this context as well.

Lemma 4.1.12. *A set of witnesses $\mathcal{V}$ of Q is closed under the application of $\tau_{\mathbf{r},\varepsilon}$ for each $\mathbf{r} \in Q$ and $\varepsilon \in [0, 1)$.*

Proof. We first claim that for $a \in \mathbb{R}$ and $\varepsilon \in [0, 1)$ we have

$$\lceil a \rceil \geq \lfloor a + \varepsilon \rfloor . \tag{4.1.2}$$

Indeed, if $a \in \mathbb{Z}$ then $\lceil a \rceil = \lfloor a \rfloor = \lfloor a + \varepsilon \rfloor$. Otherwise $\lceil a \rceil = \lfloor a \rfloor + 1 = \lfloor a + 1 \rfloor \geq \lfloor a + \varepsilon \rfloor$.

Now let $\mathbf{z} = (z_0, \ldots, z_{d-1}) \in \mathcal{V}$. By definition $(z_1, \ldots, z_{d-1}, j) \in \mathcal{V}$ for all

$$j \in I := \{\min_{\mathbf{s} \in Q} \lfloor -\mathbf{s}\mathbf{z} \rfloor, \ldots, -\min_{\mathbf{s} \in Q} \lfloor \mathbf{s}\mathbf{z} \rfloor\}.$$

Then, using (4.1.2), we have

$$\min_{\mathbf{s} \in Q} \lfloor -\mathbf{s}\mathbf{z} \rfloor = \min_{\mathbf{s} \in Q} (-\lceil \mathbf{s}\mathbf{z} \rceil) \leq \min_{\mathbf{s} \in Q} (-\lfloor \mathbf{s}\mathbf{z} + \varepsilon \rfloor) \leq -\lfloor \mathbf{r}\mathbf{z} + \varepsilon \rfloor \leq -\lfloor \mathbf{r}\mathbf{z} \rfloor \leq \max_{\mathbf{s} \in Q} (-\lfloor \mathbf{s}\mathbf{z} \rfloor) = -\min_{\mathbf{s} \in Q} \lfloor \mathbf{s}\mathbf{z} \rfloor .$$

Therefore $-\lfloor \mathbf{r}\mathbf{z} + \varepsilon \rfloor \in I$ and $\tau_{\mathbf{r},\varepsilon}(\mathbf{z}) \in \mathcal{V}$. $\square$

Next we adapt Definition 2.1.8 for our purposes.

Definition 4.1.13. Let $\mathcal{V}$ be a finite set of witnesses for some $Q \subset \mathcal{D}_{d,\varepsilon}$. Let $G(\mathcal{V}, \varepsilon) = V \times E$ be the smallest directed graph with vertices $V = \mathcal{V}$ and edges $E \subset V \times V$ such that

$$E = \{(\mathbf{x}, \tau_{\mathbf{r},\varepsilon}(\mathbf{x})) | \mathbf{x} \in V, \mathbf{r} \in Q\}.$$

The definition is meaningful because of Lemma 4.1.12. The finiteness condition on $\mathcal{V}$ assures the finiteness of the graph $G(\mathcal{V}, \varepsilon)$. Again we are interested in the (directed) graph-cycles of $G(\mathcal{V}, \varepsilon)$. After Definition 2.1.8 we have agreed upon a notation of graph cycles of $G(\mathcal{W}, Q)$. We can immediately adapt this notation for graph cycles of $G(\mathcal{V}, \varepsilon)$. This immediately yields the analogue to the Brunotte Algorithm (*cf.* Theorem 2.1.9).

Theorem 4.1.14. *Let $\varepsilon \in [0, 1)$, $Q \subset \mathcal{D}_{d,\varepsilon}$ and $\mathcal{V}$ a finite set of witnesses of Q. Furthermore let Π_Q the set of graph-cycles of $G(\mathcal{V}, \varepsilon)$ without the trivial one $\langle 0 \rangle$. Then*

$$\mathcal{D}^0_{d,\varepsilon} \cap Q = Q \setminus \bigcup_{\pi \in \Pi_Q} P_{d,\varepsilon}(\pi).$$

It is straightforward that the algorithms presented in Subsection 2.1.2 can be modified in order to use them for analysing $\mathcal{D}^0_{d,\varepsilon}$ for all $\varepsilon \in [0, 1)$. Since this is a real trivial modification we will not explicitly accomplish it here. We just denote for an $\varepsilon \in [0, 1)$ by $\mathrm{Br}_1(Q, p, \varepsilon)$ and $\mathrm{Br}_2(Q, \varepsilon)$ the particular modification of Algorithm 3 and Algorithm 4, respectively.

4.2 Characterisation results concerning $\mathcal{D}^0_{2,\varepsilon}$

4.2.1 The case $0 < \left|\varepsilon - \frac{1}{2}\right| < \frac{1}{2}$

In the following we will concentrate on the two dimensional case and show that $\mathcal{D}^0_{2,\varepsilon}$ for $\varepsilon \in (0, 1)$ can be fully characterised by cutting out finitely many polyhedra from $\mathcal{E}_d$. The result is based on the following lemma.

Lemma 4.2.1. *If there exists a closed set $D \subset \mathcal{E}_d$ with $\mathcal{D}^0_{d,\varepsilon} \subset D$ then $\mathcal{D}^0_{d,\varepsilon}$ can be obtained from D by cutting out finitely many polyhedra.*

Proof. This can be directly seen from Corollary 2.1.15 (the adaption for arbitrary values $\varepsilon \in [0, 1)$ is a triviality). $\square$

In order to prove the possibility of representing $\mathcal{D}^0_{2,\varepsilon}$ with only finitely many cutout polyhedra we will show the existence of a closed set D with $\mathcal{D}^0_{2,\varepsilon} \subset D \subset \mathcal{E}_2$ that does the job in the above lemma. We first show this for $\varepsilon \in (0, \frac{1}{2})$ and then use the symmetry described in Lemma 4.1.9 to obtain a similar result for $\varepsilon \in (\frac{1}{2}, 1)$. For $\varepsilon = \frac{1}{2}$ a full characterisation was already given in [12] (see Subsection 1.1.6).

Theorem 4.2.2. *For $\varepsilon \in (0, \frac{1}{2})$ the set $\mathcal{D}^0_{2,\varepsilon}$ can be completely characterised by cutting out finitely many polyhedra.*

Proof. For showing the theorem we will prove for each $\varepsilon \in (0, \frac{1}{2})$ the existence of a set D which satisfies the conditions made in Lemma 4.2.1.

Fix an $\varepsilon \in (0, \frac{1}{2})$. From Theorem 4.1.1 and (1.1.2) we know that $\overline{\mathcal{D}_{2,\varepsilon}}$ equals the triangle $\{(x, y) \in \mathbb{R}^2 | \, |x| \leq 1, |y| \leq x + 1\}$. Denote by $\square(Q_1, \dots, Q_k)$ the closed convex body of the points $Q_1, \dots, Q_k$ and define the following sets:

$$
\begin{aligned}
T_1 :=& \square(Q_1, Q_2, Q_3, Q_4) \text{ with} \\
& Q_1 = (-1, 0), Q_2 = (\frac{\varepsilon}{2} - 1, -\frac{\varepsilon}{2}), Q_3 = (1, 2 - \varepsilon), Q_4 = (1, 2), \\
T_2 :=& \square(Q_1, Q_2, Q_3, Q_4) \setminus \overline{Q_1 Q_4} \text{ with} \\
& Q_1 = (-1, 0), Q_2 = (1, -2), Q_3 = (1, -1 - \varepsilon), Q_4 = (\frac{-\varepsilon - 1}{2}, \frac{-\varepsilon + 1}{2}), \\
T_3 :=& \square(Q_1, Q_2, Q_3, Q_4) \text{ with} \\
& Q_1 = (1 - \varepsilon, -1 + \varepsilon), Q_2 = (1 - \varepsilon, -1), Q_3 = (1, -1 - \varepsilon), Q_4 = (1, -1 + \varepsilon), \\
T_4 :=& \square(Q_1, Q_2, Q_3, Q_4) \setminus (\overline{Q_2 Q_3} \cup \overline{Q_1 Q_4}) \text{ with} \\
& Q_1 = (1 - \varepsilon, -\varepsilon), Q_2 = (1 - \varepsilon, -1 + \varepsilon), Q_3 = (1, -1 + \varepsilon), Q_4 = (1, -\varepsilon), \\
T_5 :=& \square(Q_1, Q_2, Q_3, Q_4) \text{ with} \\
& Q_1 = (1 - \varepsilon, \varepsilon), Q_2 = (1 - \varepsilon, -\varepsilon), Q_3 = (1, -\varepsilon), Q_4 = (1, \varepsilon), \\
T_6 :=& \{(x, y) \in \mathbb{R}^2 | 1 - \varepsilon \leq x \leq 1, \varepsilon < y < x - \varepsilon\}, \\
T_7 :=& \{(x, y) \in \mathbb{R}^2 | 1 - \varepsilon \leq x \leq 1, \varepsilon < y \leq 1 + \varepsilon, x - \varepsilon \leq y < x + 1 - \varepsilon\}, \\
T_8 :=& \{(x, y) \in \mathbb{R}^2 | 1 - \varepsilon \leq x \leq 1, 1 + \varepsilon < y < 1 + x - \varepsilon\}.
\end{aligned}
$$

Figure 4.1 shows the position and shape of these sets for two different values of ε. In Lemma 4.2.3 to Lemma 4.2.10 we show that for all $i \in \{1, \dots, 8\}$ the set T_i is not contained in $\mathcal{D}^0_{2,\varepsilon}$. Now set

$$
D := \overline{\mathcal{D}_{2,\varepsilon} \setminus \bigcup_{i=1}^{8} T_i} = \{(x, y) \in \mathbb{R}^2 | -x - \varepsilon \leq y \leq x + 1 - \varepsilon, x \leq 1 - \varepsilon\} \subset \mathcal{D}_{2,\varepsilon}
$$

and observe that $\mathcal{D}^0_{2,\varepsilon} \subset D \subset \mathcal{E}_2$. Thus D satisfies the conditions of Lemma 4.2.1 which proves the theorem. $\square$

Lemma 4.2.3. *$\pi := \langle 1, -1 \rangle$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for $\mathbf{r} \in T_1$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. The set T_1 is a closed polygon. Since $P_{2,\varepsilon}(\pi)$ is a polygon, too, it is enough to show that $Q_i \in P_{2,\varepsilon}(\pi)$ for $i \in \{1, 2, 3, 4\}$. $P_{2,\varepsilon}(\pi)$ is defined by the inequalities

$$
\begin{aligned}
-\varepsilon \leq & \quad x - y + 1 & < 1 - \varepsilon, \\
-\varepsilon \leq & \; -x + y - 1 & < 1 - \varepsilon.
\end{aligned}
$$

It is easily checked that each of the four points $Q_1, \dots, Q_4$ really satisfy this system of inequalities which proves the lemma. $\square$

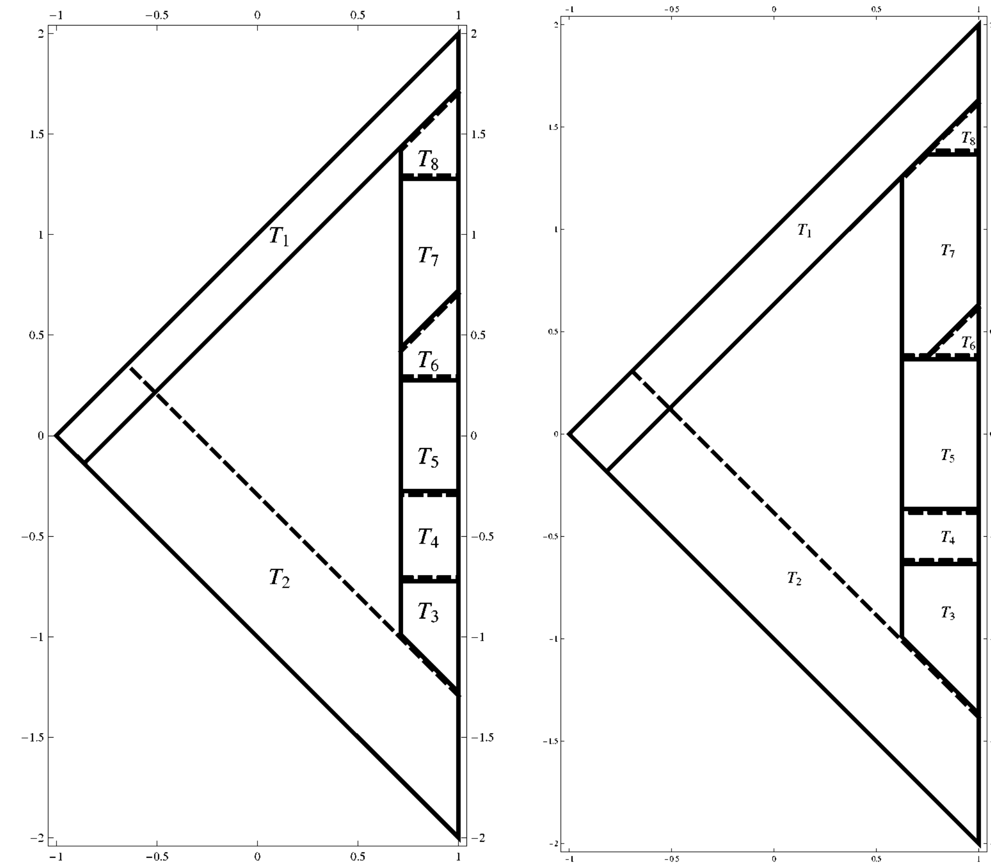


Figure 4.1: The sets T_i for $\varepsilon = \frac{2}{7}$ (left) and $\varepsilon = \frac{3}{8}$ (right)

Lemma 4.2.4. *$\pi := \langle 1 \rangle$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for $\mathbf{r} \in T_2$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. The proof runs analogously to the proof of Lemma 4.2.6[1]. □

Lemma 4.2.5. *$\pi := \langle 1, 1, 0, -1, -1, 0 \rangle$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for $\mathbf{r} \in T_3$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. The proof runs analogously to the proof of Lemma 4.2.3. □

Lemma 4.2.6. *$\pi := \langle 1, 1, 0, -1, 0 \rangle$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for $\mathbf{r} \in T_4$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. The set T_4 is a rectangle where the lines $\overline{Q_2Q_3} \subset \{(x, y) \in \mathbb{R}^2 | y = -1 + \varepsilon\}$ and $\overline{Q_1Q_4} \subset \{(x, y) \in \mathbb{R}^2 | y = -\varepsilon\}$ are not included. Note that $\square(Q_1, Q_2, Q_3, Q_4)$ is a polygon, thus

$$T_4 = \square(Q_1, Q_2, Q_3, Q_4) \setminus (\{(x, y) \in \mathbb{R}^2 | y = -1 + \varepsilon\} \cup \{(x, y) \in \mathbb{R}^2 | y = -\varepsilon\}). \tag{4.2.1}$$

$P_{2,\varepsilon}(\pi)$ is defined by the inequalities

$$\begin{array}{rcccl} -\varepsilon & \leq & x + y & < & 1 - \varepsilon, \\ -\varepsilon & \leq & x - 1 & < & 1 - \varepsilon, \\ -\varepsilon & \leq & -y & <^* & 1 - \varepsilon, \\ -\varepsilon & \leq & -x + 1 & < & 1 - \varepsilon, \\ -\varepsilon & \leq & y + 1 & <^* & 1 - \varepsilon. \end{array}$$

[1] In Lemma 4.2.4 the cycle has period 1 and we only have one double inequality. Thus the situation is easier than in Lemma 4.2.6.

Two strict "<" are tagged with $*$. Exchange them by non-strict "$\leq$" and leave the other inequalities unchanged. This modified system of inequalities defines another polygon, let us call it $P^*_{2,\varepsilon}(\pi)$. Obviously

$$P_{2,\varepsilon}(\pi) = P^*_{2,\varepsilon}(\pi) \setminus (\{(x,y) \in \mathbb{R}^2 | y = -1+\varepsilon\} \cup \{(x,y) \in \mathbb{R}^2 | y = -\varepsilon\}). \tag{4.2.2}$$

Now $\square(Q_1, Q_2, Q_3, Q_4) \subset P^*_{2,\varepsilon}(\pi)$ since all four points satisfy all inequalities of $P^*_{2,\varepsilon}(\pi)$. Observing (4.2.1) and (4.2.2) then immediately yields $T_4 \subset P_{2,\varepsilon}(\pi_4)$, which proves the lemma. $\square$

Lemma 4.2.7. *$\pi := \langle 1, 0, -1, 0\rangle$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for $\mathbf{r} \in T_5$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. The proof runs analogously to the proof of Lemma 4.2.3. $\square$

Lemma 4.2.8. *$\pi := \langle 1, 0, -1, 1, 1, -1, 0\rangle$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for $\mathbf{r} \in T_6$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. For $\varepsilon \in (0, \frac{1}{3})$ we have $T_6 = \square(Q_1, Q_2, Q_3, Q_4) \setminus (\overline{Q_2Q_3} \cup \overline{Q_1Q_4})$ with vertices $Q_1 = (1-\varepsilon, 1-2\varepsilon)$, $Q_2 = (1-\varepsilon, \varepsilon)$, $Q_3 = (1, \varepsilon)$ and $Q_4 = (1, 1-\varepsilon)$. For $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$ we have $T_6(\varepsilon) = \square(Q_1, Q_2, Q_3) \setminus (\overline{Q_1Q_2} \cup \overline{Q_1Q_3})$ with $Q_1 = (2\varepsilon, \varepsilon)$, $Q_2 = (1, \varepsilon)$ and $Q_3 = (1, 1-\varepsilon)$. However, one can easily prove that $T_6 \subset P_{2,\varepsilon}(\pi)$ in an analogue way as it was done in Lemma 4.2.6. $\square$

Lemma 4.2.9. *$\pi := \langle 1, 0, -1\rangle$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for $\mathbf{r} \in T_7$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. For $\varepsilon \in (0, \frac{1}{3})$ we have $T_7 = \square(Q_1, Q_2, Q_3, Q_4)$ with $Q_1 = (1-\varepsilon, 1+\varepsilon)$, $Q_2 = (1-\varepsilon, 1-2\varepsilon)$, $Q_3 = (1, 1-\varepsilon)$ and $Q_4 = (1, 1+\varepsilon)$. $T_7(\varepsilon) \subset P_{2,\varepsilon}(\pi)$ can be shown analogously to Lemma 4.2.3. For $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$ we have $T_7(\varepsilon) = \square(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) \setminus (\overline{Q_1Q_2} \cup \overline{Q_3Q_4})$ with $Q_1 = (2\varepsilon, 1+\varepsilon)$, $Q_2 = (1-\varepsilon, 2-2\varepsilon)$, $Q_3 = (1-\varepsilon, \varepsilon)$, $Q_4 = (2\varepsilon, \varepsilon)$, $Q_5 = (1, 1-\varepsilon)$ and $Q_6 = (1, 1+\varepsilon)$. For $\varepsilon = \frac{1}{3}$ the points Q_1 and Q_2 as well as the points Q_3 and Q_4 coincide giving a quadrangle with the points $Q_1 = Q_2 = (\frac{2}{3}, \frac{4}{3})$ and $Q_3 = Q_4 = (\frac{2}{3}, \frac{1}{3})$ missing. Analogously to Lemma 4.2.6 it can be proved that $T_6 \subset P_{2,\varepsilon}(\pi)$. $\square$

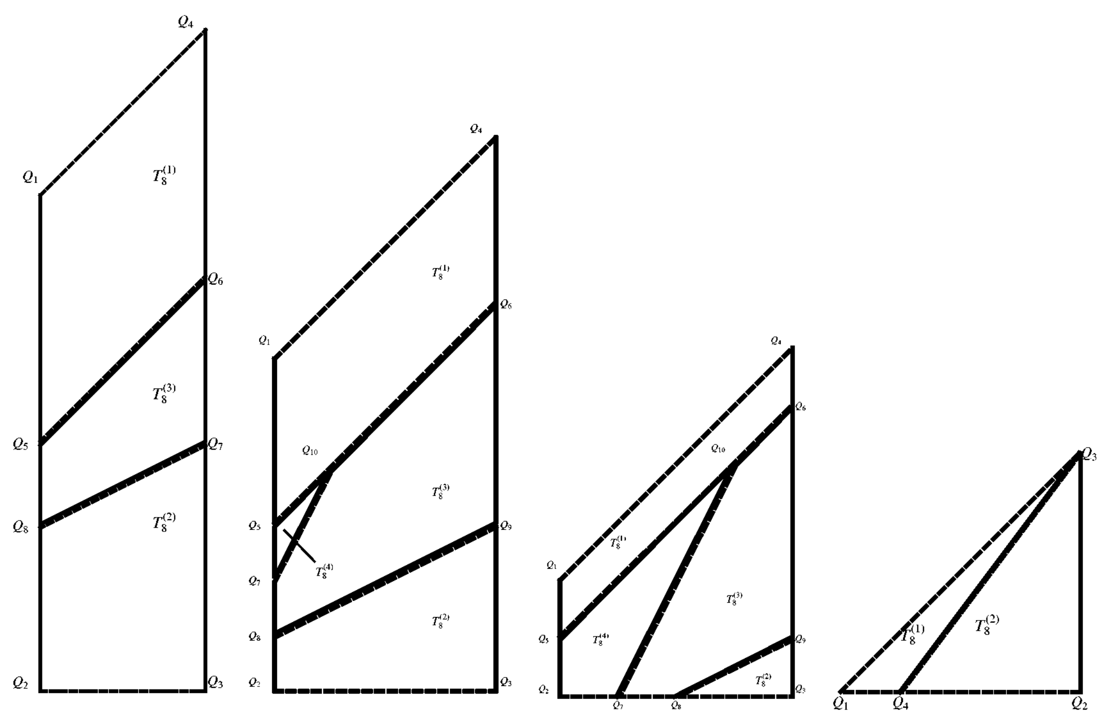


Figure 4.2: Partition of the set T_8 for $\varepsilon = \frac{1}{6}$, $\varepsilon = \frac{2}{9}$, $\varepsilon = \frac{2}{7}$, $\varepsilon = \frac{3}{8}$ (from left to right)

Lemma 4.2.10. *$T_8 \cap \mathcal{D}^0_{2,\varepsilon} = \emptyset$ for $\varepsilon \in (0, \frac{1}{2})$.*

Proof. First suppose that $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$. Then $T_8 = \square(Q_1, Q_2, Q_3) \setminus (\overline{Q_1Q_2} \cup \overline{Q_1Q_3})$ with $Q_1 = (2\varepsilon, 1+\varepsilon)$, $Q_2 = (1, 1+\varepsilon)$ and $Q_3 = (1, 2-\varepsilon)$. Let $Q_4 := (\frac{1+2\varepsilon}{2}, 1+\varepsilon) \in \overline{Q_1Q_3}$ and subdivide T_8 into the triangles

$$\begin{aligned} T_8^{(1)} &:= \square(Q_1, Q_3, Q_4) \setminus (\overline{Q_1Q_4} \cup \overline{Q_1Q_3}), \\ T_8^{(2)} &:= \square(Q_2, Q_3, Q_4) \setminus (\overline{Q_2Q_4} \cup \overline{Q_3Q_4}). \end{aligned}$$

See the rightmost example $\varepsilon = \frac{3}{8}$ in Figure 4.2. Consider the cycles $\pi_1 := \langle 1, 0, -1, 2, -2 \rangle$ and $\pi_2 := \langle 1, 0, -1, 2, -2, 1, 1, -2, 2, -1, 0, 1, -1 \rangle$. Analogously to Lemma 4.2.6 it can be shown that $T_8^{(1)} \subset P_{2,\varepsilon}(\pi_1)$ and $T_8^{(2)} \subset P_{2,\varepsilon}(\pi_2)$.

Now suppose $\varepsilon \in (0, \frac{1}{3})$. Let $Q_1 = (1-\varepsilon, 2-2\varepsilon)$, $Q_2 = (1-\varepsilon, 1+\varepsilon)$, $Q_3 = (1, 1+\varepsilon)$ and $Q_4 = (1, 2-\varepsilon)$. Then $T_8 = \square(Q_1, Q_2, Q_3, Q_4) \setminus (\overline{Q_2Q_3} \cup \overline{Q_1Q_4})$. Again we have to do some subdivision. Define $Q_5 = (1-\varepsilon, \frac{3-\varepsilon}{2})$ and $Q_6 = (1, \frac{3+\varepsilon}{2})$. Note that $Q_5 \in \overline{Q_1Q_2}$ and $Q_6 \in \overline{Q_3Q_4}$. Let

$$\begin{aligned} \pi_1 &:= \langle 1, 0, -1, 2, -2, 2, -1, 0, 1, -1 \rangle, \\ \pi_2 &:= \langle 1, 0, -1, 2, -1, 0, 1, -1 \rangle, \\ \pi_3 &:= \langle 1, 0, -1, 2, -2, 1, 1, -2, 2, -1, 0, 1, -1 \rangle, \\ \pi_4 &:= \langle 1, 0, -1, 2, -2 \rangle. \end{aligned}$$

In the same manner as in the proof of Lemma 4.2.6 one can show that

$$T_8^{(1)} := \square(Q_1, Q_5, Q_6, Q_4) \setminus (\overline{Q_5Q_6} \cup \overline{Q_1Q_4})$$

is contained in $P_{2,\varepsilon}(\pi_1)$. Set

$$T_8^* := T_8 \setminus T_8^{(1)} = \square(Q_2, Q_3, Q_7, Q_5) \setminus \overline{Q_2Q_3}.$$

We have to distinguish several cases.

$\varepsilon \in (\frac{1}{4}, \frac{1}{3})$ Define $Q_7 = (\frac{1+2\varepsilon}{2}, 1+\varepsilon)$, $Q_8 = (3\varepsilon, 1+\varepsilon)$, $Q_9 = (1, \frac{3-\varepsilon}{2})$ and $Q_{10} = (\frac{1+3\varepsilon}{2}, 1+2\varepsilon)$. We have $Q_7, Q_8 \in \overline{Q_2Q_3}$, $Q_9 \in \overline{Q_3Q_6}$ and $Q_{10} \in \overline{Q_5Q_6}$. Thus the sets

$$\begin{aligned} T_8^{(2)} &:= \square(Q_8, Q_3, Q_9) \setminus (\overline{Q_3Q_8} \cup \overline{Q_8Q_9}), \\ T_8^{(3)} &:= \square(Q_7, Q_8, Q_9, Q_6, Q_{10}) \setminus (\overline{Q_7Q_8} \cup \overline{Q_7Q_{10}}), \\ T_8^{(4)} &:= \square(Q_2, Q_7, Q_{10}, Q_5) \setminus \overline{Q_2Q_7} \end{aligned}$$

form a partition of T_8^* (see third sketch in Figure 4.2). In the style of Lemma 4.2.6 we can now show that $T_8^{(i)} \subset P_{2,\varepsilon}(\pi_i)$ for $i \in \{2, 3, 4\}$.

$\varepsilon = \frac{1}{4}$ Runs similar to the previous case with the only difference that here the points Q_7 and Q_8 coincide with Q_2. Thus $T_8^{(3)}$ is a quadrangle and $T_8^{(4)}$ is a triangle (closed, just with the point Q_2 missing).

$\varepsilon \in (\frac{1}{5}, \frac{1}{4})$ Define $Q_7 = (1-\varepsilon, 2-3\varepsilon)$, $Q_8 = (1-\varepsilon, \frac{3-2\varepsilon}{2})$, $Q_9 = (1, \frac{3-\varepsilon}{2})$ and $Q_{10} = (\frac{1+3\varepsilon}{2}, 1+2\varepsilon)$. We have $Q_7, Q_8 \in \overline{Q_2Q_5}$, $Q_9 \in \overline{Q_3Q_6}$ and $Q_{10} \in \overline{Q_5Q_6}$. Again the sets

$$\begin{aligned} T_8^{(2)} &:= \square(Q_2, Q_3, Q_9, Q_8) \setminus (\overline{Q_2Q_3} \cup \overline{Q_8Q_9}), \\ T_8^{(3)} &:= \square(Q_7, Q_8, Q_9, Q_6, Q_{10}) \setminus \overline{Q_7Q_{10}}, \\ T_8^{(4)} &:= \square(Q_5, Q_7, Q_{10}) \end{aligned}$$

form a partition of T_8^* (see second sketch in Figure 4.2) and as before we have $T_8^{(i)} \subset P_{2,\varepsilon}(\pi_i)$ for $i \in \{2, 3, 4\}$.

$\varepsilon = \frac{1}{5}$ The situation is comparable to the previous case with the difference that $Q_7 = Q_{10} = Q_5 = \left(\frac{4}{5}, \frac{7}{5}\right)$. Thus $T_8^{(3)}$ is a closed quadrangle with the only exception that Q_5 is missing and $T_8^{(4)}$ consists only of the point Q_5.

$\varepsilon \in (0, \frac{1}{5})$ Let $Q_7 = (1, \frac{3-\varepsilon}{2})$ and $Q_8 = (1-\varepsilon, \frac{3-2\varepsilon}{2})$. $Q_7 \in \overline{Q_3Q_6}$ and $Q_8 \in \overline{Q_2Q_5}$. The sets

$$\begin{aligned} T_8^{(2)} :=& \square(Q_2, Q_3, Q_7, Q_8) \setminus (\overline{Q_2Q_3} \cup \overline{Q_7Q_8}), \\ T_8^{(3)} :=& \square(Q_5, Q_8, Q_7, Q_6) \end{aligned}$$

partition T_8^* (see leftmost sketch in Figure 4.2). Further $T_8^{(i)} \subset P_{2,\varepsilon}(\pi_i)$ for $i \in \{2, 3\}$.

$\square$

We now turn to the case $\varepsilon \in (\frac{1}{2}, 1)$ and prove

Theorem 4.2.11. *For $\varepsilon \in (0, \frac{1}{2})$ the set $\mathcal{D}^0_{2,(1-\varepsilon)}$ can be completely characterised by cutting out finitely many polyhedra.*

Proof. We will show this very analogous to Theorem 4.2.2. Since by Lemma 4.1.9 for some cycle π we have $\operatorname{int}(P_{2,\varepsilon}(\pi)) = \operatorname{int}(P_{2,1-\varepsilon}(-\pi))$ for $\varepsilon \in (0,1)$ we can expect that we can use the negative cycles and quasi the same sets. Only the boundaries will change a little. In particular, we define

$$\begin{aligned} \tilde{T}_1 :=& \square(Q_1, Q_2, Q_3, Q_4) \setminus \overline{Q_2Q_3} \text{ with} \\ & Q_1 = (-1, 0), Q_2 = (\frac{\varepsilon}{2} - 1, -\frac{\varepsilon}{2}), Q_3 = (1, 2-\varepsilon), Q_4 = (1, 2), \\ \tilde{T}_2 :=& \square(Q_1, Q_2, Q_3, Q_4) \text{ with} \\ & Q_1 = (-1, 0), Q_2 = (1, -2), Q_3 = (1, -1-\varepsilon), Q_4 = (\frac{-\varepsilon - 1}{2}, \frac{-\varepsilon + 1}{2}), \\ \tilde{T}_3 :=& \square(Q_1, Q_2, Q_3, Q_4) \setminus (\overline{Q_1Q_2} \cup \overline{Q_2Q_3} \cup \overline{Q_1Q_4}) \text{ with} \\ & Q_1 = (1-\varepsilon, -1+\varepsilon), Q_2 = (1-\varepsilon, -1), Q_3 = (1, -1-\varepsilon), Q_4 = (1, -1+\varepsilon), \\ \tilde{T}_4 :=& \square(Q_1, Q_2, Q_3, Q_4) \setminus \overline{Q_1Q_2} \text{ with} \\ & Q_1 = (1-\varepsilon, -\varepsilon), Q_2 = (1-\varepsilon, -1+\varepsilon), Q_3 = (1, -1+\varepsilon), Q_4 = (1, -\varepsilon), \\ \tilde{T}_5 :=& \square(Q_1, Q_2, Q_3, Q_4) \setminus (\overline{Q_1Q_2} \cup \overline{Q_2Q_3} \cup \overline{Q_1Q_4}) \text{ with} \\ & Q_1 = (1-\varepsilon, \varepsilon), Q_2 = (1-\varepsilon, -\varepsilon), Q_3 = (1, -\varepsilon), Q_4 = (1, \varepsilon), \\ \tilde{T}_6 :=& \{(x,y) \in \mathbb{R}^2 | 1-\varepsilon < x \leq 1, \varepsilon \leq y \leq x - \varepsilon\}, \\ \tilde{T}_7 :=& \{(x,y) \in \mathbb{R}^2 | 1-\varepsilon < x \leq 1, \varepsilon \leq y < 1+\varepsilon, x - \varepsilon < y \leq x + 1 - \varepsilon\}, \\ \tilde{T}_8 :=& \{(x,y) \in \mathbb{R}^2 | 1-\varepsilon < x \leq 1, 1+\varepsilon \leq y \leq 1 + x - \varepsilon\}. \end{aligned}$$

In the same style as in Lemma 4.2.3 to Lemma 4.2.10 we can now show that

$$\begin{aligned} \tilde{T}_1 \subset& P_{2,1-\varepsilon}(\langle 1, -1 \rangle), \\ \tilde{T}_2 \subset& P_{2,1-\varepsilon}(\langle -1 \rangle), \\ \tilde{T}_3 \subset& P_{2,1-\varepsilon}(\langle 1, 1, 0, -1, -1, 0 \rangle), \\ \tilde{T}_4 \subset& P_{2,1-\varepsilon}(\langle 1, 0, -1, -1, 0 \rangle), \\ \tilde{T}_5 \subset& P_{2,1-\varepsilon}(\langle 1, 0, -1, 0 \rangle), \\ \tilde{T}_6 \subset& P_{2,1-\varepsilon}(\langle 1, 0, -1, 0, 1, -1, -1 \rangle), \\ \tilde{T}_7 \subset& P_{2,1-\varepsilon}(\langle 1, -1, 0 \rangle), \\ \tilde{T}_8 \subset& P_{2,1-\varepsilon}(\langle 1, 0, -1, 2, -2, 2, -1, 0, 1, -1 \rangle) \cup P_{2,1-\varepsilon}(\langle 1, 0, -1, 2, -1, 0, 1, -1 \rangle) \\ & \cup P_{2,1-\varepsilon}(\langle 1, 0, -1, 2, -2, 1, 1, -2, 2, -1, 0, 1, -1 \rangle) \cup P_{2,1-\varepsilon}(\langle 1, 0, -1, 2, -2 \rangle). \end{aligned}$$

Thus we have

$$D := \overline{\mathcal{D}_{2,1-\varepsilon} \setminus \bigcup_{i=1}^{8} \tilde{T}_i} = \{(x,y) \in \mathbb{R}^2 | -x-\varepsilon \le y \le x+1-\varepsilon, x \le 1-\varepsilon\} \subset \mathcal{D}_{2,1-\varepsilon}.$$

This set D is exactly the same as in Theorem 4.2.2 and we again have that $\mathcal{D}^0_{2,1-\varepsilon} \subset D$. Thus D satisfies the conditions of Lemma 4.2.1 which proves the theorem. □

If there are analogues of this result for $d > 2$ is up to now unknown.

Open question 3. Can $\mathcal{D}_{d,\varepsilon}$ for $d > 2$ and $\varepsilon \in (0,1)$ be characterised by finitely many cutout polyhedra?

We can answer this question only for $d = 3$ and $\varepsilon = \frac{1}{2}$ and give a full characterisation of $\mathcal{D}_{3,\frac{1}{2}}$ is section 4.3. For $\varepsilon = 0$ the question must by definitely answered negatively as we have already seen.

4.2.2 $\mathcal{D}^0_{2,\varepsilon}$ for concrete values of ε

At the end of this section we will use the results of Subsection 4.2.1 and the modified Algorithm 4 to give explicit characterisations of $\mathcal{D}^0_{2,\varepsilon}$ for $\varepsilon = \frac{1}{5}$ and $\varepsilon = \frac{1}{10}$. From Theorem 4.2.2 and Theorem 4.2.11 we know that

$$\mathcal{D}^0_{2,\varepsilon} \subset D^*(\varepsilon) := \{(x,y) \in \mathbb{R}^2 | -x-\varepsilon \le y < x+1-\varepsilon, x < 1-\varepsilon\} \text{ for } \varepsilon \in \left(0, \frac{1}{2}\right),$$
$$\mathcal{D}^0_{2,\varepsilon} \subset D^*(\varepsilon) := \{(x,y) \in \mathbb{R}^2 | -x-\varepsilon < y \le x+1-\varepsilon, x \le 1-\varepsilon\} \text{ for } \varepsilon \in \left(\frac{1}{2}, 1\right)$$

and that we can characterise $\mathcal{D}^0_{2,\varepsilon}$ completely for $\varepsilon \in (0,1)$. Thus we have to apply $\mathrm{Br}_2(\overline{D^*(\varepsilon)}, \varepsilon)$. The sets $\mathcal{D}^0_{2,\frac{1}{5}}$ and $\mathcal{D}^0_{2,\frac{1}{10}}$ are shown as grey areas in Figure 4.3 and Figure 4.4. The dashed lines do not belong to them. We will state these two characterisations results as theorems:

Theorem 4.2.12. *The set $D_{2,\frac{1}{5}}$ equals the set $D^*\left(\frac{1}{5}\right)$ where the polyhedra $P_{2,\frac{1}{10}}(\pi_i)$, $i \in \{1,2,3\}$, with*

$$\pi_1 = \langle 0,1 \rangle, \pi_2 = \langle -1,0,1 \rangle, \pi_3 = \langle -1,-1,1,2,1 \rangle,$$

are cut out.

Theorem 4.2.13. *The set $D_{2,\frac{1}{10}}$ equals the set $D^*\left(\frac{1}{10}\right)$ where the polyhedra $P_{2,\frac{1}{10}}(\zeta_i)$, $i \in \{1,\dots,10\}$, with*

$$\begin{array}{ll} \zeta_1 = \langle 0,1 \rangle, & \zeta_2 = \langle -3,1,3,-2,-2,3,1 \rangle, \\ \zeta_3 = \langle -4,2,1,-3,4,-2,-1,4 \rangle, & \zeta_4 = \langle -1,0,1 \rangle, \\ \zeta_5 = \langle -5,5,-4,3,-1,-1,3,-4,5 \rangle, & \zeta_6 = \langle -2,1,1,-2,3 \rangle, \\ \zeta_7 = \langle -1,-1,1,2,1 \rangle, & \zeta_8 = \langle -3,3,-2,1,1,-2,3 \rangle, \\ \zeta_9 = \langle -3,2,1,-3,3,-1,-1,3 \rangle, & \zeta_{10} = \langle -2,-1,2,2,-1,-2,1,3,1 \rangle, \end{array}$$

are cut out.

Note that the algorithm returns more cycles but the above characterisations have been minimised. There is no polygon which is covered by others. When we compare these results with the approximation of $\mathcal{D}^0_{2,0} = \mathcal{D}^0_2$ and $\mathcal{D}^0_{2,\frac{1}{2}} = \tilde{\mathcal{D}}^0_2$ we can conjecture that for ε approaching 0 more and more peaks "grow".

From Corollary 4.1.10 we know that $\mathcal{D}^0_{2,\frac{1}{5}}$ and $\mathcal{D}^0_{2,\frac{4}{5}}$ and $\mathcal{D}^0_{2,\frac{1}{10}}$ and $\mathcal{D}^0_{2,\frac{9}{10}}$, respectively, only differ by a set of measure 0. Indeed, $\mathcal{D}^0_{2,\frac{4}{5}}$ and $\mathcal{D}^0_{2,\frac{9}{10}}$ have, apart from the boundary, the same shape as $\mathcal{D}^0_{2,\frac{1}{5}}$ and $\mathcal{D}^0_{2,\frac{1}{10}}$ and we have

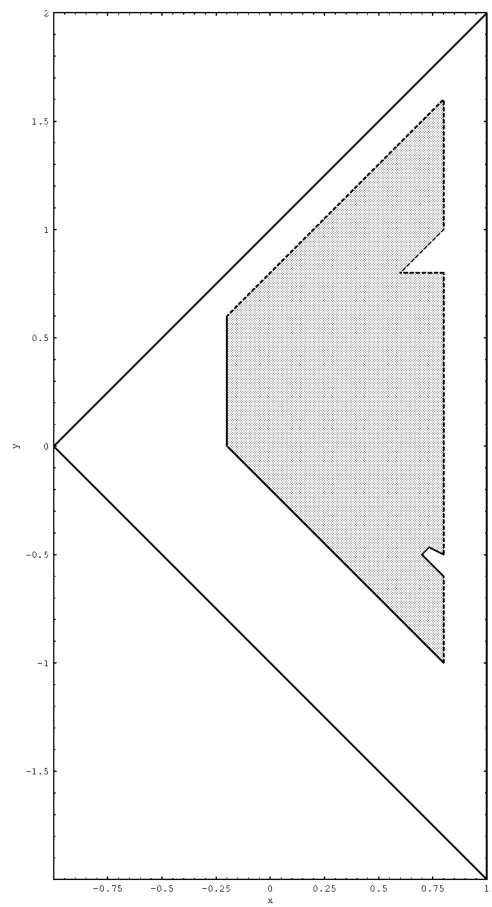


Figure 4.3: The set $\mathcal{D}^0_{2,\frac{1}{5}}$

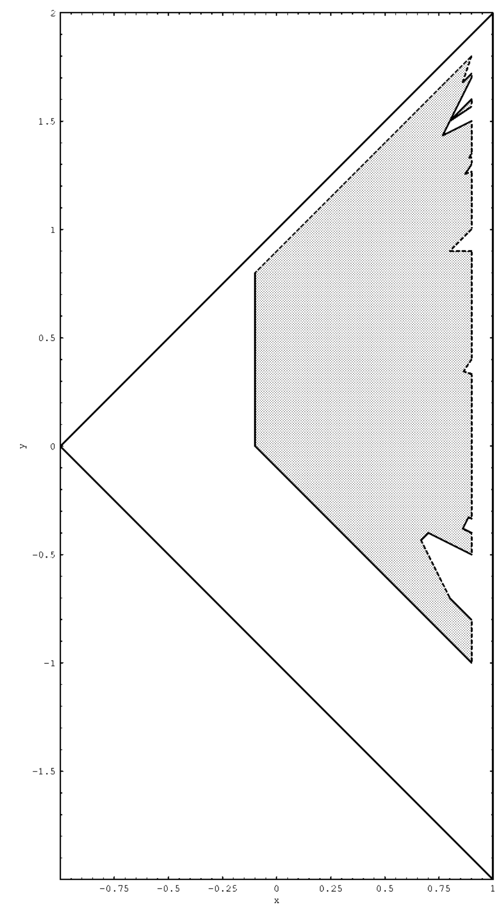


Figure 4.4: The set $\mathcal{D}^0_{2,\frac{1}{10}}$

Theorem 4.2.14. *Let π_1, π_2 and π_3 as in Theorem 4.2.12 and $\zeta_1, \ldots, \zeta_{10}$ as in Theorem 4.2.13. Then*

$$\mathcal{D}^0_{2,\frac{4}{5}} = D^*\left(\frac{4}{5}\right) \setminus \bigcup_{i=1}^{3} P_{2,\frac{4}{5}}(-\pi_i), \qquad \mathcal{D}^0_{2,\frac{9}{10}} = D^*\left(\frac{9}{10}\right) \setminus \bigcup_{i=1}^{10} P_{2,\frac{9}{10}}(-\zeta_i).$$

We omit a detailed illustration of $\mathcal{D}_{2,\frac{4}{5}}$ and $\mathcal{D}_{2,\frac{9}{10}}$ here since they equal the sets $\mathcal{D}_{2,\frac{1}{5}}$ and $\mathcal{D}_{2,\frac{1}{10}}$, only the boundaries are reversed.

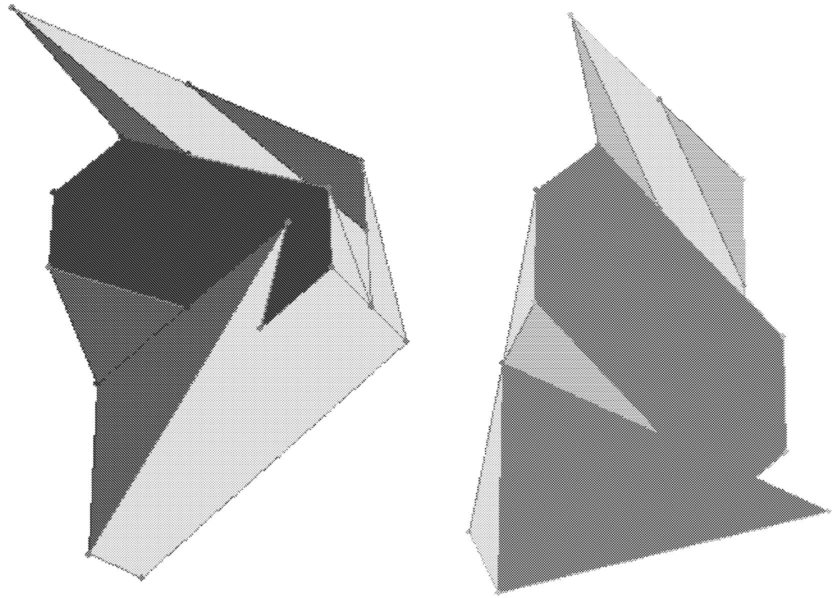


Figure 4.5: Two views of $\mathcal{D}^0_{3,\frac{1}{2}}$

4.3 Three dimensional symmetric Shift Radix Systems

4.3.1 Definitions and main result

The last section of the present thesis is dedicated to a result given by Huszti, Scheicher, Surer and Thuswaldner [30]. It provides a complete description of $\mathcal{D}^0_{3,\frac{1}{2}}$. For this reason we define the sets

$$\begin{aligned}
S_1 &:= \{(x,y,z) \mid 2x-2z \geq 1 \wedge 2x+2y+2z > -1 \wedge 2x+2y \leq 1 \\
&\qquad \wedge 2x \leq 1 \wedge 2x-2y+2z \leq 1\},\\
S_2 &:= \{(x,y,z) \mid x-z \leq -1 \wedge 2x-2y+2z \leq 1 \wedge -2x+2y \leq 1 \\
&\qquad \wedge 2x > -1\},\\
S_3 &:= \{(x,y,z) \mid x-z > -1 \wedge 2x-2y+2z \leq 1 \wedge -2x+2y < 1, 2x > -1 \\
&\qquad \wedge 2x-2z < -1 \wedge 2x+2y+2z > -1\},\\
S_4 &:= \{(x,y,z) \mid 2x-2y+2z \leq 1 \wedge -2x+2y \leq 1 \wedge 2x-2z = -1 \\
&\qquad \wedge 2x+2y+2z > -1\},\\
S_5 &:= \{(x,y,z) \mid -1 < 2x \leq 1 \wedge -1 < 2x-2z \leq 1 \wedge 2x+2y+2z > -1 \\
&\qquad \wedge 2x-2y+2z \leq 1 \wedge 2x+4y-2z < 3, 2y \leq 1\}
\end{aligned}$$

and denote their union by

$$\mathcal{S} := \bigcup_{i \in \{1,\ldots,5\}} S_i.$$

Note that S_1, S_2, S_3, S_5 are polyhedra while S_4 is a polygon. The following theorem states the main result of the present section.

Theorem 4.3.1. $\mathcal{D}^0_{3,\frac{1}{2}} = \mathcal{S}$.

Two views of the set $\mathcal{D}^0_{3,\frac{1}{2}}$ are depicted in Figure 4.5. For rotating 3D-pictures of $\mathcal{D}^0_{3,\frac{1}{2}}$ we refer the reader to the author's home pages [51].

The proof of the theorem is split into two parts that occupy the next subsections. Here we want to give an outline of the proof. In a first step we will use $\mathrm{Br}_2\left(S_i, \frac{1}{2}\right)$ for each $i \in \{1, \dots, 5\}$ in order to show that

$$\mathcal{S} \subseteq \mathcal{D}^0_{3,\frac{1}{2}}. \tag{4.3.1}$$

For showing the opposite inclusion we define the set of cycles

$$\Pi := \{\pi, \dots, \pi_{43}\},$$

where the concrete values of π_j for $j \in \{1, \dots, 43\}$ can be gathered from Table 4.1, and show that

Period	Cycles
1	$\pi_1{=}\langle -1\rangle$
2	$\pi_2{=}\langle 0,-1\rangle$ $\pi_3{=}\langle 1,-1\rangle$
3	$\pi_4{=}\langle -1,-1,0\rangle$ $\pi_5{=}\langle -1,0,1\rangle$ $\pi_6{=}\langle 0,-1,0\rangle$ $\pi_7{=}\langle 0,-1,1\rangle$
4	$\pi_8{=}\langle 0,-1,0,1\rangle$
5	$\pi_9{=}\langle -2,1,-1,-1,1\rangle$ $\pi_{10}{=}\langle -2,1,0,-1,2\rangle$ $\pi_{11}{=}\langle -1,-1,1,1,0\rangle$ $\pi_{12}{=}\langle 0,-2,-1,1,2\rangle$ $\pi_{13}{=}\langle 0,-1,1,-1,0\rangle$ $\pi_{14}{=}\langle 0,1,-1,1,0\rangle$ $\pi_{15}{=}\langle 0,1,0,-1,-1\rangle$ $\pi_{16}{=}\langle 0,1,0,-1,0\rangle$ $\pi_{17}{=}\langle 0,2,1,-1,-2\rangle$ $\pi_{18}{=}\langle 1,-1,1,-1,0\rangle$ $\pi_{19}{=}\langle 1,1,-1,-1,0\rangle$ $\pi_{20}{=}\langle 2,-1,0,1,-2\rangle$
6	$\pi_{21}{=}\langle 0,-1,0,0,1,0\rangle$ $\pi_{22}{=}\langle 1,1,0,-1,-1,0\rangle$
7	$\pi_{23}{=}\langle 0,1,-1,-1,1,0,-1\rangle$ $\pi_{24}{=}\langle 1,1,0,-1,-1,-1,0\rangle$
8	$\pi_{25}{=}\langle -1,-1,1,1,2,0,0,-2\rangle$ $\pi_{26}{=}\langle -1,0,0,1,0,0,-1,-1\rangle$ $\pi_{27}{=}\langle -1,1,0,-1,1,-1,0,1\rangle$ $\pi_{28}{=}\langle 0,0,2,1,1,-1,-1,-2\rangle$ $\pi_{29}{=}\langle 1,1,1,0,-1,-1,-1,0\rangle$ $\pi_{30}{=}\langle 2,1,-1,-2,-2,-1,1,2\rangle$
9	$\pi_{31}{=}\langle -1,0,0,1,1,1,0,-1,-1\rangle$ $\pi_{32}{=}\langle 0,1,1,1,0,-1,-2,-2,-1\rangle$
10	$\pi_{33}{=}\langle -1,-1,1,0,-1,1,1,-1,0,1\rangle$ $\pi_{34}{=}\langle 0,-2,1,1,-2,0,2,-1,-1,2\rangle$ $\pi_{35}{=}\langle 0,-1,-1,-1,0,0,1,1,1,0\rangle$ $\pi_{36}{=}\langle 1,2,1,1,-1,-1,-2,-1,-1,1\rangle$ $\pi_{37}{=}\langle 1,2,2,1,0,-1,-2,-2,-1,0\rangle$
11	$\pi_{38}{=}\langle -2,0,1,-2,1,0,-2,2,-1,-1,2\rangle$ $\pi_{39}{=}\langle 0,1,2,2,1,0,-1,-2,-2,-2,-1\rangle$
12	$\pi_{40}{=}\langle -2,2,-1,0,1,-2,2,-2,1,0,-1,2\rangle$ $\pi_{41}{=}\langle 0,1,2,2,2,1,0,-1,-2,-2,-2,-1\rangle$
13	$\pi_{42}{=}\langle 0,1,-2,2,-1,-1,2,-2,1,0,-1,1,-1\rangle$
22	$\pi_{43}{=}\langle 0,2,2,1,-1,-2,-2,0,1,2,1,0,-2,-2,-1,1,2,2,0,-1,-2,-1\rangle$

Table 4.1: The list of the 43 cycles

for $\mathcal{P} := \bigcup_{\pi\in\Pi} P_{3,\frac{1}{2}}(\pi)$ we have

$$\mathcal{S} \cup \mathcal{P} \supseteq \mathcal{D}_{3,\frac{1}{2}}.$$

From (4.3.1) we can deduce $\mathcal{S} \cap \mathcal{P} = \emptyset$. Thus,

$$\mathcal{S} \supseteq \mathcal{D}_{3,\frac{1}{2}} \setminus \mathcal{P} \supseteq \mathcal{D}^0_{3,\frac{1}{2}}.$$

Since $\mathcal{D}_{3,\frac{1}{2}} \subset \overline{\mathcal{E}_3}$ we are done if we can cover $\overline{\mathcal{E}_3}$ with $\mathcal{P} \cup \mathcal{S}$, *i.e.*, if we can show that

$$\mathcal{P} \cup \mathcal{S} \supseteq \overline{\mathcal{E}_3}.$$

In other words, the 43 cycles $\pi_1, \dots, \pi_{43}$ fully characterise $\mathcal{D}^0_{3,\frac{1}{2}}$.

We will need the following notation and definition.

Notation 4.3.2. For a logical system $\mathcal{J}$ of inequalities, which are combined by $\wedge$ and $\vee$, denote by $X(\mathcal{J})$ the set of all points that satisfy $\mathcal{J}$. Let P a set of inequalities. Then $\bigwedge P$ and $\bigvee P$ denote the systems $\bigwedge_{I\in P} I$ and $\bigvee_{I\in P} I$, respectively.

For the rest of the section denote by T_i the set of inequalities that define the set S_i for $i \in \{1, \dots, 5\}$. These sets are assembled only of single inequalities. We have

$$\begin{aligned}
T_1 &:= \{2x - 2z \geq 1, 2x + 2y + 2z > -1, 2x + 2y \leq 1, 2x \leq 1,\\
&\quad\ 2x - 2y + 2z \leq 1\},\\
T_2 &:= \{x - z \leq -1, 2x - 2y + 2z \leq 1, -2x + 2y \leq 1, 2x > -1\},\\
T_3 &:= \{x - z > -1, 2x - 2y + 2z \leq 1, -2x + 2y < 1, 2x > -1,\\
&\quad\ 2x - 2z < -1, 2x + 2y + 2z > -1\},\\
T_4 &:= \{2x - 2y + 2z \leq 1, -2x + 2y \leq 1, 2x - 2z \leq -1, 2x - 2z \geq -1,\\
&\quad\ 2x + 2y + 2z > -1\},\\
T_5 &:= \{-1 < 2x, 2x \leq 1, -1 < 2x - 2z, 2x - 2z \leq 1, 2x + 2y + 2z > -1,\\
&\quad\ 2x - 2y + 2z \leq 1, 2x + 4y - 2z < 3, 2y \leq 1\},
\end{aligned}$$

hence, the equality of S_4 and the two double inequalities of S_5 are split into inequalities. Thus, $S_i = X(\bigwedge T_i)$ for $i = 1, \dots, 5$. Denote by $\bar{T}_i$ the set T_i with all the strict inequalities changed to not strict ones. Since all occurring inequalities are linear it can easily be checked that $\overline{S_i} = X(\bigwedge \bar{T}_i)$.

Further, for each $i \in \{1, \dots, 43\}$, define Q_i as the reduced set of single inequalities such that $P_{2,\frac{1}{2}}(\pi_i) = X(\bigwedge Q_i)$. "Reduced" means that all the redundant inequalities are removed.

Remark 4.3.3. It is not really necessary to work with the reduced systems but the main algorithm works much faster and the reduction is not too difficult to realise.

Algorithm 6 RL(P), reduces a list of inequalities.

Input: P set of inequalities
Output: P reduced set of inequalities

for all inequalities $I \in P$ **do**
  $P \leftarrow P \setminus I$
  if $X(\bigwedge P \wedge \neg I) \neq \emptyset$ **then**
    $P \leftarrow P \cup I$
  end if
end for
return(P)

The algorithm simply uses the fact that an inequality I is redundant for a system $\mathcal{S} \wedge I$ if $X(\mathcal{S} \wedge I) = X(\mathcal{S})$ or, equivalently, $X(\mathcal{S} \wedge \neg I) = \emptyset$. Denote the application of Algorithm 6 with parameter P by RL(P) (RL=reduce list of inequalities). For instance, the set Q_{19} can be defined by

$$\begin{aligned}
Q_{19} := \Big\{&x + y - z - 1 < \frac{1}{2}, x - y - z < \frac{1}{2}, -\frac{1}{2} \leq -x - y + 1,\\
&- x + z + 1 < \frac{1}{2}, -\frac{1}{2} \leq y + z - 1\Big\}.
\end{aligned}$$

4.3.2 $\mathcal{S} \subset \mathcal{D}^0_{3,\frac{1}{2}}$

We start with by showing $\mathcal{S} \subset \mathcal{D}^0_{2,\frac{1}{2}}$. This is the easier direction since we can use previously defined algorithms.

Lemma 4.3.4. $\mathrm{Br}_2(\overline{S_i}, \frac{1}{2})$ *terminates for each* $i \in \{1, \dots, 5\}$.

Proof. The algorithms was implemented in Mathematica® with $c = 20$ (see the notations after Theorem 2.1.18. The program is available on the author's homepage [51]. □

For $i \in \{1, \dots, 5\}$ denote by Π_i the set of cycles computed by $\mathrm{Br}_2(\overline{S_i}, \frac{1}{2})$.

Theorem 4.3.5. *$S_i \subset \mathcal{D}^0_{3,\frac{1}{2}}$ holds for all $i \in \{1,\dots,5\}$.*

Proof. For each $i \in \{1,\dots,5\}$ we have that $X(\bigwedge \bar{T}_i)$ is a convex hull of finitely many points. Moreover, $X(\bigwedge \bar{T}_i) = \overline{S_i}$. Π_i includes all cycles associated to polyhedra having non-empty intersection with $X(\bigwedge \bar{T}_i)$. Now, according to (4.1.1), each of these cycles $\pi \in \Pi_i$ induces a set of inequalities $\mathcal{P}(\pi)$. It turns out that for each $\pi \in \Pi_i$ we have

$$X(\mathcal{P}(\pi) \wedge \bigwedge T_i) = \emptyset \text{ holds for each } i \in \{1,\dots,5\}$$

(an easy way for checking this is to apply the cylindrical algebraic decomposition algorithm). Thus there is no cycle that yields a nonempty cutout intersecting with S_i and therefore $S_i \subset \mathcal{D}^0_{3,\frac{1}{2}}$. □

4.3.3 $\mathcal{D}^0_{3,\frac{1}{2}} \subset \mathcal{S}$

As already noticed Lemma 1.1.5 does not provide a parametrisation of $\overline{\mathcal{E}_3}$. Since we will need this set now we first try to obtain a set of inequalities that fully characterise this set $\overline{\mathcal{E}_3}$. Let

$$\begin{aligned}\mathcal{E}_3' := \big\{(x,y,z) \in \mathbb{R}^3\big|\ &|x| \le 1 \wedge |y-xz| \le 1-x^2 \\ &\wedge |x+z| \le |y+1| \wedge |y-1| \le 2 \wedge |z| \le 3\big\}\end{aligned} \tag{4.3.2}$$

and consider the intersection of $\mathcal{E}_3'$ with the hyperplane

$$A_c := \big\{(x,y,z) \in \mathbb{R}^3 \mid x - c = 0\big\}$$

for constant c.

Lemma 4.3.6. *For any $|c| < 1$ the intersection of $\mathcal{E}_3'$ with the plane A_c yields the closed triangle $\triangle(A_c^{(1)}, A_c^{(2)}, A_c^{(3)})$ with $A_c^{(1)} = (c,-1,-c), A_c^{(2)} = (c, 1-2c, c-2), A_c^{(3)} = (c, 2c+1, c+2)$.*

Proof. We have

$$\begin{aligned}\mathcal{E}_3' \cap A_c = \big\{(c,y,z) \in \mathbb{R}^3\big|\ &|y-cz| \le 1-c^2 \wedge |c+z| \le |y+1| \\ &\wedge |y-1| \le 2 \wedge |z| \le 3\big\}.\end{aligned}$$

As all inequalities are linear, this is a convex set. It is quickly verified that $A_c^{(1)}, A_c^{(2)}, A_c^{(3)} \in \mathcal{E}_3' \cap A_c$. Thus $\triangle(A_c^{(1)}, A_c^{(2)}, A_c^{(3)}) \subset \mathcal{E}_3' \cap A_c$. On the other hand consider the closed convex set

$$B_c := \big\{(c,y,z) \,\big|\, y - cz \le 1 - c^2 \wedge c + z \le y + 1 \wedge -y-1 \le c+z\big\}\,.$$

Observe that for its definition we used only inequalities that occurred in the definition of $\mathcal{E}_3' \cap A_c$ and hence we have $\mathcal{E}_3' \cap A_c \subset B_c$. Pairwise intersection of the three boundary lines of B_c yields exactly the three points $A_c^{(1)}, A_c^{(2)}, A_c^{(3)}$ and therefore $\triangle(A_c^{(1)}, A_c^{(2)}, A_c^{(3)}) = B_c \supset \mathcal{E}_3' \cap A_c$. □

Theorem 4.3.7. $\overline{\mathcal{E}_3} = \mathcal{E}_3'$.

Proof. Obviously $\mathcal{E}_3'$ is a closed set while $\mathcal{E}_3$ is open. We state that $\operatorname{int} \mathcal{E}_3' = \mathcal{E}_3$. From Lemma 4.3.6 we know

$$\mathcal{E}_3' \cap A_c = \{(c,y,z) \mid y - cz \le 1 - c^2 \wedge c + z \le y + 1 \wedge -y - 1 \le c + z\}$$

and as each point of $\mathcal{E}_3$ is inside $\mathcal{E}_3' \cap A_c$ for some $|c| < 1$ we have

$$\mathcal{E}_3' = \bigcup_{|c|\le 1} (\mathcal{E}_3' \cap A_c) \supset \mathcal{E}_3$$

and therefore

$$\operatorname{int} \mathcal{E}_3' \supset \operatorname{int} \mathcal{E}_3 = \mathcal{E}_3.$$

On the other hand denote by $\operatorname{int}_{A_c}(\mathcal{E}_3' \cap A_c)$ the interior of the set $\mathcal{E}_3' \cap A_c$ (subspace topology) for $|c| < 1$, *i.e.*, the open triangle defined in Lemma 4.3.6, and observe that

$$\operatorname{int} \mathcal{E}_3' = \bigcup_{|c|<1} \operatorname{int}_{A_c}(\mathcal{E}_3' \cap A_c)$$

as we can find a neighbourhood around each point of $\operatorname{int}_{A_c}(\mathcal{E}_3' \cap A_c)$, $|c| < 1$ which is contained in $\mathcal{E}_3'$. Further each point of $\operatorname{int}_{A_c}(\mathcal{E}_3' \cap A_c)$ satisfies the conditions of $\mathcal{E}_3$ whenever $|c| < 1$. Hence

$$\operatorname{int} \mathcal{E}_3' = \bigcup_{|c|<1} \operatorname{int}(\mathcal{E}_3' \cap A_c) \subset \mathcal{E}_3.$$

Thus we have shown that $\operatorname{int} \mathcal{E}_3' = \mathcal{E}_3$.

To prove the theorem we show $\mathcal{E}_3' = \overline{\operatorname{int} \mathcal{E}_3'}$. We already have that $\operatorname{int} \mathcal{E}_3' = \bigcup_{|c|<1} \operatorname{int}_{A_c}(\mathcal{E}_3' \cap A_c)$. Hence we look at the convergent sequences of points contained in $\bigcup_{|c|<1} \operatorname{int}(\mathcal{E}_3' \cap A_c)$. Such a sequence converges either to some point within $\bigcup_{|c|<1}(\mathcal{E}_3' \cap A_c)$ or to some point within one of the sets $\lim_{c\to\pm 1}(\mathcal{E}_3' \cap A_c)$. From Lemma 4.3.6 we already have

$$\mathcal{E}_3' \cap A_c = \triangle((c, -1, -c)(c, 1-2c, c-2), (c, 2c+1, c+2))$$

and we see that

$$\lim_{c\to 1}(\mathcal{E}_3' \cap A_c) = \{(1, \lambda, \lambda) \mid -1 \le \lambda \le 3\},$$
$$\lim_{c\to -1}(\mathcal{E}_3' \cap A_c) = \{(-1, \lambda, -\lambda) \mid -1 \le \lambda \le 3\}$$

which exactly correspond to the sets $(\mathcal{E}_3' \cap A_{\pm 1})$. Thus

$$\overline{\mathcal{E}_3} = \overline{\operatorname{int} \mathcal{E}_3'} = \bigcup_{|c|\le 1} (\mathcal{E}_3' \cap A_c) = \mathcal{E}_3'$$

and we are done. □

We can state (4.3.2) as list of 12 inequalities without using absolutes and apply Algorithm 6 for reduction. This yields that $\overline{\mathcal{E}_3} = X(\bigwedge D)$ for

$$D := \{x + z \le 1 + y, -1 - y \le x + z, y - xz \le 1 - x^2, z \le 3, z \ge -3\}.$$

Let $\mathcal{P}$ be a list of sets of inequalities and G a set of inequalities. We want to verify if $\bigcup_{P\in\mathcal{P}} X(\bigwedge P)$ covers $X(\bigwedge G)$. This is equivalent to

$$X\left(\bigwedge G \wedge \neg \bigvee_{P\in\mathcal{P}} \bigwedge P\right) = \emptyset. \tag{4.3.3}$$

In principle we could do this verification directly. For computational reasons we are a little more restricted. (In fact the direct verification of (4.3.3) overcharges Mathematica®). A verification of a claim of the shape (4.3.3) can be done in a reasonable amount of time if $\#\mathcal{P} \le 3$. We give an algorithm that solves this problem for general $\mathcal{P}$ and G by a subdivision process. The idea is to split the set $X(\bigwedge G)$ into suitable subsets and hope that each of these subsets is covered by a smaller number of sets. First we state Algorithm 7 which removes those sets from $\mathcal{P}$ that do not affect G, hence a set P is removed when $X(\bigwedge G) \cap X(\bigwedge P) = \emptyset$. Denote the application of this algorithm by $\mathrm{RS}(G, \mathcal{P})$ (RS=remove inequalities with respect to a set).

The main algorithm (Algorithm 8) is recursive. As an input we have again $\mathcal{P}$ and G of the usual shape, where $\mathcal{P}$ is reduced by Algorithm 7. Whenever the algorithm recognises that a subset of $X(\bigwedge G)$ is not fully covered by the sets described in $\mathcal{P}$, it returns this subset. Denote the application by $\mathrm{VC}(G, \mathcal{P})$ (VC=verify covering). At first Algorithm 8 checks whether $\#\mathcal{P} \le 3$. If this is the case we can verify whether (4.3.3) holds, otherwise we choose an arbitrary inequality $I \in \bigcup_{P\in\mathcal{P}} P$ such that $X(\bigwedge G \wedge I) \neq X(\bigwedge G)$. There are two possibilities:

Algorithm 7 RS$(G, \mathcal{P})$, removes those lists of inequalities from $\mathcal{P}$ that do not affect a given set G.

Input: G, $\mathcal{P}$
Output: $\mathcal{P}$ reduced list of inequalities
for all sets $P \in \mathcal{P}$ **do**
  if $X(\bigwedge G \wedge \bigwedge P) = \emptyset$ **then**
    $\mathcal{P} \leftarrow \mathcal{P} \setminus \{P\}$
  end if
end for
return$(\mathcal{P})$

- There is such an inequality I. Then $X(\bigwedge G)$ is split by adding I and $\neg I$, respectively, to G and Algorithm 8 is applied (recursively) on both of these subsets. Algorithm 7 is used to possibly reduce $\mathcal{P}$ for each of the subsets. These reduced sets form the second parameter.
- There is no such I. But this would mean that all the points of $X(\bigwedge G)$ suffice all inequalities of $\bigcup_{P \in \mathcal{P}} P$. This is equivalent to $X(\bigwedge G) \subset X(P)$ for any $P \in \mathcal{P}$ and this implies that G and $\mathcal{P}$ suffice the condition (4.3.3).

Now, whenever (4.3.3) is not fulfilled, the set $X(\bigwedge G)$ is not covered by $X(\bigvee_{P \in \mathcal{P}} \bigwedge P)$ and the algorithm returns the set $X(\bigwedge G)$. The application of Algorithm 8 terminates without returning an output if $X(\bigvee_{P \in \mathcal{P}} \bigwedge P)$ covers $X(\bigwedge G)$.

Algorithm 8 VC$(G, \mathcal{P})$, checks if a set is covered by the union of others (recursively).

Input: G, $\mathcal{P}$
Output: subsets of $X(\bigwedge G)$ that are not fully covered by $X(\bigvee_{P \in \mathcal{P}} \bigwedge P)$
if $\#\mathcal{P} \leq 3$ **then**
  if $X(G \wedge \neg \bigvee_{P \in \mathcal{P}} \bigwedge P) \neq \emptyset$ **then**
    return($X(\bigwedge G)$ is not fully covered)
  end if
else
  if $\exists I \in \bigcup_{P \in \mathcal{P}} P : X(\bigwedge G \wedge I) \neq \emptyset$ **then**
    $\mathrm{VC}(\mathrm{RL}(G \cap \{I\}), \mathrm{RS}(G \cap \{I\}, \mathcal{P})$
    $\mathrm{VC}(\mathrm{RL}(G \cap \{\neg I\}), \mathrm{RS}(G \cap \{\neg I\}, \mathcal{P})$
  end if
end if

We can now state the main theorem of this subsection.

Theorem 4.3.8. *The algorithm* VC$(D, \mathcal{P})$ *terminates without yielding any output for*

$$\mathcal{P} = \{Q_1, \ldots, Q_{43}, T_1, \ldots, T_5\}.$$

Proof. The algorithms was implemented in `Mathematica`®. The program is available on the author's homepage [51]. □

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