Shift Radix Systems and Their Generalizations

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DISSERTATION

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AFFIDAVIT

I declare in lieu of oath, that I wrote this thesis and performed the associated research myself, using only literature cited in this volume.

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Acknowledgements

The first of what I consider to be the two biggest mysteries of the universe is the fact that there are absolute truths: dependable, immutable, a priori truths. It is by these truths that we - the mathematicians - are intrigued and motivated. The second of the two biggest mysteries is that of consciousness and conscious experiences. Cosmology, geology, and biology provide a remarkable understanding on how the universe, our solar system and planet earth were formed and how life evolved. On the other hand there is not even the most basic understanding on why or how the biological computers that are our brains work. Why they would create the conscious experience of blue while processing the visual input of a clear sky - or why we would experience curiosity, fascination, and satisfaction in the process of explaining the world. I don't know which, undoubtedly remarkable, laws and mechanisms are responsible for these mysteries, but I am deeply thankful for them! I am thankful to be a part of an ongoing process that began more than thirteen billion years ago and not only placed me in a world full of beauty and wonders, but also equipped me with the ability consciously to enjoy this beauty and to appreciate these wonders!

Working as a natural scientist and as a mathematician in particular is the most wonderful job on earth and I am very thankful that our brilliant Doctoral Program and the Austrian Science Fund gave me the opportunity to do so in the past four years. I feel privileged to have met many great thinkers and wonderful people among my colleagues in Leoben and in the DK and during my research stays in Debrecen, Calgary, Rennes, and Montpellier. I would like to thank my DK mentors Jörg Thuswaldner and Alfred Geroldinger for their helpful advice on many occasions. Also I would like to thank Attila Pethő for several inspiring and stimulating discussions, which motivated a good deal of the material of this thesis. My deepest thanks go to my supervisor Peter Kirschenhofer for his constant support, his patience, and his thorough advice on scientific matters and beyond!

As beautiful and satisfying it can be to work as a mathematician, it does not come without its downsides. Fascinating problems don't stop being fascinating just because the work day is over and it happens more than not that work is brought home. Being a mathematician therefore takes a highly understanding and supportive family and I am more than lucky that mine is both. While I named the two biggest scientific wonders of the universe above, my personal four biggest wonders are the love of my life Elisabeth and our wonderful children Pascal, Shiloh, and Mia, all four of whom I love more than I can tell. You are the meaning of my life and the main subject of this section: What I am most thankful for!

Contents

Conventions

- $\mathbb{N}$ is defined as the set of all positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- $\mathbb{Z}_e$ and $\mathbb{Z}_o$ are the sets of even and odd integers and $\mathbb{N}_e := \mathbb{Z}_e \cap \mathbb{N}$, $\mathbb{N}_o := \mathbb{Z}_o \cap \mathbb{N}$.
- $\mathcal{P}(M)$ denotes the power set of a set M.
- For $x \in \mathbb{R}$, $\lfloor x \rfloor$ (floor) denotes the largest integer less than or equal to x, $\lceil x \rceil$ (ceiling) denotes the smallest integer greater than or equal to x, and $\{x\} := x - \lfloor x \rfloor$ (fractional part). The floor, ceiling, and fractional part of a complex number is obtained by applying the respective function separately to its real and imaginary parts.
- The modulo function % is defined as $a\%b := \{a/b\}\, b$ for all $a \in \mathbb{Z}$, $b \in \mathbb{Z} \setminus \{0\}$, and has precedence over addition and subtraction.
- $\mathbf{0}$ means the zero vector of suitable type and length.
- The image of a vector under a function which naturally accepts only the vectors entries as its arguments is obtained by elementwise application of the function.
- The Pochhammer symbol for $x \in \mathbb{C}$ and $j \in \mathbb{Z}$ is defined as $(x)_j := \prod_{i=0}^{j-1}(x - i)$.
- id_M is the identity map on a set M.
- For a subset M of a topological space, $\mathrm{int}\,(M)$ denotes the interior, $\overline{M}$ the closure, and ∂M the boundary of M.

Introduction

In the present thesis we will be mostly concerned with Shift Radix Systems in different settings and with the Schur-Cohn region and its generalizations. For $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ the mapping

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$$
$$\mathbf{a} = (a_1, \ldots, a_d) \mapsto (a_2, \ldots, a_d, -\lfloor \mathbf{ra} \rfloor)$$

where $\mathbf{ra} = \sum_{i=1}^{d} r_i a_i$ is the scalar product of $\mathbf{r}$ and $\mathbf{a}$, is called the d-dimensional Shift Radix System (SRS for short) associated with $\mathbf{r}$ and $\mathbf{r}$ is called the parameter of $\tau_{\mathbf{r}}$. Shift Radix Systems were introduced by Akiyama, Borbély, Brunotte, Pethő, and Thuswaldner in [**Akiyama et al., 2005**] to generalize two important notions of number systems: β-expansions and Canonical Number Systems. Since their introduction Shift Radix System found great interest for their own sake and were subject of many publications (cf. e.g. [**Kirschenhofer and Thuswaldner, 2014**] for a recent survey).

In this thesis new algorithms, characterization results, and topological results related to Shift Radix Systems as well as results on the Lebesgue measure of a generalized Schur-Cohn region are presented. Furthermore it includes related results on a special type of multiple integral due to Selberg and Aomoto. The material has appeared or will appear in parts in the following papers:

- Characterization algorithms for shift radix systems with finiteness property [**Weitzer, 2015a**] (cf. Chapter 3 and Chapter 4)
- On the characterization of Pethő's Loudspeaker [**Weitzer, 2015b**] (cf. Chapter 5)
- A number theoretic problem on the distribution of polynomials with bounded roots [**Kirschenhofer and Weitzer, 2015**] (cf. Chapter 2)

Furthermore, joint work with Attila Pethő and Peter Varga is currently in preparation:

- [**Pethő et al., IP**] (cf. Chapter 6)

Additional results that have not been published outside of this thesis are marked by (**Weitzer**) to clearly emphasize the author's original contributions.

The thesis comes with a CD which contains annotated versions of the C++ program which computed the results presented in Chapter 4. The content of the CD can also be found at:

`http://institute.unileoben.ac.at/mathstat/personal/weitzer.htm`

In the following the six chapters of this thesis will be introduced.

Chapter 1. Selberg and Aomoto integrals

In this chapter certain generalizations of Euler's beta function known as Selberg and Aomoto integrals are introduced and generalized. In the most general form considered by Aomoto these multiple integrals involve an arbitrary number of specific polynomial factors of degree two or less. The original formulas given by Selberg and Aomoto are presented (cf. Section 1.2) and generalized to allow polynomial factors of degree up to and including four (cf. Section 1.3, in particular Theorem 1.3.4). The results are achieved by adaptation of Aomoto's original method of partial derivation and integration by one of the variables to find a recurrence relation for the respective integral, which can then be solved. The formulas, while interesting for their own sake, are needed in Chapter 2 to compute the volumes of parts of a certain subdivision of the Schur-Cohn region, which is treated there.

Chapter 2. The Schur-Cohn region and its generalizations

The d-dimensional, real Schur-Cohn region $\mathcal{E}_d^{(\mathbb{R})}$ is defined as the set of all d-dimensional, real coefficient vectors (constant term first), the corresponding polynomial of which is contractive (i.e. all of its roots lie in the open complex unit disk). Next to many applications in science and engineering problems (cf. beginning of Section 2.1) the Schur-Cohn region is also intimately related to an important dynamical property of Shift Radix Systems (cf. Chapter 3). In this chapter a recent conjecture of Akiyama and Pethő on the volumes of parts of a certain subdivision is considered and proved for the instance $s = 1$ (cf. Section 2.6): The parts $\mathcal{E}_{d,s}^{(\mathbb{R})}$ of this subdivision contain those elements of $\mathcal{E}_d^{(\mathbb{R})}$ the corresponding polynomials of which have exactly s pairs of complex conjugate roots. Akiyama and Pethő proved the surprising fact that these sets $\mathcal{E}_{d,s}^{(\mathbb{R})}$ have a rational Lebesgue measure $v_d^{(s)}$ (cf. Table 1 in Section 2.5 and Theorem 2.5.5). Based on numerical evidence (cf. Table 2 in Section 2.5 and Conjecture 2.5.7) they furthermore formulated the even more surprising conjecture that the quotient of $v_d^{(s)}$ and $v_d^{(0)}$ is always an integer. We prove this conjecture for the special case of $s = 1$. The result is achieved by a series of transformations of combinatorial sums originating from an involved integral formula for $v_d^{(1)}$ given by Akiyama and Pethő. This treatment, which also includes techniques of hypergeometric summation leads to the remarkably simple result

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{P_d(3) - 2d - 1}{4}$$

where $P_d(x)$ are the Legendre polynomials (cf. Theorem 2.6.1). As a byproduct we are able to analyze the asymptotic behavior of the above quotients for $d \to \infty$ (cf. Corollary 2.6.3). Furthermore we derive a formula for $v_d^{(2)}$ (cf. Theorem 2.6.6) by applying the generalization of Selberg's and Aomoto' original integral formulas found in Chapter 1. Unfortunately this formula could not be simplified as far as to prove the conjecture by Akyiama and Pethő for $s = 2$.

Chapter 3. Shift Radix Systems and the finiteness property

This chapter summarizes well-known results on Shift Radix Systems needed in the subsequent chapters. Let $\mathcal{D}_d$ consist of all parameters in $\mathbb{R}^d$ for which all orbits of the corresponding Shift Radix System are ultimately periodic, and let $\mathcal{D}_d^{(0)}$ consist of all parameters in $\mathbb{R}^d$ for which all orbits of the corresponding Shift Radix System end up in $\mathbf{0}$.

For the characterization of $\mathcal{D}_d^{(0)}$ two important tools are known to be of importance: Cutout polyhedra (cf. Section 3.5) and sets of witnesses (cf. Section 3.6). Cutout polyhedra allow a characterization of $\mathcal{D}_d^{(0)}$ in terms of $\mathcal{D}_d$ by "cutting out" certain regions (which can be shown to be of polyhedral shape) which consist of all parameters the corresponding Shift Radix System of which admits a given cycle. While in the interior of $\mathcal{D}_d$ a finite number of cutout polyhedra always suffices to completely characterize $\mathcal{D}_d^{(0)}$ (cf. Theorem 3.5.6), on the boundary of $\mathcal{D}_d$ this needs not to be the case. Due to the existence of critical points, $\mathcal{D}_d^{(0)}$ has a very complicated structure even for $d = 2$. We will identify six infinite families of cycles for $d = 2$ the corresponding cutout polygons of which are threaded on a line segment on the boundary of $\mathcal{D}_2$ and which tend to either of the two existing critical points of $\mathcal{D}_2^{(0)}$ (cf. Section 3.5, in particular Theorem 3.5.10). This generalizes results by Surer, who already found two of these families.

In order to prove that a parameter (or a region of parameters) belongs to $\mathcal{D}_d^{(0)}$ so-called sets of witnesses can be used, which form the basis of what is known as Brunotte's algorithm. They are subsets of $\mathbb{Z}^d$ constructed in such a way that they contain all cycles that might be admitted by the Shift Radix System of the given parameter (or any of the given paramters). If the set of witnesses is finite this provides a method to test for the finiteness property in finite time. The sets of witnesses found by Brunotte's algorithm for regions need not to be finite, but at least under mild conditions (positive distance from the boundary of $\mathcal{D}_d$) we show that there exists a finite partition of the given region where the associated set of witnesses of each part is finite (cf. Theorem 3.6.9).

Chapter 4. New algorithms and topological results

Two new algorithms which allow the characterization of Shift Radix Systems with finiteness property in a given region are presented here. Even if a finite set of witnesses for a region of parameters is found by Brunotte's algorithm for regions, the computation of all cutout polyhedra corresponding to the occurring cycles can be very time consuming. The more general "graphs of witnesses" overcome this problem (cf. Section 4.1). Just as for cycles and their corresponding cutout polyhedra it is possible to compute the set of all parameters the corresponding Shift Radix System of which admits a given graph of witnesses. And just as for cutout polyhedra it can be proven that these sets have a very simple geometric structure: They are the intersection of a nondegenerate, open, convex polyhedron and an affine subspace of $\mathbb{R}^d$ (cf. Lemma 4.1.4). In duality to cutout polyhedra which characterize $\mathcal{D}_d^{(0)}$ by being subtracted from $\mathcal{D}_d$, the sets corresponding to graphs of witnesses characterize $\mathcal{D}_d^{(0)}$ in terms of their disjoint union (cf. Lemma 4.1.5). This fact forms the basis of our Algorithm 1 which takes as input a convex hull of finitely many points in $\mathcal{D}_d$ which is completely contained in the interior of $\mathcal{D}_d$ and outputs the intersection of $\mathcal{D}_d^{(0)}$ and the given convex hull. Algorithm 1 is guaranteed to terminate for all inputs (cf. Theorem 4.2.1) whereas Brunotte's algorithm for regions needs not do so. Also Algorithm 1 turned out to be considerably faster in all applications and there are heuristic reasons to believe that it is faster in general (cf. Section 4.2).

We furthermore present a second algorithm (Algorithm 2) which again performed much faster than Algorithm 1 in all applications (cf. Section 4.3 and Section 4.4). By its nature Algorithm 1 computes a decomposition of its input (a convex region of parameters) into finitely many disjoint polyhedra (from which it selects those which are contained in $\mathcal{D}_d^{(0)}$ in the final step). Algorithm 2 computes a refinement of this decomposition which is given by the classes of a certain equivalence relation (cf. Definition 4.3.1). It can be shown that there is a geometric interpretation of this equivalence relation which allows a fast computation of a complete list of its classes (cf. Theorem 4.3.3). If the set of equivalence classes is known one can use Brunotte's algorithm to decide whether or not a given class belongs to $\mathcal{D}_d^{(0)}$. The definition of the equivalence relation guarantees that the result will be the same for all parameters in the class. But instead of treating all classes independently and in random order, decisive optimizations can be made to speed up the process considerably (cf. Theorem 4.3.5). The output of Algorithm 2 is a minimal set (with respect to set inclusion but not necessarily cardinality) of cutout polyhedra which characterizes $\mathcal{D}_d^{(0)}$ inside of the given convex hull. Using this new algorithm a region of $\mathcal{D}_2^{(0)}$ which is considerably larger than those in previous results could be characterized (cf. Theorem 4.5.1). A careful analysis of the characterized region also settled two previously open questions on the topology of $\mathcal{D}_2^{(0)}$: It is shown that it is disconnected and that the largest connected component has a non-trivial fundamental group (cf. Corollary 4.5.2).

Chapter 5. Gaussian Shift Radix Systems and Pethő's Loudspeaker

In this chapter a generalization of Shift Radix Systems to Gaussian integers due to Brunotte, Kirschenhofer, and Thuswaldner is considered. Most concepts of Shift Radix Systems translate to the complex case (cf. Section 5.1), including the sets $\mathcal{D}_d$ and $\mathcal{D}_d^{(0)}$ which are denoted by $\mathcal{G}_d$ and $\mathcal{G}_d^{(0)}$ respectively. Our main interest again lies in these two sets. A conjecture on $\mathcal{G}_1^{(0)}$ (which, in honor of Attila Pethő and because of its shape, is called Pethő's Loudspeaker) is formulated (cf. Section 5.2, in particular Conjecture 5.2.2) and proved in parts. By identification of certain infinite families of cutout polygons it is shown that $\mathcal{G}_1^{(0)}$ is contained in the conjectured characterizing set $\mathcal{G}_C$ (cf. Section 5.3, in particular Theorem 5.3.1). With respect to the other inclusion partial results could be achieved using the complex analogues of Algorithm 1 and Algorithm 2 from Chapter 4 (cf. Section 5.5, in particular Theorem 5.5.1). In addition to these computational results further steps towards a proof of $\mathcal{G}_C \subseteq \mathcal{G}_1^{(0)}$ were taken. This led to two stronger conjectures on the behavior of cycles and a partial result (cf. Theorem 5.5.7).

The question on critical points and the related weakly critical points is completely settled for $\mathcal{G}_1^{(0)}$ (cf. Section 5.4). Furthermore the perimeter and the area of $\mathcal{G}_C$ are computed (cf. Section 5.6). Finally a kind of "self-similarity" of $\mathcal{G}_1^{(0)}$ that is revealed by the complex analogue of Algorithm 1 introduced in Chapter 4 is explained (cf. Section 5.7).

Chapter 6. Shift Radix Systems over imaginary quadratic Euclidean domains

Very recently Pethő and Varga considered a generalization of Shift Radix Systems for imaginary quadratic Euclidean domains. The key to this generalization is the definition of a floor function $\lfloor r \rfloor_D$ on $\mathbb{E}_D := \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$ for $D \in \{-1,-2,-3,-7,-11\}$. If such a floor function is fixed, Shift Radix Systems can be defined as usual and several notions of real Shift Radix Systems translate to the new situation as expected (cf. Section 6.2). The sets which correspond to $\mathcal{D}_d$ and $\mathcal{D}_d^{(0)}$ are denoted by $\mathcal{F}_{D,d}$ and $\mathcal{F}_{D,d}^{(0)}$ respectively. Surprisingly, depending on D, the sets $\mathcal{F}_{D,d}^{(0)}$ seem to differ in terms of their complexity. While $\mathcal{F}_{D,1}^{(0)}$ appears to have at least weakly critical points for $D \in \{-1,-3,-7\}$, it is shown that $\mathcal{F}_{-2,1}^{(0)}$ and $\mathcal{F}_{-11,1}^{(0)}$ have none (cf. Section 6.3, in particular Theorem 6.3.2). The floor functions which are used to define Shift Radix Systems in the five Euclidean cases can also be applied to define Shift Radix Systems if the discriminant D is any real number in a certain interval.

CHAPTER 1

Selberg and Aomoto integrals

1.1. Introduction and definitions

This chapter contains results on special types of integrals known as Selberg and Aomoto integrals which are generalizations of Euler's beta function. The formulas derived in this chapter will be needed in order to compute the volumes of a generalization of the so-called Schur-Cohn region due to Akyiama and Pethő which is the main subject of Chapter 2. The definition and original formulas on Selberg and Aomoto integrals derived in [**Selberg, 1944**] and [**Aomoto, 1987**] as well as generalizations from [**Andrews et al., 1999**] will be given in this chapter. Furthermore we will adapt the methods used in [**Andrews et al., 1999**] to generalize Aomoto integrals to contain polynomial factors of degrees higher than two. A survey on Selberg and Aomoto integrals can be found in [**Forrester and Warnaar, 2008**]. For the whole chapter let $C_n := [0,1]^n$, $\mathbf{x} := (x_1, \ldots, x_n)$, and $d\mathbf{x} := dx_1 \cdots dx_n$ for all $n \in \mathbb{N}$.

Definition 1.1.1. [**Andrews et al., 1999**] *For $n \in \mathbb{N}$, $\alpha, \beta, \gamma \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $\Re(\gamma) > -\min\{1/n, \Re(\alpha)/(n-1), \Re(\beta)/(n-1)\}$ we define*

$$\begin{aligned} \omega_n(\alpha,\beta,\gamma,\mathbf{x}) &:= \prod_{i=1}^{n} \left(x_i^{\alpha-1}(1-x_i)^{\beta-1}\right) \prod_{i=1}^{n}\prod_{j=i+1}^{n} |x_i - x_j|^{2\gamma} \\ S_n(\alpha,\beta,\gamma) &:= \int_{C_n} \omega_n(\alpha,\beta,\gamma,\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Any integral of this form shall be denoted as Selberg integral. Furthermore, for $k, l, m \in [\![0, n]\!]$ with $m \le k, l$ we define the following generalizations known as Aomoto integrals:

$$\begin{aligned} A_{n,k}(\alpha,\beta,\gamma) &:= \int_{C_n} \prod_{i=1}^{k} x_i \omega_n(\alpha,\beta,\gamma,\mathbf{x}) d\mathbf{x} \\ A_{n,k,l}(\alpha,\beta,\gamma) &:= \int_{C_n} \prod_{i=1}^{k} x_i \prod_{i=k+1}^{k+l} (1-x_i) \omega_n(\alpha,\beta,\gamma,\mathbf{x}) d\mathbf{x} \qquad (k+l \le n) \\ A_{n,k,l,m}(\alpha,\beta,\gamma) &:= \int_{C_n} \prod_{i=1}^{k} x_i \prod_{i=k+1-m}^{k+l-m} (1-x_i) \omega_n(\alpha,\beta,\gamma,\mathbf{x}) d\mathbf{x} \qquad (k+l-m \le n). \end{aligned}$$

It is clear that the different integrals in the previous definition increase in generality. While the original Selberg integrals only contain n factors of the form $\left(x_i^{\alpha-1}(1-x_i)^{\beta-1}\right)$, the Aomoto integrals gradually introduce additional factors of the form x_i, $(1-x_i)$, and $x_i(1-x_i)$ the numbers of which are given by $k-m$, $l-m$, and m respectively.

1.2. Known results

The Selberg integrals were introduced in [**Selberg, 1944**] along with the following closed formula.

Theorem 1.2.1. [**Selberg, 1944**] *Let n, α, β, γ be as in Definition 1.1.1. Then*

$$S_n(\alpha,\beta,\gamma) = \prod_{i=1}^{n} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma+1)}{\Gamma(\alpha+\beta+(n+i-2)\gamma)\Gamma(\gamma+1)}.$$

The original proof by Selberg was quite complicated but two easier and essentially different proofs have been found more than 40 years later. The first one is given in [**Anderson, 1991**] and the second one is due to Aomoto who actually proved the more general formula:

THEOREM 1.2.2. [**Aomoto, 1987**] *Let $n, k, \alpha, \beta, \gamma$ be as in Definition 1.1.1. Then*

$$A_{n,k}(\alpha,\beta,\gamma) = \prod_{i=1}^{k} \frac{\alpha+(n-i)\gamma}{\alpha+\beta+(2n-i-1)\gamma} S_n(\alpha,\beta,\gamma).$$
$$= \prod_{i=1}^{k} \frac{\alpha+(n-i)\gamma}{\alpha+\beta+(2n-i-1)\gamma} \prod_{i=1}^{n} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma+1)}{\Gamma(\alpha+\beta+(n+i-2)\gamma)\Gamma(\gamma+1)}$$

The formula found by Aomoto generalizes Selberg's formula by introducing additional factors of the form x_i. A natural question to ask is whether it can be generalized even further by adding more types of factors. Indeed it can as the following theorem shows.

THEOREM 1.2.3. [**Andrews et al., 1999**] *Let $n, k, l, m, \alpha, \beta, \gamma$ be as in Definition 1.1.1. Then*

(i) $$A_{n,k,l}(\alpha,\beta,\gamma) = \frac{\prod_{i=1}^{k}(\alpha+(n-i)\gamma)\prod_{i=1}^{l}(\beta+(n-i)\gamma)}{\prod_{i=1}^{k+l}(\alpha+\beta+(2n-i-1)\gamma)} S_n(\alpha,\beta,\gamma)$$

(ii) $$A_{n,k,l,m}(\alpha,\beta,\gamma) = \prod_{i=1}^{m} \frac{\alpha+\beta+(n-i-1)\gamma}{\alpha+\beta+(2n-i-1)\gamma+1}$$
$$\frac{\prod_{i=1}^{k}(\alpha+(n-i)\gamma)\prod_{i=1}^{l}(\beta+(n-i)\gamma)}{\prod_{i=1}^{k+l}(\alpha+\beta+(2n-i-1)\gamma)} S_n(\alpha,\beta,\gamma)$$

1.3. Generalizations

In the previous section we presented known generalizations of Aomotos original integral by introducing additional factors of the form x_i and $(1-x_i)$ and also allowed these factors to overlap. Another interpretation of the overlapping would be that we introduced new factors of the forms x_i, $(1-x_i)$, and $x_i(1-x_i)$ but disallowed overlappings. This last interpretation will fit our purposes better and so we will further generalize Aomoto's original integral by introducing new factors (i.e. polynomial factors of higher degree) but don't allow them to overlap. Furthermore we will only consider the case where $\alpha = \beta = 1$ and $\gamma = \frac{1}{2}$ which simplifies $\omega_n(\alpha, \beta, \gamma, \mathbf{x})$ to

$$\omega_n(\mathbf{x}) \ := \ \omega_n(1,1,1/2,\mathbf{x}) = \prod_{i=1}^{n}\prod_{j=i+1}^{n} |x_i - x_j|$$

One question we need to answer is of which form our additional factors should be. We will postpone this question for now and assume general polynomials with one restriction: They have to be multiples of $x(1-x)$. This limitation will allow us to apply a strategy of partial derivative and integration by one of the x_i to derive recurrence relations for the integral, which is also used in the proofs of Theorem 1.2.2 and Theorem 1.2.3 as given in [**Andrews et al., 1999**]. Our general integrals are given in the following definition.

DEFINITION 1.3.1. *For $m \in \mathbb{N}_0$, $\mathcal{P} = (P^{(1)}(x), \ldots, P^{(k)}(x)) \in \{x\} \times \{x(1-x)\} \times (x(1-x)\mathbb{R}[x])^{k-2}$ with $m \le k \in \mathbb{N}_0 \cup \{\infty\}$, and $\mathcal{D} = (d_0, \ldots, d_l) \in \mathbb{N}_0^{l+1}$ with $m \le l \in \mathbb{N}_0 \cup \{\infty\}$ and $\sum_{i=1}^{l} d_i \le d_0$ let*

$$I_m(\mathcal{P},\mathcal{D}) \ := \int_{C_{d_0}} \prod_{i=1}^{m} \prod_{j=1+\sum_{k=1}^{i-1} d_k}^{\sum_{k=1}^{i} d_k} P^{(i)}(x_j)\omega_{d_0}(\mathbf{x})d\mathbf{x}$$

Note that the number of variables is given by d_0 and the multiples of the factors are given by $d_1, \ldots, d_l$. The notation allows the vector of polynomials $\mathcal{P}$ (size k) and the vector of multiples $\mathcal{D}$ (size $l+1$, where the 0th entry is d_0) to be of larger size than the number of factors actually in use (m). The following lemma summarizes the relation between this new type of integrals and Aomoto integrals from the previous section.

LEMMA 1.3.2. (**Weitzer**)

(i) $S_{d_0}(1,1,1/2) = I_0((),(d_0))$

(ii) $A_{d_0,d_1}(1,1,1/2) = I_1((x),(d_0,d_1))$

(iii) $A_{d_0,d_1,d_2}(1,1,1/2) = I_2((x,1-x),(d_0,d_1,d_2))$

(iv) $A_{d_0,d_1+d_2,d_2,d_2}(1,1,1/2) = I_2((x,x(1-x)),(d_0,d_1,d_2))$

(v) $A_{d_0,d_1+d_3,d_2+d_3,d_3}(1,1,1/2) = I_3((x,1-x,x(1-x)),(d_0,d_1,d_2,d_3))$.

PROOF. Immediate from the definitions. □

Next we will establish recurrence relations for some integrals of the new type.

LEMMA 1.3.3. (**Weitzer**) *Assume the notions used in Definition 1.3.1 above and let* $P^{(i)}(x) = x(1-x)\sum_{j=0}^{i-2} a_j^{(i)} x^j$ *with* $\deg P^{(i)} = i$ *for all* $i \in [\![3,k]\!]$. *Furthermore let* $\mathcal{D}' = (d'_1,\ldots,d'_{l'}) \in \mathbb{Z}^{l'}$ *with* $m \le l' \in \mathbb{N}\cup\{\infty\}$, $l' \le l$, $d_i + d'_i \ge 0$ *for all* $i \in [\![1,l']\!]$, *and* $\sum_{i=1}^{l'}(d_i+d'_i) \le d_0$. *Then* $J_m(\mathcal{D}') := I_m(\mathcal{P},(d_0,d_1+d'_1,\ldots,d_l+d'_{l'}))$ *satisfies the recurrence relations (for suitable bounds):*

$$
\begin{aligned}
&J_2(\ \ 0,-1)(1(d_0-d_1-d_2+2))+\\
&J_2(\ \ 1,-1)(1(-2d_0+d_1+2d_2-4))=0
\end{aligned}
$$

$$
\begin{aligned}
&J_3(\ \ 0,\ \ 0,-1)(a_0^{(3)}(d_0-d_1-d_2-d_3+2))+\\
&J_3(\ \ 0,\ \ 1,-1)(a_1^{(3)}(2d_0-d_2-2d_3+6))+\\
&J_3(\ \ 1,\ \ 0,-1)(a_0^{(3)}(-2d_0+d_1+2d_2+2d_3-4)+a_1^{(3)}(-d_1-2))+\\
&J_3(\ \ 2,\ \ 0,-1)(a_1^{(3)}(-d_0+d_1+d_2+d_3))=0
\end{aligned}
$$

$$
\begin{aligned}
&J_4(-1,\ \ 2,\ \ 0,-1)(a_1^{(3)}a_2^{(4)}(-d_1))+\\
&J_4(\ \ 0,\ \ 0,\ \ 0,-1)(a_1^{(3)}a_0^{(4)}(d_0-d_1-d_2-d_3-d_4+2))+\\
&J_4(\ \ 0,\ \ 0,\ \ 1,-1)(a_2^{(4)}(2d_0-d_3-2d_4+8))+\\
&J_4(\ \ 0,\ \ 1,\ \ 0,-1)(a_0^{(3)}a_2^{(4)}(-2d_0+2d_3+2d_4-8)+a_1^{(3)}a_1^{(4)}(2d_0-d_2-2d_3-2d_4+6)+\\
&\qquad\qquad a_1^{(3)}a_2^{(4)}(2d_1+2))+\\
&J_4(\ \ 0,\ \ 2,-1,-1)(((a_0^{(3)})^2a_2^{(4)}+(a_1^{(3)})^2a_0^{(4)})(-d_3)+a_0^{(3)}a_1^{(3)}a_1^{(4)}(d_3))+\\
&J_4(\ \ 1,\ \ 0,\ \ 0,-1)(a_1^{(3)}a_0^{(4)}(-2d_0+d_1+2d_2+2d_3+2d_4-4)+(a_1^{(3)}a_1^{(4)}+a_1^{(3)}a_2^{(4)})(-d_1-2))+\\
&J_4(\ \ 1,\ \ 1,\ \ 0,-1)(a_1^{(3)}a_2^{(4)}(2d_0-2d_1-2d_2-2d_3-2d_4))+\\
&J_4(\ \ 2,\ \ 0,\ \ 0,-1)((a_1^{(3)}a_1^{(4)}+a_1^{(3)}a_2^{(4)})(-d_0+d_1+d_2+d_3+d_4))=0
\end{aligned}
$$

$$
\begin{aligned}
&J_5(-1,\ \ 1,\ \ 1,\ \ 0,-1)(a_1^{(3)}a_2^{(4)}a_3^{(5)}(-2d_1))+\\
&J_5(-1,\ \ 2,\ \ 0,\ \ 0,-1)(a_0^{(3)}a_1^{(3)}a_2^{(4)}a_3^{(5)}(2d_1)+((a_1^{(3)})^2a_2^{(4)}a_2^{(5)}+(a_1^{(3)})^2a_2^{(4)}a_3^{(5)})(-d_1))+\\
&J_5(\ \ 0,-1,\ \ 2,\ \ 0,-1)(a_2^{(4)}a_3^{(5)}(d_2))+\\
&J_5(\ \ 0,\ \ 0,\ \ 0,\ \ 0,-1)((a_1^{(3)})^2a_2^{(4)}a_0^{(5)}(d_0-d_1-d_2-d_3-d_4-d_5+2))+\\
&J_5(\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1)((a_1^{(3)})^2a_3^{(5)}(2d_0-d_4-2d_5+10))+\\
&J_5(\ \ 0,\ \ 0,\ \ 1,\ \ 0,-1)(a_0^{(3)}a_2^{(4)}a_3^{(5)}(d_3-2d_2)+a_1^{(3)}a_1^{(4)}a_3^{(5)}(-2d_0+2d_4+2d_5-10)+\\
&\qquad\qquad a_1^{(3)}a_2^{(4)}a_2^{(5)}(2d_0-d_3-2d_4-2d_5+8)+a_1^{(3)}a_2^{(4)}a_3^{(5)}(2d_1+2))+\\
&J_5(\ \ 0,\ \ 0,\ \ 2,-1,-1)((a_0^{(4)}a_2^{(4)}a_3^{(5)}+a_1^{(4)}a_2^{(4)}a_2^{(5)})(d_4)+((a_1^{(4)})^2a_3^{(5)}+(a_2^{(4)})^2a_1^{(5)})(-d_4))+
\end{aligned}
$$

$$
\begin{aligned}
&J_5(\ 0,\ \ 1,\ \ 0,\ \ 0,-1)((a_0^{(3)})^2a_2^{(4)}a_3^{(5)}(d_2-2d_3)+a_0^{(3)}a_1^{(3)}a_1^{(4)}a_3^{(5)}(2d_0-2d_4-2d_5+10)+\\
&\qquad a_0^{(3)}a_1^{(3)}a_2^{(4)}a_2^{(5)}(-2d_0+2d_3+2d_4+2d_5-8)+\\
&\qquad a_0^{(3)}a_1^{(3)}a_2^{(4)}a_3^{(5)}(-2d_1-2)+(a_1^{(3)})^2a_0^{(4)}a_3^{(5)}(-2d_0+2d_4+2d_5-10)+\\
&\qquad (a_1^{(3)})^2a_2^{(4)}a_1^{(5)}(2d_0-d_2-2d_3-2d_4-2d_5+6)+\\
&\qquad ((a_1^{(3)})^2a_2^{(4)}a_2^{(5)}+(a_1^{(3)})^2a_2^{(4)}a_3^{(5)})(2d_1+2))+\\
&J_5(\ 0,\ \ 1,\ \ 1,-1,-1)((a_0^{(3)}a_0^{(4)}a_2^{(4)}a_3^{(5)}+a_0^{(3)}a_1^{(4)}a_2^{(4)}a_2^{(5)}+a_1^{(3)}a_0^{(4)}a_1^{(4)}a_3^{(5)}+\\
&\qquad a_1^{(3)}(a_2^{(4)})^2a_0^{(5)})(-2d_4)+\\
&\qquad (a_0^{(3)}(a_1^{(4)})^2a_3^{(5)}+a_0^{(3)}(a_2^{(4)})^2a_1^{(5)}+\\
&\qquad a_1^{(3)}a_0^{(4)}a_2^{(4)}a_2^{(5)})(2d_4))+\\
&J_5(\ 0,\ \ 2,-1,\ \ 0,-1)(((a_0^{(3)})^2a_1^{(3)}a_2^{(4)}a_2^{(5)}+(a_1^{(3)})^3a_2^{(4)}a_0^{(5)})(-d_3)+((a_0^{(3)})^3a_2^{(4)}a_3^{(5)}+\\
&\qquad a_0^{(3)}(a_1^{(3)})^2a_2^{(4)}a_1^{(5)})(d_3))+\\
&J_5(\ 0,\ \ 2,\ \ 0,-1,-1)(((a_0^{(3)})^2a_0^{(4)}a_2^{(4)}a_3^{(5)}+(a_0^{(3)})^2a_1^{(4)}a_2^{(4)}a_2^{(5)}+(a_1^{(3)})^2a_0^{(4)}a_2^{(4)}a_1^{(5)})(d_4)+\\
&\qquad ((a_0^{(3)})^2(a_1^{(4)})^2a_3^{(5)}+(a_0^{(3)})^2(a_2^{(4)})^2a_1^{(5)}+(a_1^{(3)})^2(a_0^{(4)})^2a_3^{(5)}+\\
&\qquad (a_1^{(3)})^2a_1^{(4)}a_2^{(4)}a_0^{(5)})(-d_4)+\\
&\qquad (a_0^{(3)}a_1^{(3)}a_0^{(4)}a_1^{(4)}a_3^{(5)}+a_0^{(3)}a_1^{(3)}(a_2^{(4)})^2a_0^{(5)})(2d_4)+\\
&\qquad a_0^{(3)}a_1^{(3)}a_0^{(4)}a_2^{(4)}a_2^{(5)}(-2d_4))+\\
&J_5(\ 0,\ \ 2,\ \ 0,\ \ 0,-1)((a_1^{(3)})^2a_2^{(4)}a_3^{(5)}(-d_0+d_1+d_2+d_3+d_4+d_5))+\\
&J_5(\ 1,\ \ 0,\ \ 0,\ \ 0,-1)((a_1^{(3)})^2a_2^{(4)}a_0^{(5)}(-2d_0+d_1+2d_2+2d_3+2d_4+2d_5-4)+\\
&\qquad ((a_1^{(3)})^2a_2^{(4)}a_1^{(5)}+(a_1^{(3)})^2a_2^{(4)}a_2^{(5)}+(a_1^{(3)})^2a_2^{(4)}a_3^{(5)})(-d_1-2))+\\
&J_5(\ 1,\ \ 0,\ \ 1,\ \ 0,-1)(a_1^{(3)}a_2^{(4)}a_3^{(5)}(2d_0-2d_1-2d_2-2d_3-2d_4-2d_5))+\\
&J_5(\ 1,\ \ 1,\ \ 0,\ \ 0,-1)(a_0^{(3)}a_1^{(3)}a_2^{(4)}a_3^{(5)}(-2d_0+2d_1+2d_2+2d_3+2d_4+2d_5)+\\
&\qquad ((a_1^{(3)})^2a_2^{(4)}a_2^{(5)}+(a_1^{(3)})^2a_2^{(4)}a_3^{(5)})(2d_0-2d_1-2d_2-2d_3-2d_4-2d_5))+\\
&J_5(\ 2,\ \ 0,\ \ 0,\ \ 0,-1)(((a_1^{(3)})^2a_2^{(4)}a_1^{(5)}+(a_1^{(3)})^2a_2^{(4)}a_2^{(5)}+(a_1^{(3)})^2a_2^{(4)}a_3^{(5)})\\
&\qquad (-d_0+d_1+d_2+d_3+d_4+d_5))=0
\end{aligned}
$$

Proof. At first we will show the claimed recurrence relation for J_2 by adapting a method of partial derivation and integration by one of the x_i that was used in the proofs of Theorem 1.2.2 and Theorem 1.2.3 as given in [**Andrews et al., 1999**]. For that we set

$$
I(a,b) \ :=\ I_2(\mathcal{P},(d_0,a,b))=\int_{C_{d_0}}\prod_{i=1}^{b}x_i(1-x_i)\prod_{i=b+1}^{b+a}x_i\omega_{d_0}(\mathbf{x})d\mathbf{x}
$$

for all $a,b\in\mathbb{N}_0$ with $a+b\le d_0$ and start with proving the following auxiliary statement:

$$
\begin{aligned}
J \ :=\ &\int_{C_{d_0}}\frac{1}{x_1-x_j}\prod_{i=1}^{d_2}x_i(1-x_i)\prod_{i=d_2+1}^{d_2+d_1}x_i\omega_{d_0}(\mathbf{x})d\mathbf{x}\\
=\ &\begin{cases}0 & \text{if } j\in[\![1,d_2]\!]\\ -\frac{1}{2}I(d_1+1,d_2-1) & \text{if } j\in[\![d_2+1,d_2+d_1]\!]\\ -I(d_1+1,d_2-1)+\frac{1}{2}I(d_1,d_2-1) & \text{if } j\in[\![d_2+d_1+1,d_0]\!]\end{cases}
\end{aligned}
$$

Note that for symmetry reasons the value of the integral will not change if x_1 and x_j are interchanged.

If $j \in [\![1, d_2]\!]$ we have

$$\begin{aligned}
J &= \int_{C_{d_0}} \frac{x_1(1-x_1)x_j(1-x_j)}{x_1 - x_j} \prod_{i \in [\![2,d_2]\!] \setminus \{j\}} x_i(1-x_i) \prod_{i=d_2+1}^{d_2+d_1} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&= \int_{C_{d_0}} \frac{x_j(1-x_j)x_1(1-x_1)}{x_j - x_1} \prod_{i \in [\![2,d_2]\!] \setminus \{j\}} x_i(1-x_i) \prod_{i=d_2+1}^{d_2+d_1} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&= -J.
\end{aligned}$$

If $j \in [\![d_2+1, d_2+d_1]\!]$ we have

$$\begin{aligned}
J &= \int_{C_{d_0}} \frac{x_1(1-x_1)x_j}{x_1 - x_j} \prod_{i=2}^{d_2} x_i(1-x_i) \prod_{i \in [\![d_2+1,d_2+d_1]\!] \setminus \{j\}} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&= \int_{C_{d_0}} \frac{x_j(1-x_j)x_1}{x_j - x_1} \prod_{i=2}^{d_2} x_i(1-x_i) \prod_{i \in [\![d_2+1,d_2+d_1]\!] \setminus \{j\}} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&= \int_{C_{d_0}} \left(-\frac{x_1(1-x_1)x_j}{x_1 - x_j} - x_1 x_j \right) \prod_{i=2}^{d_2} x_i(1-x_i) \prod_{i \in [\![d_2+1,d_2+d_1]\!] \setminus \{j\}} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&= -J - I(d_1+1, d_2-1).
\end{aligned}$$

If $j \in [\![d_2+d_1+1, d_0]\!]$ we have

$$\begin{aligned}
J &= \int_{C_{d_0}} \frac{x_1(1-x_1)}{x_1 - x_j} \prod_{i=2}^{d_2} x_i(1-x_i) \prod_{i=d_2+1}^{d_2+d_1} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&= \int_{C_{d_0}} \frac{x_j(1-x_j)}{x_j - x_1} \prod_{i=2}^{d_2} x_i(1-x_i) \prod_{i=d_2+1}^{d_2+d_1} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&= \int_{C_{d_0}} \left(-\frac{x_1(1-x_1)}{x_1 - x_j} - x_1 - x_j + 1 \right) \prod_{i=2}^{d_2} x_i(1-x_i) \prod_{i=d_2+1}^{d_2+d_1} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&= -J - 2I(d_1+1, d_2-1) + I(d_1, d_2-1).
\end{aligned}$$

Differentiation and integration of the integrand of $I(d_1, d_2)$ by x_1 leads to the same function again, and plugging in the upper and lower bounds 1 and 0 causes the term $x_1(1-x_1)$ to be zero in both cases which makes the whole integral equal to zero. On the other hand the derivative can be computed by the product rule and by using the fact that $\frac{d}{dx}|x|^a = a|x|^{a-1}\operatorname{sgn}(x) = \frac{a|x|^a}{x}$ for $x \neq 0$ and $a \in \mathbb{R}$. We get

$$\begin{aligned}
0 &= \int_{C_{d_0}} \frac{\partial}{\partial x_1} \left(x_1(1-x_1) \prod_{i=2}^{d_2} x_i(1-x_i) \prod_{i=d_2+1}^{d_2+d_1} x_i \omega_{d_0}(\mathbf{x}) \right) d\mathbf{x} \\
&= \int_{C_{d_0}} (1-x_1) \prod_{i=2}^{d_2} x_i(1-x_i) \prod_{i=d_2+1}^{d_2+d_1} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} - \int_{C_{d_0}} x_1 \prod_{i=2}^{d_2} x_i(1-x_i) \prod_{i=d_2+1}^{d_2+d_1} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&\quad + \sum_{j=2}^{d_0} \int_{C_{d_0}} \frac{1}{x_1 - x_j} \prod_{i=1}^{d_2} x_i(1-x_i) \prod_{i=d_2+1}^{d_2+d_1} x_i \omega_{d_0}(\mathbf{x}) d\mathbf{x} \\
&= I(d_1, d_2-1)((d_0 - d_1 - d_2)/2 + 1) + I(d_1+1, d_2-1)(-d_0 + d_1/2 + d_2 - 2)
\end{aligned}$$

which proves the recurrence relation for J_2.

The same method can be applied to get the other recurrence relations. This can be done automatically with Mathematica using the function fR from below. The argument m has the very meaning of the m of the lemma and should be a natural number. The arguments a and d will be used to represent the coefficients of the polynomials and their multiplicities in the output and should be clear symbols. The last argument p determines of which form the factor should be, that will be used for derivation and integration and should be an element of $[\![1, m]\!]$. Note that if p is chosen to be 1, i.e. a factor of the form x should be used, an additional factor of the form $(1 - x)$ will be introduced during computation. Otherwise the factor itself is suitable without modification due to the requirement for all factors of degrees greater than 2 to be multiples of $x(1 - x)$.

```
fR[m_,a_,d_,p_]:=Module[{L1={},L2={},L={},LT={},i=0,j=0,T=0,c,R={},x},
  L1=Join[{1,x[1],x[1](1-x[1])},
    Table[(x[1](1-x[1])Sum[a[j,k]x[1]^k,{k,0,j-2}]),{j,3,m}]];
  L2=Join[{1,x[2],x[2](1-x[2])},
    Table[(x[2](1-x[2])Sum[a[j,k]x[2]^k,{k,0,j-2}]),{j,3,m}]];
  L=Flatten[Table[L1[[i]]L2[[j]],{i,1,m+1},{j,1,m+1}]];
  LT=Flatten[Table[{L1[[i]]->d[i-1],L2[[i]]->d[i-1]},{i,m+1,2,-1}],1];
  T=L1[[p+1]];
  If[p==1,T*=(1-x[1])];
  T=D[T,x[1]];
  T=FullSimplify[Expand[FullSimplify[
    (T+(T/.{x[1]->x[2],x[2]->x[1]}))/2]2]];
  T=(Table[c[i],{i,1,(m+1)^2}]/.
    Solve[ForAll[{x[1],x[2]},L.Table[c[i],{i,1,(m+1)^2}]==T],
    Table[c[i],{i,1,(m+1)^2}]][[1]]).L;
  For[j=1,j<=Length[LT],{T=(T/.LT[[j]]),j++}];
  T=CoefficientRules[T,Table[d[i],{i,1,m}]]/.Rule->List;
  T=Map[{#[[1]]-Table[If[j==p,1,0],{j,1,m}],#[[2]]}&,T];
  AppendTo[R,T];
  For[i=0,i<=m,{
    T=L1[[p+1]]L2[[i+1]]/(x[1]-x[2]),
    If[p==1,T*=(1-x[1])],
    T=FullSimplify[Expand[FullSimplify[
      (T+(T/.{x[1]->x[2],x[2]->x[1]}))/2]2]],
    T=(Table[c[i],{i,1,(m+1)^2}]/.
      Solve[ForAll[{x[1],x[2]},L.Table[c[i],{i,1,(m+1)^2}]==T],
      Table[c[i],{i,1,(m+1)^2}]][[1]]).L,
    For[j=1,j<=Length[LT],{T=(T/.LT[[j]]),j++}],
    T=CoefficientRules[T,Table[d[i],{i,1,m}]]/.Rule->List,
    T=Map[{#[[1]]-Table[If[j==p,1,0],{j,1,m}],#[[2]]}&,T],
    T=Map[{#[[1]]-Table[If[j==i,1,0],{j,1,m}],#[[2]]
      If[i>=1,If[i==p,d[i]-1,d[i]],d[0]-Sum[d[j],{j,1,m}]]}&,T],
    AppendTo[R,T],
  i++}];
  R=Flatten[R,1];
  T=Union[Map[#[[1]]&,R]];
  R=Table[{T[[i]],FullSimplify[Apply[Plus,Map[#[[2]]&,
    Select[R,#[[1]]==T[[i]]&]]]]},{i,1,Length[T]}];
  Return[R];
];
```

The function can be used by the commands below to yield the claimed recurrence relations. Note that the output will look slightly different as it was post processed manually.

```
Clear[a,d];
For[m=1,m<=5,{Print[TableForm[fR[m,a,d,m],TableDepth->1]],Print[],m++}]
```

□

Now it is time to make a specific choice for the extra factors. The choice we will make is motivated by two aspects. The first fact we have to consider is the actual goal we have in mind which is the computation of the volume of a generalization of the so-called Schur-Cohn region in Chapter 2. It will turn out that not all types of extra factors are equally usable. The second fact to consider is that some choices notably simplify the recurrences as some of terms might cancel out. With these two aspects in mind we find the choice in the following theorem to be adequate.

THEOREM 1.3.4. (**Weitzer**) *Let* $\mathcal{P} = (x, x(1-x), x^2(1-x), x^2(1-x)^2, x^3(1-x)^2, x^3(1-x)^3, \ldots)$, $\mathcal{D} = (d_i)_{i\in\mathbb{N}_0} \in \mathbb{N}_0^{\mathbb{N}_0}$, *and* $(x)_j := \prod_{i=0}^{j-1}(x-i)$ *for* $x \in \mathbb{C}$ *and* $j \in \mathbb{Z}$. *Then*

(i)

$$I_3(\mathcal{P},\mathcal{D}) = \frac{1}{(2d_0+6)_{d_3}(2d_0-d_2-d_3+4)_{d_3}} \sum_{i=0}^{d_3}\binom{d_3}{i}\frac{(-d_0+d_1+d_2+d_3+i-1)_i(4d_0-d_1-2d_2-3d_3+9)_i}{(2d_0-d_1-d_2-2d_3+3)_{-d_3+i}(2d_0-d_1-d_2-2d_3+4)_{-d_3+i}} A_{d_0,d_1+d_2+d_3+i,d_2+d_3,d_2+d_3}(1,1,1/2)$$

(ii)

$$\begin{aligned}
I_4(\mathcal{P},\mathcal{D}) = &\sum_{A=(a_0,\ldots,a_{d_4-1})\in\{1,\ldots,5\}^{d_4}} I_3\Big(\mathcal{P},\Big(d_0, d_1+c^{(1)}_{A,d_4}+c^{(5)}_{A,d_4}, d_2+c^{(1)}_{A,d_4}-c^{(2)}_{A,d_4}+2c^{(4)}_{A,d_4}, d_3+c^{(1)}_{A,d_4}+2c^{(2)}_{A,d_4}+c^{(3)}_{A,d_4}+c^{(5)}_{A,d_4}\Big)\Big)\\
&\prod_{i=0}^{d_4-1}\frac{1}{2d_0-d_4+i+9}\\
&\prod_{i=0}^{d_4-1}\left\{\begin{matrix}
-2d_1+2c^{(1)}_{A,i}-2c^{(5)}_{A,i} & \textit{if } a_i=1\\
d_2+c^{(1)}_{A,i}-c^{(2)}_{A,i}+2c^{(4)}_{A,i} & \textit{if } a_i=2\\
2d_1+d_3-c^{(1)}_{A,i}+2c^{(2)}_{A,i}+c^{(3)}_{A,i}+3c^{(5)}_{A,i}+4 & \textit{if } a_i=3\\
-d_0+d_1+d_2+d_3+d_4+c^{(1)}_{A,i}+c^{(2)}_{A,i}+c^{(3)}_{A,i}+2c^{(4)}_{A,i}+2c^{(5)}_{A,i}-i & \textit{if } a_i=4\\
2d_0-2d_1-2d_2-2d_3-2d_4-2c^{(1)}_{A,i}-2c^{(2)}_{A,i}-2c^{(3)}_{A,i}-4c^{(4)}_{A,i}-4c^{(5)}_{A,i}+2i & \textit{if } a_i=5
\end{matrix}\right\}\\
= &\sum_{A=(a_0,\ldots,a_{d_4-1})\in\{1,\ldots,4\}^{d_4}} I_3\Big(\mathcal{P},\Big(d_0, d_1-c^{(1)}_{A,d_4}+c^{(4)}_{A,d_4}, d_2+c^{(1)}_{A,d_4}+2c^{(3)}_{A,d_4}, d_3+c^{(1)}_{A,d_4}+c^{(2)}_{A,d_4}+c^{(4)}_{A,d_4}\Big)\Big)\\
&\prod_{i=0}^{d_4-1}\frac{1}{(-2d_0+d_3+2d_4+c^{(1)}_{A,i}+c^{(2)}_{A,i}+c^{(4)}_{A,i}-2i-7)(2d_0-d_4+i+9)}\\
&\prod_{i=0}^{d_4-1}\left\{\begin{matrix}
(d_1-c^{(1)}_{A,i}+c^{(4)}_{A,i}) & \\
(4d_0-d_2-2d_3-4d_4-3c^{(1)}_{A,i}-2c^{(2)}_{A,i}-2c^{(3)}_{A,i}-2c^{(4)}_{A,i}+4i+14) & \textit{if } a_i=1\\
(2d_1+d_2+d_3+c^{(2)}_{A,i}+2c^{(3)}_{A,i}+3c^{(4)}_{A,i}+4) & \\
(-2d_0+d_2+d_3+2d_4+2c^{(1)}_{A,i}+c^{(2)}_{A,i}+2c^{(3)}_{A,i}+c^{(4)}_{A,i}-2i-7) & \textit{if } a_i=2\\
(-d_0+d_1+d_2+d_3+d_4+c^{(1)}_{A,i}+c^{(2)}_{A,i}+2c^{(3)}_{A,i}+2c^{(4)}_{A,i}-i) & \\
(-2d_0+d_3+2d_4+c^{(1)}_{A,i}+c^{(2)}_{A,i}+c^{(4)}_{A,i}-2i-7) & \textit{if } a_i=3\\
(-d_0+d_1+d_2+d_3+d_4+c^{(1)}_{A,i}+c^{(2)}_{A,i}+2c^{(3)}_{A,i}+2c^{(4)}_{A,i}-i) & \\
(4d_0-2d_2-2d_3-4d_4-4c^{(1)}_{A,i}-2c^{(2)}_{A,i}-4c^{(3)}_{A,i}-2c^{(4)}_{A,i}+4i+14) & \textit{if } a_i=4
\end{matrix}\right\}
\end{aligned}$$

where $c^{(n)}_{A,i} := |\{j \in [\![0, i-1]\!] \mid a_j = n\}|$.

Proof. Let J be defined as in Lemma 1.3.3. Then for the specific choice of $\mathcal{P}$ the recurrence relations given in the lemma simplify to

$$\begin{aligned}
&J_2(\ \ 0,-1)(d_0-d_1-d_2+2)+\\
&J_2(\ \ 1,-1)(-2d_0+d_1+2d_2-4)=0,\\[2ex]
&J_3(\ \ 0,\ \ 1,-1)(2d_0-d_2-2d_3+6)+\\
&J_3(\ \ 1,\ \ 0,-1)(-d_1-2)+\\
&J_3(\ \ 2,\ \ 0,-1)(-d_0+d_1+d_2+d_3)=0,\\[2ex]
&J_4(-1,\ \ 2,\ \ 0,-1)(d_1)+\\
&J_4(\ \ 0,\ \ 0,\ \ 1,-1)(-2d_0+d_3+2d_4-8)+\\
&J_4(\ \ 0,\ \ 1,\ \ 0,-1)(2d_0-2d_1-d_2-2d_3-2d_4+4)+\\
&J_4(\ \ 1,\ \ 1,\ \ 0,-1)(-2d_0+2d_1+2d_2+2d_3+2d_4)=0,\\[2ex]
&J_5(-1,\ \ 1,\ \ 1,\ \ 0,-1)(-2d_1)+\\
&J_5(\ \ 0,-1,\ \ 2,\ \ 0,-1)(d_2)+\\
&J_5(\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1)(-2d_0+d_4+2d_5-10)+\\
&J_5(\ \ 0,\ \ 0,\ \ 1,\ \ 0,-1)(2d_1+d_3+4)+\\
&J_5(\ \ 0,\ \ 2,\ \ 0,\ \ 0,-1)(-d_0+d_1+d_2+d_3+d_4+d_5)+\\
&J_5(\ \ 1,\ \ 0,\ \ 1,\ \ 0,-1)(2d_0-2d_1-2d_2-2d_3-2d_4-2d_5)=0.
\end{aligned}$$

Assuming suitable bounds the recurrence in J_4 for $d_4=1$ and $d_3\to d_3-1$ implies

$$\begin{aligned}
J_3(0,0,0)=J_3(-1,2,-1)&\frac{d_1}{2d_0-d_3+7}+\\
J_3(0,1,-1)&\frac{2d_0-2d_1-d_2-2d_3+4}{2d_0-d_3+7}+\\
J_3(1,1,-1)&\frac{-2d_0+2d_1+2d_2+2d_3}{2d_0-d_3+7}.
\end{aligned}$$

The recurrence in J_3 for $d_1\to d_1-1$ and $d_2\to d_2+1$ implies

$$\begin{aligned}
J_3(-1,2,-1)=J_3(0,1,-1)&\frac{d_1+1}{2d_0-d_2-2d_3+5}+\\
J_3(1,1,-1)&\frac{d_0-d_1-d_2-d_3}{2d_0-d_2-2d_3+5}.
\end{aligned}$$

Therefore

$$\begin{aligned}
J_3(0,0,0)=J_3(0,1,-1)&\frac{(2d_0-d_1-d_2-2d_3+5)(2d_0-d_1-d_2-2d_3+4)}{(2d_0-d_2-2d_3+5)(2d_0-d_3+7)}+\\
J_3(1,1,-1)&\frac{(4d_0-d_1-2d_2-4d_3+10)(-d_0+d_1+d_2+d_3)}{(2d_0-d_2-2d_3+5)(2d_0-d_3+7)}.
\end{aligned}$$

By induction one can verify that the solution of the recurrence above is

$$\begin{aligned}
J_3(0,0,0)=&\frac{1}{(2d_0+6)_{d_3}(2d_0-d_2-d_3+4)_{d_3}}\\
&\sum_{i=0}^{d_3}\binom{d_3}{i}\frac{(-d_0+d_1+d_2+d_3+i-1)_i(4d_0-d_1-2d_2-3d_3+9)_i}{(2d_0-d_1-d_2-2d_3+3)_{-d_3+i}(2d_0-d_1-d_2-2d_3+4)_{-d_3+i}}\\
&\quad J_3(i,d_3,-d_3)
\end{aligned}$$

which, by Lemma 1.3.2 (vi), is equal to

$$\frac{1}{(2d_0+6)_{d_3}(2d_0-d_2-d_3+4)_{d_3}}$$
$$\sum_{i=0}^{d_3}\binom{d_3}{i}\frac{(-d_0+d_1+d_2+d_3+i-1)_i(4d_0-d_1-2d_2-3d_3+9)_i}{(2d_0-d_1-d_2-2d_3+3)_{-d_3+i}(2d_0-d_1-d_2-2d_3+4)_{-d_3+i}}$$
$$A_{d_0,d_1+d_2+d_3+i,d_2+d_3,d_2+d_3}(1,1,1/2)$$

which proves (i).

The recurrence in J_5 for $d_5=1$ and $d_4\to d_4-1$ implies

$$\begin{aligned}J_4(0,0,0,0) = {} & J_4(-1,1,1,-1)\frac{-2d_1}{2d_0-d_4+9}+\\ & J_4(0,-1,2,-1)\frac{d_2}{2d_0-d_4+9}+\\ & J_4(0,0,1,-1)\frac{2d_1+d_3+4}{2d_0-d_4+9}+\\ & J_4(0,2,0,-1)\frac{-d_0+d_1+d_2+d_3+d_4}{2d_0-d_4+9}+\\ & J_4(1,0,1,-1)\frac{2d_0-2d_1-2d_2-2d_3-2d_4}{2d_0-d_4+9}.\end{aligned}$$

Summing over all possible terms in the above recurrence relation gives the first formula for $I_4(\mathcal{P},\mathcal{D})$. The recurrence in J_4 for $d_2\to d_2-1$ and $d_3\to d_3+1$ implies

$$\begin{aligned}J_4(0,-1,2,-1) = {} & J_4(-1,1,1,-1)\frac{-d_1}{-2d_0+d_3+2d_4-7}+\\ & J_4(0,0,1,-1)\frac{-2d_0+2d_1+d_2+2d_3+2d_4-3}{-2d_0+d_3+2d_4-7}+\\ & J_4(1,0,1,-1)\frac{2d_0-2d_1-2d_2-2d_3-2d_4}{-2d_0+d_3+2d_4-7}.\end{aligned}$$

Therefore

$$\begin{aligned}J_4(0,0,0,0) = {} & J_4(-1,1,1,-1)\frac{d_1(4d_0-d_2-2d_3-4d_4+14)}{(-2d_0+d_3+2d_4-7)(2d_0-d_4+9)}+\\ & J_4(0,0,1,-1)\frac{(2d_1+d_2+d_3+4)(-2d_0+d_2+d_3+2d_4-7)}{(-2d_0+d_3+2d_4-7)(2d_0-d_4+9)}+\\ & J_4(0,2,0,-1)\frac{(-d_0+d_1+d_2+d_3+d_4)(-2d_0+d_3+2d_4-7)}{(-2d_0+d_3+2d_4-7)(2d_0-d_4+9)}+\\ & J_4(1,0,1,-1)\frac{(-d_0+d_1+d_2+d_3+d_4)(4d_0-2d_2-2d_3-4d_4+14)}{(-2d_0+d_3+2d_4-7)(2d_0-d_4+9)}.\end{aligned}$$

Summing over all possible terms in the above recurrence relation gives the second formula for $I_4(\mathcal{P},\mathcal{D})$ which proves (ii). □

The formulas given in (ii) of the previous theorem are of course not in closed form in the strict sense of the words as they involve sums over certain vectors and counting of occurrences of certain values within these vectors. Despite best efforts the recurrences for $I_4(\mathcal{P},\mathcal{D})$ so far could not be combined in any way to give a nicer result. However, for $I_3(\mathcal{P},\mathcal{D})$ an expression in terms of a 1-balanced hypergeometric ${}_4F_3$ function could be found.

DEFINITION 1.3.5. [**Graham et al., 1994**, Chapter 5] *For $p,q\in\mathbb{N}_0$ and $a_1,\ldots,a_p,b_1,\ldots,b_q$, $z\in\mathbb{C}$ let*

$${}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right] := \sum_{n=0}^{\infty}\frac{(a_1+n-1)_n\ldots(a_p+n-1)_n}{(b_1+n-1)_n\ldots(b_q+n-1)_n}\frac{z^n}{n!}$$

denote the generalized hypergeometric function (where $(x)_j := \prod_{i=0}^{j-1}(x-i)$ *for* $x \in \mathbb{C}$ *and* $j \in \mathbb{Z}$*). If* $s := \sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i$ *then the corresponding hypergeometric function is said to be s-balanced.*

COROLLARY 1.3.6. (**Weitzer**) *Let* $\mathcal{P} = (x, x(1-x), x^2(1-x), x^2(1-x)^2, x^3(1-x)^2, x^3(1-x)^3, \dots)$ *and* $\mathcal{D} = (d_i)_{i\in\mathbb{N}_0} \in \mathbb{N}_0^{\mathbb{N}_0}$*. Then*

$$\begin{aligned}I_3(\mathcal{P},\mathcal{D}) = {} & (2d_0-d_1-d_2-d_3+4)_{d_3}(2d_0-d_1-d_2-d_3+3)_{d_3}\\ & \frac{(d_0+2)_{d_2+d_3}(d_0+1)_{d_2+d_3}(d_0+1)_{d_1+d_2+d_3}}{(2d_0+6)_{d_3}(2d_0+4)_{d_2+2d_3}(2d_0+2)_{d_1+2d_2+2d_3}}\frac{1}{\prod_{i=0}^{d_0-1}\binom{2i+1}{i}}\\ & {}_4F_3\left[\begin{matrix}-d_3, -d_0+d_1+d_2+d_3-1, -d_0+d_1+d_2+d_3, -4d_0+d_1+2d_2+3d_3-9\\ -2d_0+d_1+d_2+d_3-4, -2d_0+d_1+d_2+d_3-3, -2d_0+d_1+2d_2+2d_3-2\end{matrix};1\right]\end{aligned}$$

PROOF. It follows from Theorem 1.3.4 (i), Theorem 1.2.3, and Theorem 1.2.1 that:

$$\begin{aligned}I_3(\mathcal{P},\mathcal{D}) = {} & \frac{1}{(2d_0+6)_{d_3}(2d_0-d_2-d_3+4)_{d_3}}\\ & \sum_{i=0}^{d_3}\binom{d_3}{i}\frac{(-d_0+d_1+d_2+d_3+i-1)_i(4d_0-d_1-2d_2-3d_3+9)_i}{(2d_0-d_1-d_2-2d_3+3)_{-d_3+i}(2d_0-d_1-d_2-2d_3+4)_{-d_3+i}}\\ & \quad A_{d_0,d_1+d_2+d_3+i,d_2+d_3,d_2+d_3}(1,1,1/2)\\ = {} & \frac{1}{(2d_0+6)_{d_3}(2d_0-d_2-d_3+4)_{d_3}}\\ & \sum_{i=0}^{d_3}\binom{d_3}{i}\frac{(-d_0+d_1+d_2+d_3+i-1)_i(4d_0-d_1-2d_2-3d_3+9)_i}{(2d_0-d_1-d_2-2d_3+3)_{-d_3+i}(2d_0-d_1-d_2-2d_3+4)_{-d_3+i}}\\ & \quad \prod_{j=1}^{d_2+d_3}\frac{d_0-j+3}{2d_0-j+5}\frac{\prod_{j=1}^{d_1+d_2+d_3+i}(d_0-j+2)\prod_{j=1}^{d_2+d_3}(d_0-j+2)}{\prod_{j=1}^{d_1+2d_2+2d_3+i}(2d_0-j+3)}\frac{1}{\prod_{j=0}^{d_0-1}\binom{2j+1}{j}}\\ = {} & \frac{1}{(2d_0+6)_{d_3}(2d_0-d_2-d_3+4)_{d_3}}\\ & \sum_{i=0}^{d_3}\frac{(d_3)_i}{i!}\frac{(-d_0+d_1+d_2+d_3+i-1)_i(4d_0-d_1-2d_2-3d_3+9)_i}{(2d_0-d_1-d_2-2d_3+3)_{-d_3+i}(2d_0-d_1-d_2-2d_3+4)_{-d_3+i}}\\ & \quad \frac{(d_0+2)_{d_2+d_3}(d_0+1)_{d_1+d_2+d_3+i}(d_0+1)_{d_2+d_3}}{(2d_0+4)_{d_2+d_3}(2d_0+2)_{d_1+2d_2+2d_3+i}}\frac{1}{\prod_{j=0}^{d_0-1}\binom{2j+1}{j}}\\ = {} & (2d_0-d_1-d_2-d_3+4)_{d_3}(2d_0-d_1-d_2-d_3+3)_{d_3}\\ & \frac{(d_0+2)_{d_2+d_3}(d_0+1)_{d_2+d_3}(d_0+1)_{d_1+d_2+d_3}}{(2d_0+6)_{d_3}(2d_0+4)_{d_2+2d_3}(2d_0+2)_{d_1+2d_2+2d_3}}\frac{1}{\prod_{i=0}^{d_0-1}\binom{2i+1}{i}}\\ & \sum_{i=0}^{d_3}\frac{(d_3)_i(d_0-d_1-d_2-d_3+1)_i(d_0-d_1-d_2-d_3)_i}{(2d_0-d_1-d_2-d_3+4)_i(2d_0-d_1-d_2-d_3+3)_i}\\ & \quad \frac{(4d_0-d_1-2d_2-3d_3+9)_i}{(2d_0-d_1-2d_2-2d_3+2)_i}(-1)^i\frac{1}{i!}\\ = {} & (2d_0-d_1-d_2-d_3+4)_{d_3}(2d_0-d_1-d_2-d_3+3)_{d_3}\\ & \frac{(d_0+2)_{d_2+d_3}(d_0+1)_{d_2+d_3}(d_0+1)_{d_1+d_2+d_3}}{(2d_0+6)_{d_3}(2d_0+4)_{d_2+2d_3}(2d_0+2)_{d_1+2d_2+2d_3}}\frac{1}{\prod_{i=0}^{d_0-1}\binom{2i+1}{i}}\\ & {}_4F_3\left[\begin{matrix}-d_3, -d_0+d_1+d_2+d_3-1, -d_0+d_1+d_2+d_3, -4d_0+d_1+2d_2+3d_3-9\\ -2d_0+d_1+d_2+d_3-4, -2d_0+d_1+d_2+d_3-3, -2d_0+d_1+2d_2+2d_3-2\end{matrix};1\right]\end{aligned}$$

□

Note that the hypergeometric function in the previous corollary is 1-balanced.

CHAPTER 2

The Schur-Cohn region and its generalizations

2.1. Introduction and definitions

The Schur-Cohn region was first introduced in [**Schur, 1918**]. Its characterization there is a consequence of a convergence criterion for power series expansions of rational functions given in [**Schur, 1917**] which found widespread application in science and engineering problems alike. These applications include speech analysis and synthesis, inverse scattering, decoding of error-correcting codes, synthesis of digital filters, modeling of random signals, and Padé approximation for linear systems (cf. [**Kailath, 1986**]). The Schur-Cohn region is closely related to contractive polynomials which play an important role in dynamical systems in general, including Shift Radix Systems where they characterize a certain dynamical property almost everywhere in a measure-theoretic sense. We will discuss the relation to Shift Radix Systems in Chapter 3 and will dedicate the present chapter to the Schur-Cohn region and to a generalization due to Akyiama and Pethő. The material of this chapter will be published in parts in [**Kirschenhofer and Weitzer, 2015**].

Definition 2.1.1. [**Schur, 1918**] *A normed polynomial $P(x) \in \mathbb{C}[x]$ is said to be contractive iff all of its roots (real and complex) lie in the open complex unit disk.*

For $d \in \mathbb{N}$ let

$$\mathcal{E}_d^{(\mathbb{R})} := \left\{(r_0, \ldots, r_{d-1}) \in \mathbb{R}^d \mid x^d + x^{d-1}r_{d-1} + \ldots + r_0 \text{ contractive}\right\}$$

be the real, d-dimensional Schur-Cohn region and

$$\mathcal{E}_d^{(\mathbb{C})} := \left\{(r_0, \ldots, r_{d-1}) \in \mathbb{C}^d \mid x^d + x^{d-1}r_{d-1} + \ldots + r_0 \text{ contractive}\right\}$$

the complex, d-dimensional Schur-Cohn region.

Definition 2.1.2. [**Lancaster and Tismenetsky, 1985**, Chapter 2] *For a normed polynomial $P(x) = x^d + x^{d-1}r_{d-1} + \ldots + r_0 \in \mathbb{C}[x]$ let*

$$C(P) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_0 & \cdots & \cdots & \cdots & -r_{d-1} \end{pmatrix} \in \mathbb{C}^{d\times d}$$

be the companion matrix of P.

For any matrix $M \in \mathbb{C}^{d\times d}$ let $\rho(M)$ denote the spectral radius of M. The matrix M is said to be contractive iff $\rho(M) < 1$.

Remark 2.1.3. *The characteristic polynomial of the companion matrix of a polynomial P is P again. Therefore a polynomial is contractive iff its companion matrix is contractive.*

Several topological results on the Schur-Cohn region have been achieved since its introduction in 1918. In the remaining parts of this sections we shall give a brief summary. The first basic observation is that the Schur-Cohn region[1] is the interior of its closure. This follows immediately from the fact that the coefficients of a polynomial depend continuously on its roots

[1]Note that whenever we speak of "the Schur-Cohn region" we mean any of the sets $\mathcal{E}_d^{(\mathbb{R})}$ or $\mathcal{E}_d^{(\mathbb{C})}$. We might choose to specify further by assigning the attributes "real", "complex", and "d-dimensional" in any valid combination.

[**Naulin and Pabst, 1994**]. Already in [**Schur, 1918**] it was proven that the boundary of the Schur-Cohn region is contained in the union of finitely many algebraic surfaces. This result was improved in [**Fam and Meditch, 1978**] where it is shown that the boundary can be described in terms of exactly two hyperplanes and one hypersurface and also that the Schur-Cohn region is simply connected. Further results on the boundary have been achieved e.g. in [**Kirschenhofer et al., 2010**]. In [**Schur, 1918**] the following theorem was given which characterizes the Schur-Cohn region by determinants of certain matrices.

THEOREM 2.1.4. [**Schur, 1918**] *Let* $d \in \mathbb{N}$ *and for all* $k \in [\![0, d-1]\!]$, $(r_0, \dots, r_{d-1}) \in \mathbb{C}^d$ *let*

$$M_k(r_0,\dots,r_{d-1}) := \begin{pmatrix} 1 & 0 & \cdots & 0 & r_0 & \cdots & \cdots & r_k \\ r_{d-1} & \ddots & \ddots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ r_{d-k} & \cdots & r_{d-1} & 1 & 0 & \cdots & 0 & r_0 \\ \overline{r_0} & 0 & \cdots & 0 & 1 & \overline{r_{d-1}} & \cdots & \overline{r_{d-k}} \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & \vdots & \ddots & \ddots & \overline{r_{d-1}} \\ \overline{r_k} & \cdots & \cdots & \overline{r_0} & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{C}^{2(k+1)\times 2(k+1)}.$$

Then

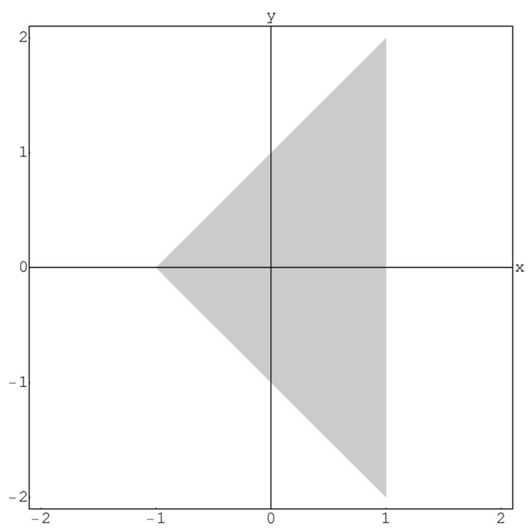
(i) $\mathcal{E}_d^{(\mathbb{R})} = \left\{(r_0,\dots,r_{d-1}) \in \mathbb{R}^d \mid \forall\, k \in [\![0,d-1]\!] : \det(M_k(r_0,\dots,r_{d-1})) > 0\right\}$

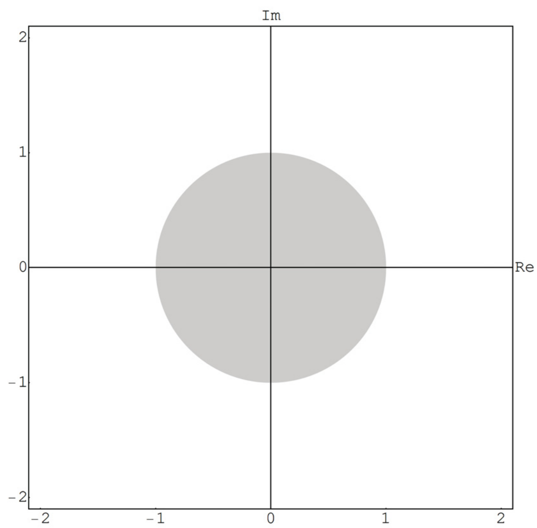
(ii) $\mathcal{E}_d^{(\mathbb{C})} = \left\{(r_0,\dots,r_{d-1}) \in \mathbb{C}^d \mid \forall\, k \in [\![0,d-1]\!] : \det(M_k(r_0,\dots,r_{d-1})) > 0\right\}$.

2.2. Examples

Theorem 2.1.4 from the previous section allows to compute the Schur-Cohn region for a given dimension explicitly. In the following section we will give some real and complex examples of low dimensions which are direct consequences of the theorem (cf. [**Akiyama et al., 2005**] and [**Brunotte et al., 2011**]).

- $\mathcal{E}_1^{(\mathbb{R})} = \{x \in \mathbb{R} \mid |x| < 1\}$
- $\mathcal{E}_2^{(\mathbb{R})} = \left\{(x,y) \in \mathbb{R}^2 \mid |x| < 1 \wedge |y| < x+1\right\}$
- $\mathcal{E}_3^{(\mathbb{R})} = \left\{(x,y,z) \in \mathbb{R}^3 \mid |x| < 1 \wedge |y - xz| < 1 - x^2 \wedge |x+z| < y+1\right\}$
- $\mathcal{E}_1^{(\mathbb{C})} = \{x \in \mathbb{C} \mid |x| < 1\}$
- $\mathcal{E}_2^{(\mathbb{C})} = \left\{(x,y) \in \mathbb{C}^2 \mid |x| < 1 \wedge \left(1 - |x|^2\right)^2 + 2\Re\left(\overline{x}y^2\right) > \left(1 + |x|^2\right)|y|^2\right\}$





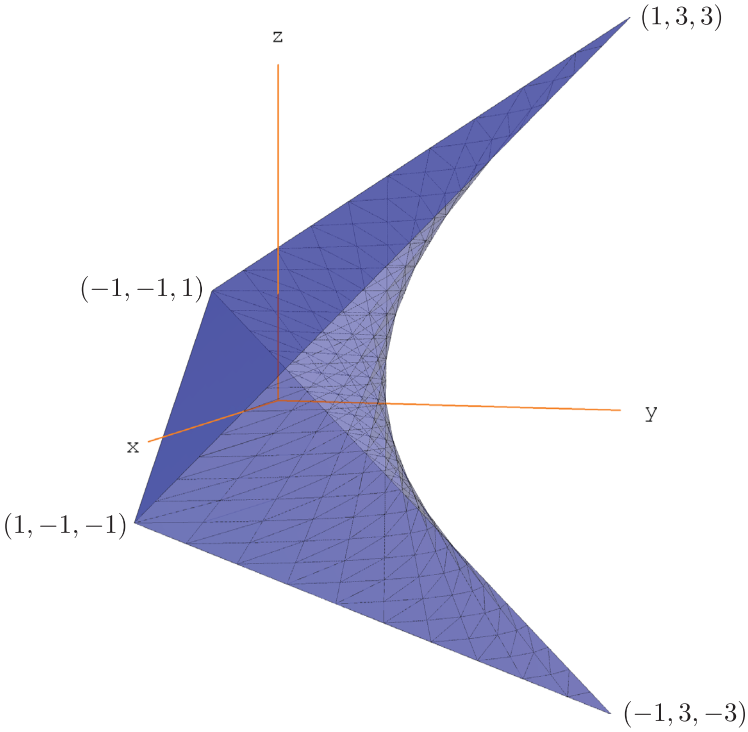


FIGURE 1. $\mathcal{E}_2^{(\mathbb{R})}$, $\mathcal{E}_1^{(\mathbb{C})}$, and $\mathcal{E}_3^{(\mathbb{R})}$.

2.3. Subdividing the Schur-Cohn region

In [**Akiyama and Pethő, 2014**] a generalization of the Schur-Cohn region is considered. The original real Schur-Cohn region is subdivided into disjoint parts each of which corresponds to polynomials with a given root signature (i.e. specific number of real roots and pairs of complex roots). In this way the distribution of contractive polynomials with a given root signature among all contractive polynomials can be studied by comparing the Lebesgue measures of the corresponding sets. Especially the relation between polynomials with real roots only and polynomials with any other given signature holds several surprises which will be discussed in the upcoming sections.

DEFINITION 2.3.1. [**Akiyama and Pethő, 2014**] *For $d \in \mathbb{N}$ and $s \in \mathbb{N}_0$ let*

$$\mathcal{E}_{d,s}^{(\mathbb{R})} := \Big\{(r_0, \ldots, r_{d-1}) \in \mathcal{E}_d^{(\mathbb{R})} \mid x^d + x^{d-1}r_{d-1} + \ldots + r_0 \textit{ has exactly } 2s \textit{ complex roots}\Big\}.$$

It is clear that $\mathcal{E}_d^{(\mathbb{R})}$ is the disjoint union of all $\mathcal{E}_{d,s}^{(\mathbb{R})}$, $s \in [\![0, \lfloor d/2 \rfloor]\!]$. As for the Schur-Cohn region it can be proven that the boundary of any such set is contained in the union of finitely many algebraic surfaces [**Akiyama and Pethő, 2014**].

2.4. Examples

No analogue of Theorem 2.1.4 for $\mathcal{E}_{d,s}^{(\mathbb{R})}$ has been found yet, which makes it more difficult to actually compute the sets. At least for $d = 2$ and $d = 3$ it can be given explicitly as for polynomials of these degrees there is a one-to-one correspondence between the existence of complex roots and the sign of its discriminant. Indeed, in both cases the polynomial does have complex roots iff its discriminant is negative.

- $\mathcal{E}_{2,0}^{(\mathbb{R})} = \mathcal{E}_2^{(\mathbb{R})} \cap \big\{(x, y) \in \mathbb{R}^2 \mid y^2 - 4x \geq 0\big\}$
- $\mathcal{E}_{2,1}^{(\mathbb{R})} = \mathcal{E}_2^{(\mathbb{R})} \cap \big\{(x, y) \in \mathbb{R}^2 \mid y^2 - 4x < 0\big\}$
- $\mathcal{E}_{3,0}^{(\mathbb{R})} = \mathcal{E}_3^{(\mathbb{R})} \cap \big\{(x, y) \in \mathbb{R}^2 \mid -27x^2 + 18xyz - 4xz^3 - 4y^3 + y^2z^2 \geq 0\big\}$
- $\mathcal{E}_{2,1}^{(\mathbb{R})} = \mathcal{E}_3^{(\mathbb{R})} \cap \big\{(x, y) \in \mathbb{R}^2 \mid -27x^2 + 18xyz - 4xz^3 - 4y^3 + y^2z^2 < 0\big\}$

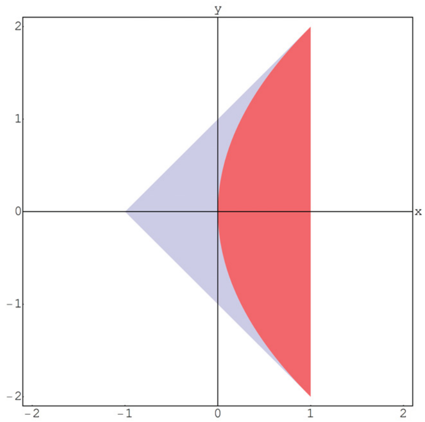


FIGURE 2. $\mathcal{E}_{2,0}^{(\mathbb{R})}$ and $\mathcal{E}_{2,1}^{(\mathbb{R})}$.

2.5. Volumes and quotients

More than 70 years after its first description in 1918 the volume of the Schur-Cohn region has been computed in [**Fam, 1989**]. Another 25 years later the volume of $\mathcal{E}_{d,s}^{(\mathbb{R})}$ has been given in [**Akiyama and Pethő, 2014**] by a rather complicated integral which could be solved explicitly at least for two special cases. In the following section we will repeat the results on volumes from those two papers as well as the proofs of the integral formula for $\mathcal{E}_{d,s}^{(\mathbb{R})}$ and its simplification for special cases. We will then adapt the methods used in these proofs to solve the integral for another special case. In any case we will make heavy use of the formulas for Selberg and Aomoto integrals and their generalizations which we derived in Chapter 1. Finally we will discuss quotients of volumes first considered in [**Akiyama and Pethő, 2014**] and the surprising discovery that they are always rational numbers and in some cases even integers. This probably indicates that there is a combinatorial enumeration problem the integer quotients are the answer to. Unfortunately a suitable problem setting has not been found, yet.

DEFINITION 2.5.1. [**Akiyama and Pethő, 2014**] *For $d \in \mathbb{N}$ and $s \in \mathbb{N}_0$ let*

$$\begin{aligned} v_d &:= \lambda_d\left(\mathcal{E}_d^{(\mathbb{R})}\right) \\ v_d^{(s)} &:= \lambda_d\left(\mathcal{E}_{d,s}^{(\mathbb{R})}\right) \end{aligned}$$

where λ_d is the d-dimensional Lebesgue-measure.

The following theorems summarize previous results on the above defined volumes.

THEOREM 2.5.2. [**Fam, 1989**] *Let $d \in \mathbb{N}$. Then*

$$v_d = \begin{cases} 2^{2m^2} \prod_{i=1}^{m} \frac{(i-1)!^4}{(2i-1)!^2} & \text{if } d = 2m \\ 2^{2m^2+2m+1} \prod_{i=1}^{m} \frac{i!^2(i-1)!^2}{(2i-1)!(2i+1)!} & \text{if } d = 2m+1 \end{cases}$$

LEMMA 2.5.3. [**Akiyama and Pethő, 2014**] *Let $d \in \mathbb{N}$, $r, s \in \mathbb{N}_0$ such that $d = r + 2s$, $R_i(x) := x^2 - y_{r+2i-1}x + y_{r+2i}$ for $i \in [\![1, s]\!]$, $S_i(x_1, \ldots, x_d) := \sum_{1 \le j_1 < \ldots < j_i \le d)} x_{j_1} \ldots x_{j_i}$ for $i \in [\![1, d]\!]$, the elementary symmetric functions, and $J := \left(\frac{\partial S_i(x_1,\ldots,x_d)}{\partial y_j}\right)_{1 \le i,j \le d}$. Then*

$$\det(J) = \prod_{j=1}^{r} \prod_{k=j+1}^{r} (y_j - y_k) \prod_{j=1}^{r} \prod_{k=1}^{s} R_k(y_j) \prod_{j=1}^{s} \prod_{k=j+1}^{s} \operatorname{Res}_x(R_j(x), R_k(x))$$

where $\operatorname{Res}_x(P(x), Q(x))$ is the resultant of two polynomials $P, Q \in \mathbb{R}[x]$.

THEOREM 2.5.4. [**Akiyama and Pethő, 2014**] *Let* $d \in \mathbb{N}$ *and* $r, s \in \mathbb{N}_0$ *such that* $d = r + 2s$. *Then*

$$v_d^{(s)} = \frac{1}{r!s!} \int_{D_{r,s}} |\Delta_r| \, \Delta_s \Delta_{r,s} dX$$

where

$$\begin{aligned} D_{r,s} &= [-1,1]^r \times \prod_{i=1}^{s} ([-2\sqrt{z_i}, 2\sqrt{z_i}] \times [0,1]) \\ dX &= dx_1 \dots dx_r dy_1 dz_1 \dots dy_s dz_s \\ \Delta_r &= \prod_{j=1}^{r} \prod_{k=j+1}^{r} (x_j - x_k) \\ \Delta_s &= \prod_{j=1}^{s} \prod_{k=j+1}^{s} \operatorname{Res}_x(R_j(x), R_k(x)) \\ \Delta_{r,s} &= \prod_{j=1}^{r} \prod_{k=1}^{s} R_k(x_j) \\ R_j(x) &= x^2 - y_j x + z_j. \end{aligned}$$

Note that $\operatorname{Res}_x(R_j(x), R_k(x)) = -y_j y_k (z_j + z_k) + y_j^2 z_k + y_k^2 z_j + (z_j - z_k)^2$.

SKETCH OF THE PROOF. At first we observe that

$$v_d^{(s)} = \lambda_d(\mathcal{E}_{d,s}^{(\mathbb{R})}) = \int_{\mathcal{E}_{d,s}^{(\mathbb{R})}} dr_0 \dots dr_{d-1}.$$

To solve this integral we perform integration by substitution. For a polynomial $P(x) = x^d + r_{d-1}x^{d-1} + \ldots + r_0 \in \mathbb{R}[x]$ with roots $(x_1, \ldots, x_d) \in \mathbb{R}^r \times \mathbb{C}^{2s}$ we use Vieta's formulas to express its coefficients in terms of its roots and substitute by them. We have

$$r_j = (-1)^{d-j} S_{d-j}(x_1, \ldots, x_d)$$

where $S_i(x_1, \ldots, x_d)$ is defined as in Lemma 2.5.3 for $i \in [\![1, d]\!]$. We further substitute $x_1, \ldots, x_d$ by $y_1, \ldots, y_d$ where $y_i := x_i$ for $i \in \{1, \ldots, r\}$ and $y_{r+2i-1} := x_{r+2i-1} + x_{r+2i}$, $y_{r+2i} := x_{r+2i-1} x_{r+2i}$ for $i \in \{1, \ldots, s\}$. The boundaries of the integral change to unit intervals (for real roots) and unit disks (for complex roots) and we are left with the computation of the determinant of the Jacobian matrix $J = \left(\frac{\partial S_i(x_1, \ldots, x_d)}{\partial y_j}\right)_{1 \le i,j \le d}$. But this determinant is given in Lemma 2.5.3 which completes the proof. □

d	$v_d^{(0)}$	$v_d^{(1)}$	$v_d^{(2)}$	$v_d^{(3)}$	$v_d^{(4)}$
2	$\frac{4}{3}$	$\frac{8}{3}$			
3	$\frac{16}{45}$	$\frac{224}{45}$			
4	$\frac{64}{1575}$	$\frac{1664}{525}$	$\frac{2048}{525}$		
5	$\frac{1024}{496125}$	$\frac{428032}{496125}$	$\frac{3334144}{496125}$		
6	$\frac{16384}{343814625}$	$\frac{1114112}{10418625}$	$\frac{93519872}{22920975}$	$\frac{268435456}{68762925}$	
7	$\frac{524288}{1032475318875}$	$\frac{2124414976}{344158439625}$	$\frac{379792130048}{344158439625}$	$\frac{6491843067904}{1032475318875}$	
8	$\frac{16777216}{6643978676960625}$	$\frac{1114476904448}{6643978676960625}$	$\frac{313947815149568}{2214659558986875}$	$\frac{693972225753088}{189827962198875}$	$\frac{562949953421312}{189827962198875}$
9	$\frac{4294967296}{7268180473661075718 75}$	$\frac{92376156602368}{4275400278624162 1875}$	$\frac{12626155878219776}{1433566168374965625}$	$\frac{70817769017175336 5504}{7268180473661075718 75}$	$\frac{32803926951790913 78176}{7268180473661075718 75}$

TABLE 1. Values of $v_d^{(s)}$ as given in [**Akiyama and Pethő, 2014**].

d	$v_d/v_d^{(0)}$	$v_d^{(1)}/v_d^{(0)}$	$v_d^{(2)}/v_d^{(0)}$	$v_d^{(3)}/v_d^{(0)}$	$v_d^{(4)}/v_d^{(0)}$
2	3	2			
3	15	14			
4	175	78	96		
5	3675	418	3256		
6	169785	2244	85620	81920	
7	14567553	12156	2173188	12382208	
8	2678348673	66428	56138244	1447738880	1174405120
9	930152232009	365636	1490456292	164885467424	763775942656

TABLE 2. Values of $v_d/v_d^{(0)}$ and $v_d^{(s)}/v_d^{(0)}$ as given in [**Akiyama and Pethő, 2014**].

As can be seen in the previous tables, not only do the volumes $v_d^{(s)}$ seem to be rational numbers, the quotients $v_d/v_d^{(0)}$ and $v_d^{(s)}/v_d^{(0)}$ appear to be integers even. These observations led to the following theorems and conjecture.

THEOREM 2.5.5. [**Akiyama and Pethő, 2014**] *Let $d \in \mathbb{N}$ and $s \in \mathbb{N}_0$. Then v_d and $v_d^{(s)}$ are rational numbers.*

THEOREM 2.5.6. [**Akiyama and Pethő, 2014**] *Let $d \in \mathbb{N}$. Then the quotient $v_d/v_d^{(0)}$ is an integer.*

CONJECTURE 2.5.7. [**Akiyama and Pethő, 2014**] *Let $d \in \mathbb{N}$ and $s \in \mathbb{N}_0$. Then the quotient $v_d^{(s)}/v_d^{(0)}$ is an integer.*

The proofs of the two theorems will not be performed here. Instead we will repeat the proofs of the following two formulas for $v_d^{(0)}$ and $v_d^{(1)}$ (the first of which is essential for the proof of Theorem 2.5.6) as the proof of the upcoming Theorem 2.6.6 will follow a similar strategy.

THEOREM 2.5.8. [**Akiyama and Pethő, 2014**] *Let $d \in \mathbb{N}$. Then*

(i) $$v_d^{(0)} = \frac{2^{d(d+1)/2}}{d!} S_d(1,1,1/2) = \frac{2^{d(d+1)/2}}{d!} \frac{1}{\prod_{i=0}^{d-1} \binom{2i+1}{i}}$$

(ii) $$\begin{aligned} v_d^{(1)} &= 2^{(d-1)(d-2)/2-2} \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^{d-k} 2^{2d-2-2k-j}}{j!k!(d-2-j-k)!} A_{d-2,d-2-k,d-2-k-j,d-2-k-j}(1,1,1/2) \\ &\qquad \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dydz \\ &= 2^{(d-1)(d-2)/2-2} \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^{d-k} 2^{2d-2-2k-j}}{j!k!(d-2-j-k)!} \prod_{i=1}^{d-2-k-j} \frac{2+\frac{d-2-i-1}{2}}{3+\frac{2(d-2)-i-1}{2}} \\ &\qquad \frac{\prod_{i=1}^{d-2-k}(1+\frac{d-2-i}{2}) \prod_{i=1}^{d-2-k-j}(1+\frac{d-2-i}{2})}{\prod_{i=1}^{d-2-k+d-2-k-j}(2+\frac{2(d-2)-i-1}{2})} \frac{1}{\prod_{i=0}^{d-2-1}\binom{2i+1}{i}} \\ &\qquad \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dydz. \end{aligned}$$

Note that in [**Akiyama and Pethő, 2014**] the formula for $v_d^{(1)}$ is wrong by a factor of 4.

PROOF. For the proof of (i) we set $r := d$ and $s := 0$. Then by Theorem 2.5.4 we have

$$v_d^{(0)} = \frac{1}{d!} \int_{[-1,1]^d} \prod_{j=1}^{d} \prod_{k=j+1}^{d} |x_j - x_k| \, dx_1 \dots dx_d$$

and if we rearrange $x_1, \ldots, x_d$ such that they are in decreasing order and consider that there are $d!$ orderings we thus have

$$v_d^{(0)} = \int_{-1}^{1} \int_{x_1}^{1} \cdots \int_{x_{d-1}}^{1} \prod_{j=1}^{d} \prod_{k=j+1}^{d} (x_j - x_k) dx_1 \ldots dx_d.$$

By substituting x_i with $2x_i - 1$ for $i \in [\![1, d]\!]$ we get

$$v_d^{(0)} = 2^{d(d+1)/2} \int_{x_1}^{1} \cdots \int_{x_{d-1}}^{1} \prod_{j=1}^{d} \prod_{k=j+1}^{d} (x_j - x_k) dx_1 \ldots dx_d$$

and if we undo the very first manipulation we finally have

$$\begin{aligned} v_d^{(0)} &= \frac{2^{d(d+1)/2}}{d!} \int_{[-1,1]^d} \prod_{j=1}^{d} \prod_{k=j+1}^{d} |x_j - x_k| \, dx_1 \ldots dx_d \\ &= \frac{2^{d(d+1)/2}}{d!} S_d(1, 1, 1/2) = \frac{2^{d(d+1)/2}}{d!} \frac{1}{\prod_{i=0}^{d-1} \binom{2i+1}{i}} \quad \text{(Theorem 1.2.1)}. \end{aligned}$$

For the proof of (ii) we set $r := d - 2$ and $s := 1$ and Theorem 2.5.4 gives

$$v_d^{(1)} = \frac{1}{4(d-2)!} \int_{[-1,1]^{d-2}} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} \prod_{j=1}^{d-2} \prod_{k=j+1}^{d-2} |x_j - x_k| \prod_{j=1}^{d-2} (x_j^2 - y x_j + z) dy dz dx_1 \ldots dx_d.$$

Substituting x_i with $2x_i - 1$ for $i \in [\![1, d]\!]$ again we get

$$\begin{aligned} v_d^{(1)} = \frac{2^{(d-1)(d-2)/2-2}}{(d-2)!} \int_{[0,1]^{d-2}} \prod_{j=1}^{d-2} \prod_{k=j+1}^{d-2} |x_j - x_k| \\ \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} \prod_{j=1}^{d-2} ((2x_j - 1)^2 - y(2x_j - 1) + z) dy dz dx_1 \ldots dx_d. \end{aligned}$$

Next we will deal with the innermost product and try to separate the variables x_j from y and z. We will do so by performing the multiplications.

$$\begin{aligned} \prod_{j=1}^{d-2} ((2x_j - 1)^2 - y(2x_j - 1) + z) &= \prod_{j=1}^{d-2} (-4x_j(1 - x_j) - 2x_j y + (y + z + 1)) \\ &= \sum_{j=0}^{d-2} (-2)^j \sum_{k=0}^{d-j-2} (-4)^{d-j-k-2} y^j (y + z + 1)^k \\ &\qquad \sum_{\substack{L \subseteq [\![1, d-2]\!] \\ |L| = j}} \sum_{\substack{M \subseteq [\![1, d-2]\!] \setminus L \\ |M| = d-j-k-2}} \prod_{l \in L} x_l \prod_{m \in M} x_l(1 - x_l). \end{aligned}$$

Plugging in gives

$$\begin{aligned} v_d^{(1)} = \frac{2^{(d-1)(d-2)/2-2}}{(d-2)!} \sum_{j=0}^{d-2} (-2)^j \sum_{k=0}^{d-j-2} (-4)^{d-j-k-2} \\ \sum_{\substack{L \subseteq [\![1, d-2]\!] \\ |L| = j}} \sum_{\substack{M \subseteq [\![1, d-2]\!] \setminus L \\ |M| = d-j-k-2}} \int_{[0,1]^{d-2}} \prod_{l \in L} x_l \prod_{m \in M} x_l(1 - x_l) \prod_{j=1}^{d-2} \prod_{k=j+1}^{d-2} |x_j - x_k| \, dx_1 \ldots dx_d \\ \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y + z + 1)^k dy dz \end{aligned}$$

and after counting the number of summands of the two innermost constant sums we get

$$v_d^{(1)} = 2^{(d-1)(d-2)/2-2} \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^{d-k} 2^{2d-2-2k-j}}{j!k!(d-2-j-k)!} A_{d-2,d-2-k,d-2-k-j,d-2-k-j}(1,1,1/2) \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k dydz$$

$$= 2^{(d-1)(d-2)/2-2} \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^{d-k} 2^{2d-2-2k-j}}{j!k!(d-2-j-k)!} \prod_{i=1}^{d-2-k-j} \frac{2+\frac{d-2-i-1}{2}}{3+\frac{2(d-2)-i-1}{2}} \frac{\prod_{i=1}^{d-2-k}(1+\frac{d-2-i}{2}) \prod_{i=1}^{d-2-k-j}(1+\frac{d-2-i}{2})}{\prod_{i=1}^{d-2-k+d-2-k-j}(2+\frac{2(d-2)-i-1}{2})} \frac{1}{\prod_{i=0}^{d-2-1}\binom{2i+1}{i}} \int_0^1 \int_{-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k dydz \quad \text{(Theorem 1.2.3 (ii))}.$$

□

2.6. Main results on the Schur-Cohn region

In this last section of the chapter we will prove Conjecture 2.5.7 for the special case of $s = 1$ and deduce several consequences. Furthermore we will adapt the proof of Theorem 2.5.8 (ii) to find a formula for $v_d^{(2)}$. Most of the results have been published in [**Kirschenhofer and Weitzer, 2015**].

THEOREM 2.6.1. [**Kirschenhofer and Weitzer, 2015**] *The quotient $v_d^{(1)}/v_d^{(0)}$ is an integer for all $d \in \mathbb{N}_{\geq 2}$. Furthermore we have*

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{P_d(3) - 2d - 1}{4}$$

where

$$P_d(x) := 2^{-d} \sum_{k=0}^{\lfloor d/2 \rfloor} (-1)^k \binom{d-k}{k} \binom{2d-2k}{d-k} x^{d-2k} = \sum_{k=0}^{d} \binom{d+k}{2k} \binom{2k}{k} \left(\frac{x-1}{2}\right)^k$$

are the Legendre polynomials (cf. [**Riordan, 1968**, p. 66]*).*

PROOF. We will use the formulas given in Theorem 2.5.8. First we solve the double integral in the formula of $v_d^{(1)}$. Let $j, k \in \mathbb{N}_0$. Then

$$\int_{z=0}^{1} \int_{y=-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz = \int_{y=-2}^{2} \int_{z=y^2/4}^{1} y^j (y+z+1)^k \, dz \, dy$$

$$= \frac{1}{k+1} \left(\int_{-2}^{2} y^j (y+2)^{k+1} \, dy - \int_{-2}^{2} y^j (y/2+1)^{2k+2} \, dy \right)$$

$$(y/2 \to y) \qquad = \frac{1}{k+1} \left(2^{j+k+2} \int_{-1}^{1} y^j (y+1)^{k+1} \, dy - 2^{j+1} \int_{-1}^{1} y^j (y+1)^{2k+2} \, dy \right)$$

$$\text{(partial integration)} \qquad = \frac{2^{j+2k+4}}{k+1} \left(\sum_{r=1}^{j+1} \frac{(-2)^{r-1}(j)_{r-1}}{(k+r+1)_r} - \sum_{r=1}^{j+1} \frac{(-2)^{r-1}(j)_{r-1}}{(2k+r+2)_r} \right)$$

We plug in the result and get

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \left(2^{\frac{(d-1)(d-2)}{2}-2} \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^{d+k} 2^{2d-2-2k-j}}{j!k!(d-2-j-k)!} \prod_{i=1}^{d-2-k-j} \frac{2+\frac{d-2-i-1}{2}}{3+\frac{2(d-2)-i-1}{2}}\right.$$

$$\frac{\prod_{i=1}^{d-2-k}(1+\frac{d-2-i}{2}) \prod_{i=1}^{d-2-k-j}(1+\frac{d-2-i}{2})}{\prod_{i=1}^{d-2-k+d-2-k-j}(2+\frac{2(d-2)-i-1}{2})} \frac{1}{\prod_{i=0}^{d-2-1}\binom{2i+1}{i}}$$

$$\left.\frac{2^{j+2k+4}}{k+1}\left(\sum_{r=1}^{j+1} \frac{(-2)^{r-1}(j)_{r-1}}{(k+r+1)_r} - \sum_{r=1}^{j+1} \frac{(-2)^{r-1}(j)_{r-1}}{(2k+r+2)_r}\right)\right) \Big/ \left(\frac{2^{d(d+1)/2}}{d! \prod_{i=0}^{d-1}\binom{2i+1}{i}}\right)$$

$$= \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} (-1)^{d+k+1} \frac{d!}{j!(k+1)!(d-j-k-2)!} \prod_{i=1}^{d-j-k-2} \frac{d-i+1}{2d-i+1}$$

$$\frac{\prod_{i=1}^{d-k-2}(d-i) \prod_{i=1}^{d-j-k-2}(d-i)}{\prod_{i=1}^{2d-j-2k-4}(2d-i-1)} \frac{\prod_{i=0}^{d-1}\binom{2i+1}{i}}{\prod_{i=0}^{d-3}\binom{2i+1}{i}}$$

$$\left(\sum_{r=1}^{j+1} \frac{(-2)^r(j)_{r-1}}{(k+r+1)_r} - \sum_{r=1}^{j+1} \frac{(-2)^r(j)_{r-1}}{(2k+r+2)_r}\right)$$

$$= \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} (-1)^{d+k+1} \frac{d!}{j!(k+1)!(d-j-k-2)!} \frac{\frac{d!}{(j+k+2)!}}{\frac{(2d)!}{(d+j+k+2)!}} \frac{\frac{(d-1)!}{(k+1)!}\frac{(d-1)!}{(j+k+1)!}}{\frac{(2d-2)!}{(j+2k+2)!}}$$

$$\frac{(2d-3)!}{(d-2)!(d-1)!} \frac{(2d-1)!}{(d-1)!d!} \left(\sum_{r=1}^{j+1} \frac{(-2)^r \frac{j!}{(j-r+1)!}}{\frac{(k+r+1)!}{(k+1)!}} - \sum_{r=1}^{j+1} \frac{(-2)^r \frac{j!}{(j-r+1)!}}{\frac{(2k+r+2)!}{(2k+2)!}}\right)$$

$$= \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} (-1)^{d+k+1} \frac{(d+j+k+2)!(j+2k+2)!}{(d-j-k-2)!j!(j+k+2)!(j+k+1)!(k+1)!^2}$$

$$\left(\sum_{r=1}^{j+1} \frac{(-2)^{r-2}j!(k+1)!}{(j-r+1)!(k+r+1)!} - \sum_{r=1}^{j+1} \frac{(-2)^{r-2}j!(2k+2)!}{(j-r+1)!(2k+r+2)!}\right).$$

$$= \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} (-1)^{d+k+1} \binom{d}{j+k+2} \binom{d+j+k+2}{d} \frac{j+k+2}{j+2k+3}$$

$$\left(\sum_{r=1}^{j+1}(-2)^{r-2}\binom{j+2k+3}{2k+r+2}\binom{2k+r+2}{k+1} - \sum_{r=1}^{j+1}(-2)^{r-2}\binom{j+2k+3}{2k+r+2}\binom{2k+2}{k+1}\right).$$

$$= \sum_{a=2}^{d} \sum_{b=0}^{a-1} \sum_{r=1}^{a-b} (-1)^{d+b}(-2)^{r-2} \frac{a}{a+b} \binom{d}{a} \binom{d+a}{d} \binom{a+b}{2b+r} \binom{2b+r}{b} -$$

$$\sum_{a=2}^{d} \sum_{b=0}^{a-1} \sum_{r=1}^{a-b} (-1)^{d+b}(-2)^{r-2} \frac{a}{a+b} \binom{d}{a} \binom{d+a}{d} \binom{a+b}{2b+r} \binom{2b}{b} \qquad (j+k+2 \to a, k+1 \to b)$$

$$= \sum_{a=2}^{d} (-1)^d a \binom{d}{a} \binom{d+a}{d} \sum_{r=1}^{a} (-2)^{r-2}$$

$$\left(\sum_{b=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} - \sum_{b=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b}{b}\right).$$

In the following we will simplify the two innermost sums. We start with the first sum. If $r = a$ the sum trivially equals $\frac{1}{a}$. Let us assume $1 \le r \le a-1$ now. From

$$(-1)^k \binom{k-n-1}{k} = \binom{n}{k} \qquad (n \in \mathbb{Z}, k \ge 0)$$

and from Vandermonde's identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{s}{k+t}=\sum_{k=0}^{n}\binom{n}{k}\binom{s}{n+t-k}=\binom{n+s}{n+t}\qquad(s\in\mathbb{Z},n,t\geq 0)$$

it follows that

$$\begin{aligned}\sum_{b=0}^{a-r}(-1)^b\frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b}&=\frac{1}{a-r}\sum_{b=0}^{a-r}(-1)^b\binom{a-r}{b}\binom{a+b-1}{b+r}\\&=\frac{(-1)^r}{a-r}\sum_{b=0}^{a-r}\binom{a-r}{b}\binom{r-a}{b+r}=\frac{(-1)^r}{a-r}\binom{0}{a}=0.\end{aligned}$$

Altogether we have established

$$\sum_{b=0}^{a-r}(-1)^b\frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b+r}{b}=\frac{1}{a}\delta_{r,a}\qquad(1\leq r\leq a)$$

where $\delta_{r,a}$ denotes the Kronecker symbol.

Next we try to simplify the second sum which is very similar to a sum that has been treated in [**Graham et al., 1994**, Section 5.2, Problem 7]. Therefore we first try to adopt the strategy followed there and use [**Graham et al., 1994**, Section 5.1, Identity 5.26]

$$\binom{l+q+1}{m+n+1}=\sum_{0\leq k\leq l}\binom{l-k}{m}\binom{q+k}{n}\qquad(l,m\geq 0,n\geq q\geq 0)$$

with $l=a+b-1,q=0,m=2b,n=r-1$ and $k=s$ to get

$$\begin{aligned}\sum_{b=0}^{a-r}(-1)^b\frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b}&=\sum_{b=0}^{a-r}\sum_{s=0}^{a+b-1}\frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{s}{r-1}\binom{2b}{b}\\&=\sum_{s=r-1}^{2a-r-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1}\frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b}\\&=\sum_{s=r-1}^{a-1}\binom{s}{r-1}\sum_{b=0}^{a-s-1}\frac{(-1)^b}{a+b}\binom{a+b-s-1}{2b}\binom{2b}{b}.\end{aligned}$$

Now we apply sum S_m from [**Graham et al., 1994**, Section 5.2, Problem 8]

$$S_m=\sum_{k=0}^{n}(-1)^k\frac{1}{k+m+1}\binom{n+k}{2k}\binom{2k}{k}=(-1)^n\frac{m!n!}{(m+n+1)!}\binom{m}{n}\qquad(m,n\geq 0)$$

with $m=a-1,n=a-s-1$ and $k=b$ which gives

$$\begin{aligned}&\sum_{s=r-1}^{a-1}\binom{s}{r-1}\frac{(-1)^{a+s+1}(a-1)!(a-s-1)!}{(2a-s-1)!}\binom{a-1}{a-s-1}\\&=\frac{(-1)^{a+1}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=r-1}^{a-1}(-1)^s\binom{2a-r}{s-r+1}\\&=\frac{(-1)^{a+r}(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\sum_{s=0}^{a-r}(-1)^s\binom{2a-r}{s}\qquad(s-r+1\to s).\end{aligned}$$

Using the basic identity

$$\sum_{j=0}^{k}(-1)^j\binom{n}{j}=(-1)^k\binom{n-1}{k}\qquad(n,k\geq 0)$$

we finally get

$$\sum_{b=0}^{a-r}(-1)^b \frac{1}{a+b}\binom{a+b}{2b+r}\binom{2b}{b} = \frac{(a-1)!(a-1)!}{(2a-1)!}\binom{2a-1}{r-1}\binom{2a-r-1}{a-r}$$
$$= \frac{1}{2a-r}\binom{a-1}{a-r}.$$

We plug in the results for the two sums and find

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d}(-1)^d a\binom{d}{a}\binom{d+a}{d}\left(\frac{(-2)^{a-2}}{a} - \sum_{r=1}^{a}(-2)^{r-2}\frac{1}{2a-r}\binom{a-1}{a-r}\right)$$
$$= \sum_{a=2}^{d}(-1)^{d+1} a\binom{d}{a}\binom{d+a}{d}\left(\sum_{r=0}^{a-1}(-2)^{r-1}\frac{1}{2a-r-1}\binom{a-1}{r} - (-2)^{a-2}\frac{1}{a}\right).$$

In order to get rid of the inner sum we use an identity that may be proved as an application of the classical reflection law

$$\frac{1}{(1-z)^a} F\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| \frac{-z}{1-z}\right) = F\left(\left.\begin{matrix} a, c-b \\ c \end{matrix}\right| z\right)$$

for hypergeometric functions [**Pfaff, 1797**], namely

$$\sum_{k=0}^{m}(-2)^k \frac{2m+1}{2m-k+1}\binom{m}{k} = \frac{(-1)^m 2^{2m}}{\binom{2m}{m}} \qquad (m \geq 0),$$

compare [**Graham et al., 1994**, Identity (5.104)]. In this way we find

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d}(-1)^{d+a+1} a\binom{d}{a}\binom{d+a}{d}\left(2^{2a-3}\frac{1}{2a-1}\frac{1}{\binom{2a-2}{a-1}} - 2^{a-2}\frac{1}{a}\right)$$
$$= \sum_{a=2}^{d}(-1)^{d+a}\binom{d}{a}\binom{d+a}{d}\left(2^{a-2} - 2^{2a-2}\frac{1}{\binom{2a}{a}}\right)$$
$$= \sum_{a=2}^{d}(-1)^{d+a}2^{a-2}\binom{d+a}{2a}\left(\binom{2a}{a} - 2^a\right)$$

which proves that the quotient $v_d^{(1)}/v_d^{(0)}$ is an integer.

To prove the formula for the quotient let

$$\rho_d(x) := \sum_{k=0}^{d}\binom{d+k}{d-k}x^k$$

denote the associated Legendre polynomials (cf. [**Riordan, 1968**, p. 66]). Then

$$\frac{v_d^{(1)}}{v_d^{(0)}} = (-1)^d \frac{P_d(-3) - \rho_d(-4)}{4}.$$

Since $P_d(-x) = (-1)^d P_d(x)$ (cf. [**Rainville, 1960**, p. 158]) and ρ_d satisfies the recursive formula

$$\rho_d(x) = (x+2)\rho_{d-1}(x) - \rho_{d-2}(x)$$
$$\rho_0(x) = 0, \rho_1(x) = x+1$$

(cf. [**Riordan, 1968**, p. 66]) we get

$$(-1)^d \rho_d(-4) = 2d+1$$

and therefore

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{P_d(3) - 2d - 1}{4}.$$

□

COROLLARY 2.6.2. [**Kirschenhofer and Weitzer, 2015**] $v_d^{(1)}/v_d^{(0)}$ *satisfies the second order linear recurrence relation*

$$d\frac{v_d^{(1)}}{v_d^{(0)}} - 3(2d-1)\frac{v_{d-1}^{(1)}}{v_{d-1}^{(0)}} + (d-1)\frac{v_{d-2}^{(1)}}{v_{d-2}^{(0)}} = 2d(d-1) \text{ for } d \geq 2, \frac{v_0^{(1)}}{v_0^{(0)}} = \frac{v_1^{(1)}}{v_1^{(0)}} = 0.$$

PROOF. The recurrence relation is a direct consequence of the following recurrence relation for Legendre polynomials (cf. [**Rainville, 1960**, p. 160]):

$$dP_d(x) - (2d-1)xP_{d-1}(x) + (d-1)P_{d-2}(x) = 0 \quad (d \geq 2)$$
$$P_0(x) = 1, P_1(x) = x.$$

□

In the next part of this section we study the asymptotic behavior of the quotients for $d \to \infty$.

COROLLARY 2.6.3. [**Kirschenhofer and Weitzer, 2015**] *The generating function of the ratios* $v_d^{(1)}/v_d^{(0)}$ *is given by*

$$V_1(z) := \sum_{d\geq 0} \frac{v_d^{(1)}}{v_d^{(0)}} z^d = \frac{1}{4}\left(\frac{1}{\sqrt{z^2-6z+1}} - \frac{z+1}{(z-1)^2}\right).$$

PROOF. This follows directly from the generating function of the Legendre polynomials which is given by (cf. [**Riordan, 1968**, p. 78])

$$\sum_{d\geq 0} P_d(x) z^d = \frac{1}{\sqrt{z^2-2xz+1}}.$$

□

THEOREM 2.6.4. [**Kirschenhofer and Weitzer, 2015**] *For* $d \to \infty$

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{1}{8\sqrt[4]{2}\sqrt{\pi d}}(3+2\sqrt{2})^{d+\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{d}\right)\right).$$

PROOF. We adopt the usual technique of singularity analysis of generating functions, compare e.g. [**Flajolet and Sedgewick, 2009**, Chapter IV] or [**Szpankowski, 2001**, Chapter 8]. The dominating singularity of the generating function $V_1(z)$ is given by the zero $3-2\sqrt{2}$ of z^2-6z+1 closest to the origin, whereas the other zero of z^2-6z+1 as well as the term $\frac{1+z}{(1-z)^2}$ will give a contribution that is exponentially smaller than the contribution of the main term. The local expansion of $V_1(z)$ about the dominating singularity is given by

$$\begin{aligned}
V_1(z) &= \frac{1}{4}\left(\frac{1}{\sqrt{z^2-6z+1}} - \frac{z+1}{(z-1)^2}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{3-2\sqrt{2}-z}}\frac{1}{\sqrt{3+2\sqrt{2}-z}} - \frac{z+1}{(z-1)^2}\right) \\
&= \frac{1}{4}\left(\left(\frac{3-2\sqrt{2}-z}{3-2\sqrt{2}}\right)^{-1/2}\frac{1}{\sqrt{3-2\sqrt{2}}}\frac{1}{2\sqrt[4]{2}}\left(1-\frac{z-(3-2\sqrt{2})}{4\sqrt{2}}\right)^{-1/2} - \frac{z+1}{(z-1)^2}\right) \\
&= \frac{1}{8\sqrt[4]{2}\sqrt{3-2\sqrt{2}}}\left(1-\frac{z}{3-2\sqrt{2}}\right)^{-1/2}\left(1+\mathcal{O}\left(z-\left(3-2\sqrt{2}\right)\right)\right) \\
&\quad -\frac{1}{4}\left(\frac{2+\sqrt{2}}{2} + \mathcal{O}\left(z-\left(3-2\sqrt{2}\right)\right)\right) \\
&= \frac{1}{8\sqrt[4]{2}\sqrt{3-2\sqrt{2}}}\left(1-\frac{z}{3-2\sqrt{2}}\right)^{-1/2}\left(1+\mathcal{O}\left(1-\frac{z}{3-2\sqrt{2}}\right)\right) - \frac{2+\sqrt{2}}{8}
\end{aligned}$$

for $z \to 3-2\sqrt{2}$

from which the asymptotics is immediate by the theorem

$$[z^n](1-z)^{-\alpha} = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)$$

from [**Flajolet and Odlyzko, 1990**], where $\alpha \in \mathbb{R}\setminus(-\mathbb{N}_0)$, $n \in \mathbb{N}_0$, and $[z^n]f(z)$ is the coefficient of z^n in the Taylor expansion of $f(z)$. □

In [**Akiyama and Pethő, 2014**] the probability $p_d^{(s)} := v_d^{(s)}/v_d$ for a contractive normed polynomial of degree d in $\mathbb{R}[x]$ to have exactly s pairs of complex conjugate roots is discussed. In particular they derived

$$\log p_d^{(0)} = -\frac{\log 2}{2}d^2 + \frac{1}{8}\log d + \mathcal{O}(1), \quad \text{for } d \to \infty$$

for the probability of totally real polynomials and conjectured that

$$\log p_d^{(1)} \leq -\frac{\log 2}{2}d^2 + d\log q$$

for some constant q. Now, obviously, $p_d^{(1)} = \frac{v_d^{(1)}}{v_d^{(0)}}p_d^{(0)}$, so that from Theorem 2.6.4 we gain

Corollary 2.6.5. [**Kirschenhofer and Weitzer, 2015**] *The probability $p_d^{(1)}$ for a contractive normed polynomial of degree d in $\mathbb{R}[x]$ to have exactly one pair of complex conjugate roots satisfies*

$$\log p_d^{(1)} = -\frac{\log 2}{2}d^2 + d\log(3+2\sqrt{2}) + \mathcal{O}(\log d) \quad \textit{for } d \to \infty.$$

We close the chapter by computing a formula for $v_d^{(2)}$.

Theorem 2.6.6. (**Weitzer**) *Let $d \in \mathbb{N}_{\geq 2}$, $\mathcal{P} = (x, x(1-x), x^2(1-x), x^2(1-x)^2, x^3(1-x)^2, x^3(1-x)^3, \ldots)$, and $F := [-2\sqrt{z_1}, 2\sqrt{z_1}] \times [0,1] \times [-2\sqrt{z_2}, 2\sqrt{z_2}] \times [0,1]$. Then*

$$v_d^{(2)} = \frac{2^{4d+(d-3)(d-4)/2-17}}{(d-4)!}\sum_{j_1=0}^{d-4}\sum_{k_1=0}^{d-j_1-4}\sum_{j_2=0}^{d-4}\sum_{k_2=0}^{d-j_2-4}\frac{(-1)^{k_1+k_2}}{2^{j_1+j_2+2k_1+2k_2}}$$
$$\sum_{m_{22}=\max(0,d-j_1-k_1-j_2-k_2-4)}^{\min(d-j_1-k_1-4,d-j_2-k_2-4)}$$
$$\sum_{l_{22}=\max(0,-d+j_1+j_2+m_{22}+4)}^{\min(j_1,j_2,-d+j_1+j_2+k_1+k_2+m_{22}+4)}$$
$$\sum_{m_{11}=\max(0,d-j_1-j_2-k_1-m_{22}+l_{22}-4)}^{\min(d-j_1-k_1-m_{22}-4,k_2,d-j_1-j_2-m_{22}+l_{22}-4)}$$
$$\sum_{m_{21}=\max(0,d-j_1-j_2-k_2-m_{22}+l_{22}-4}^{\min(d-j_2-k_2-m_{22}-4,d-m_{22}-l_{22}-m_{11}-4,d-j_1-j_2-m_{22}+l_{22}-m_{11}-4)}$$
$$\sum_{l_{14}=0}^{l_{22}}(-1)^{l_{22}-l_{14}}$$
$$\begin{pmatrix} d-4 \\ m_{22}, m_{11}, m_{21}, l_{14}, l_{22}-l_{14}, \\ d-j_1-k_1-m_{22}-m_{11}-4, d-j_2-k_2-m_{22}-m_{21}-4, \\ -d+j_1+j_2+k_1+m_{22}-l_{22}+m_{11}+4, \\ -d+j_1+j_2+k_2+m_{22}-l_{22}+m_{21}+4, \\ d-j_1-j_2-m_{22}+l_{22}-m_{11}-m_{21}-4 \end{pmatrix}$$
$$I_4(\mathcal{P}, (-2d+2j_1+2j_2+k_1+k_2-2l_{22}+2m_{22}+m_{11}+m_{21}+l_{14}+8,$$
$$m_{11}+m_{21}+l_{22}-l_{14},$$
$$2d-j_1-j_2-k_1-k_2-2m_{22}-m_{11}-m_{21}-8,$$
$$m_{22}))$$
$$\int_F \left(y_2^2 z_1 + (z_1-z_2)^2 + y_1^2 z_2 - y_1 y_2(z_1+z_2)\right)$$
$$y_1^{j_1}(y_1+z_1+1)^{k_1} y_2^{j_2}(y_2+z_2+1)^{k_2}\, dy_1 dz_1 dy_2 dz_2,$$

where $I_4(\mathcal{P}, \mathcal{D})$ is given by Theorem 1.3.4 and the large brackets denote a multinomial coefficient.

PROOF. We adopt the proof of Theorem 2.5.8 (ii). Let

$$\begin{aligned} D_d &:= [-1,1]^{d-4} \times [-2\sqrt{z_1}, 2\sqrt{z_1}] \times [0,1] \times [-2\sqrt{z_2}, 2\sqrt{z_2}] \times [0,1] \\ E_d &:= [0,1]^{d-4} \times [-2\sqrt{z_1}, 2\sqrt{z_1}] \times [0,1] \times [-2\sqrt{z_2}, 2\sqrt{z_2}] \times [0,1]. \end{aligned}$$

Then

$$v_d^{(2)} =$$

$$\frac{1}{2(d-4)!} \int_{D_d} \prod_{i=1}^{d-4} \prod_{j=i+1}^{d-4} |x_i - x_j| \prod_{i=1}^{2} \prod_{j=i+1}^{2} \left(-y_i y_j (z_i + z_j) + y_i^2 z_j + y_j^2 z_i + (z_i - z_j)^2\right)$$

$$\prod_{i=1}^{d-4} \prod_{j=1}^{2} \left(x_i^2 - y_j x_i + z_j\right) dx_1 \dots dx_{d-4} dy_1 dz_1 dy_2 dz_2 =$$

$$\frac{1}{2(d-4)!} \int_{D_d} \left(y_2^2 z_1 + (z_1 - z_2)^2 + y_1^2 z_2 - y_1 y_2 (z_1 + z_2)\right)$$

$$\prod_{i=1}^{d-4} \left(x_i^2 - x_i y_1 + z_1\right) \prod_{i=1}^{d-4} \left(x_i^2 - x_i y_2 + z_2\right) dx_1 \dots dx_{d-4} dy_1 dz_1 dy_2 dz_2 =$$

$$\frac{2^{(d-3)(d-4)/2}}{2(d-4)!} \int_{E_d} \left(y_2^2 z_1 + (z_1 - z_2)^2 + y_1^2 z_2 - y_1 y_2 (z_1 + z_2)\right)$$

$$\left(\sum_{j=0}^{d-4} (-2)^j \sum_{k=0}^{d-j-4} (-4)^{d-j-k-4} y_1^j (y_1 + z_1 + 1)^k \right.$$

$$\left. \sum_{\substack{L \subseteq \{1,\dots,d-4\} \\ |L| = j}} \sum_{\substack{M \subseteq \{1,\dots,d-4\} \setminus L \\ |M| = d-j-k-4}} \prod_{l \in L} x_l \prod_{m \in M} x_m (1 - x_m) \right)$$

$$\left(\sum_{j=0}^{d-4} (-2)^j \sum_{k=0}^{d-j-4} (-4)^{d-j-k-4} y_2^j (y_2 + z_2 + 1)^k \right.$$

$$\left. \sum_{\substack{L \subseteq \{1,\dots,d-4\} \\ |L| = j}} \sum_{\substack{M \subseteq \{1,\dots,d-4\} \setminus L \\ |M| = d-j-k-4}} \prod_{l \in L} x_l \prod_{m \in M} x_m (1 - x_m) \right)$$

$$\prod_{i=1}^{d-4} \prod_{j=i+1}^{d-4} |x_i - x_j| \, dx_1 \dots dx_{d-4} dy_1 dz_1 dy_2 dz_2 =$$

$$\frac{2^{(d-3)(d-4)/2}}{2(d-4)!} \int_{E_d} \left(y_2^2 z_1 + (z_1 - z_2)^2 + y_1^2 z_2 - y_1 y_2 (z_1 + z_2)\right)$$

$$\sum_{j_1=0}^{d-4} \sum_{k_1=0}^{d-j_1-4} \sum_{j_2=0}^{d-4} \sum_{k_2=0}^{d-j_2-4} (-2)^{j_1+j_2} (-4)^{2d-j_1-k_1-j_2-k_2-8} y_1^{j_1} (y_1 + z_1 + 1)^{k_1} y_2^{j_2} (y_2 + z_2 + 1)^{k_2}$$

$$\sum_{\substack{L_1 \subseteq \{1,\dots,d-4\} \\ |L_1| = j_1}} \sum_{\substack{M_1 \subseteq \{1,\dots,d-4\} \setminus L_1 \\ |M_1| = d-j_1-k_1-4}} \sum_{\substack{L_2 \subseteq \{1,\dots,d-4\} \\ |L_2| = j_2}} \sum_{\substack{M_2 \subseteq \{1,\dots,d-4\} \setminus L_2 \\ |M_2| = d-j_2-k_2-4}}$$

$$\prod_{l \in L_1} x_l \prod_{l \in M_1} x_l (1 - x_l) \prod_{l \in L_2} x_l \prod_{l \in M_2} x_l (1 - x_l)$$

$$\prod_{i=1}^{d-4} \prod_{j=i+1}^{d-4} |x_i - x_j| \, dx_1 \dots dx_{d-4} dy_1 dz_1 dy_2 dz_2 =$$

$$\frac{2^{(d-3)(d-4)/2}}{2(d-4)!}\int_{E_d}\left(y_2^2z_1+(z_1-z_2)^2+y_1^2z_2-y_1y_2(z_1+z_2)\right)$$
$$\sum_{j_1=0}^{d-4}\sum_{k_1=0}^{d-j_1-4}\sum_{j_2=0}^{d-4}\sum_{k_2=0}^{d-j_2-4}(-2)^{j_1+j_2}(-4)^{2d-j_1-k_1-j_2-k_2-8}y_1^{j_1}(y_1+z_1+1)^{k_1}y_2^{j_2}(y_2+z_2+1)^{k_2}$$
$$\sum_{\substack{L_1\subseteq\{1,\ldots,d-4\}\\|L_1|=j_1}}\sum_{\substack{M_1\subseteq\{1,\ldots,d-4\}\setminus L_1\\|M_1|=d-j_1-k_1-4}}\sum_{\substack{L_2\subseteq\{1,\ldots,d-4\}\\|L_2|=j_2}}\sum_{\substack{M_2\subseteq\{1,\ldots,d-4\}\setminus L_2\\|M_2|=d-j_2-k_2-4}}$$
$$\prod_{l\in(L_1\setminus(L_2\cup M_2))\cup(L_2\setminus(L_1\cup M_1))}x_l\prod_{l\in(M_1\setminus(L_2\cup M_2))\cup(M_2\setminus(L_1\cup M_1))}x_l(1-x_l)$$
$$\prod_{l\in(L_1\cap M_2)\cup(L_2\cap M_1)}x_l^2(1-x_l)\prod_{l\in M_1\cap M_2}x_l^2(1-x_l)^2\prod_{l\in L_1\cap L_2}x_l^2$$
$$\prod_{i=1}^{d-4}\prod_{j=i+1}^{d-4}|x_i-x_j|\,dx_1\ldots dx_{d-4}dy_1dz_1dy_2dz_2=$$

$$\frac{2^{(d-3)(d-4)/2}}{2(d-4)!}\int_{E_d}\left(y_2^2z_1+(z_1-z_2)^2+y_1^2z_2-y_1y_2(z_1+z_2)\right)$$
$$\sum_{j_1=0}^{d-4}\sum_{k_1=0}^{d-j_1-4}\sum_{j_2=0}^{d-4}\sum_{k_2=0}^{d-j_2-4}(-2)^{j_1+j_2}(-4)^{2d-j_1-k_1-j_2-k_2-8}y_1^{j_1}(y_1+z_1+1)^{k_1}y_2^{j_2}(y_2+z_2+1)^{k_2}$$
$$\sum_{\substack{L_1\subseteq\{1,\ldots,d-4\}\\|L_1|=j_1}}\sum_{\substack{M_1\subseteq\{1,\ldots,d-4\}\setminus L_1\\|M_1|=d-j_1-k_1-4}}\sum_{\substack{L_2\subseteq\{1,\ldots,d-4\}\\|L_2|=j_2}}\sum_{\substack{M_2\subseteq\{1,\ldots,d-4\}\setminus L_2\\|M_2|=d-j_2-k_2-4}}$$
$$\prod_{l\in(L_1\setminus(L_2\cup M_2))\cup(L_2\setminus(L_1\cup M_1))}x_l\prod_{l\in(M_1\setminus(L_2\cup M_2))\cup(M_2\setminus(L_1\cup M_1))}x_l(1-x_l)$$
$$\prod_{l\in(L_1\cap M_2)\cup(L_2\cap M_1)}x_l^2(1-x_l)\prod_{l\in M_1\cap M_2}x_l^2(1-x_l)^2$$
$$\sum_{S\subseteq L_1\cap L_2}\left(\prod_{l\in S}x_l\prod_{l\in(L_1\cap L_2)\setminus S}(-x_l(1-x_l))\right)$$
$$\prod_{i=1}^{d-4}\prod_{j=i+1}^{d-4}|x_i-x_j|\,dx_1\ldots dx_{d-4}dy_1dz_1dy_2dz_2=$$

$$\frac{2^{(d-3)(d-4)/2}}{2(d-4)!}\int_{E_d}\left(y_2^2z_1+(z_1-z_2)^2+y_1^2z_2-y_1y_2(z_1+z_2)\right)$$
$$\sum_{j_1=0}^{d-4}\sum_{k_1=0}^{d-j_1-4}\sum_{j_2=0}^{d-4}\sum_{k_2=0}^{d-j_2-4}(-2)^{j_1+j_2}(-4)^{2d-j_1-k_1-j_2-k_2-8}y_1^{j_1}(y_1+z_1+1)^{k_1}y_2^{j_2}(y_2+z_2+1)^{k_2}$$
$$\sum_{\substack{L_1\subseteq\{1,\ldots,d-4\}\\|L_1|=j_1}}\sum_{\substack{M_1\subseteq\{1,\ldots,d-4\}\setminus L_1\\|M_1|=d-j_1-k_1-4}}\sum_{\substack{L_2\subseteq\{1,\ldots,d-4\}\\|L_2|=j_2}}\sum_{\substack{M_2\subseteq\{1,\ldots,d-4\}\setminus L_2\\|M_2|=d-j_2-k_2-4}}\sum_{S\subseteq L_1\cap L_2}$$
$$(-1)^{|(L_1\cap L_2)\setminus S|}\prod_{l\in(L_1\setminus(L_2\cup M_2))\cup(L_2\setminus(L_1\cup M_1))\cup S}x_l\prod_{l\in(M_1\setminus(L_2\cup M_2))\cup(M_2\setminus(L_1\cup M_1))\cup((L_1\cap L_2)\setminus S)}x_l(1-x_l)$$
$$\prod_{l\in(L_1\cap M_2)\cup(L_2\cap M_1)}x_l^2(1-x_l)\prod_{l\in M_1\cap M_2}x_l^2(1-x_l)^2$$
$$\prod_{i=1}^{d-4}\prod_{j=i+1}^{d-4}|x_i-x_j|\,dx_1\ldots dx_{d-4}dy_1dz_1dy_2dz_2=$$

$$\frac{2^{(d-3)(d-4)/2}}{2(d-4)!}\int_F \left(y_2^2 z_1 + (z_1-z_2)^2 + y_1^2 z_2 - y_1 y_2(z_1+z_2)\right) dy_1 dz_1 dy_2 dz_2$$

$$\sum_{j_1=0}^{d-4}\sum_{k_1=0}^{d-j_1-4}\sum_{j_2=0}^{d-4}\sum_{k_2=0}^{d-j_2-4}(-2)^{j_1+j_2}(-4)^{2d-j_1-k_1-j_2-k_2-8} y_1^{j_1}(y_1+z_1+1)^{k_1} y_2^{j_2}(y_2+z_2+1)^{k_2}$$

$$\sum_{\substack{L_1\subseteq\{1,\ldots,d-4\}\\ |L_1|=j_1}}\ \sum_{\substack{M_1\subseteq\{1,\ldots,d-4\}\setminus L_1\\ |M_1|=d-j_1-k_1-4}}\ \sum_{\substack{L_2\subseteq\{1,\ldots,d-4\}\\ |L_2|=j_2}}\ \sum_{\substack{M_2\subseteq\{1,\ldots,d-4\}\setminus L_2\\ |M_2|=d-j_2-k_2-4}}\ \sum_{S\subseteq L_1\cap L_2}$$

$$(-1)^{|(L_1\cap L_2)\setminus S|}$$

$$\begin{aligned}I_4(\mathcal{P},(&|(L_1\setminus(L_2\cup M_2))\cup(L_2\setminus(L_1\cup M_1))\cup S|,\\ &|(M_1\setminus(L_2\cup M_2))\cup(M_2\setminus(L_1\cup M_1))\cup((L_1\cap L_2)\setminus S)|,\\ &|(L_1\cap M_2)\cup(L_2\cap M_1)|,\\ &|M_1\cap M_2|)) =\end{aligned}$$

$$\frac{2^{(d-3)(d-4)/2}}{2(d-4)!}\sum_{j_1=0}^{d-4}\sum_{k_1=0}^{d-j_1-4}\sum_{j_2=0}^{d-4}\sum_{k_2=0}^{d-j_2-4}(-2)^{j_1+j_2}(-4)^{2d-j_1-k_1-j_2-k_2-8}$$

$$\sum_{\substack{L_1\subseteq\{1,\ldots,d-4\}\\ |L_1|=j_1}}\ \sum_{\substack{M_1\subseteq\{1,\ldots,d-4\}\setminus L_1\\ |M_1|=d-j_1-k_1-4}}\ \sum_{\substack{L_2\subseteq\{1,\ldots,d-4\}\\ |L_2|=j_2}}\ \sum_{\substack{M_2\subseteq\{1,\ldots,d-4\}\setminus L_2\\ |M_2|=d-j_2-k_2-4}}\ \sum_{S\subseteq L_1\cap L_2}$$

$$(-1)^{|(L_1\cap L_2)\setminus S|}$$

$$\begin{aligned}I_4(\mathcal{P},(&|(L_1\setminus(L_2\cup M_2))\cup(L_2\setminus(L_1\cup M_1))\cup S|,\\ &|(M_1\setminus(L_2\cup M_2))\cup(M_2\setminus(L_1\cup M_1))\cup((L_1\cap L_2)\setminus S)|,\\ &|(L_1\cap M_2)\cup(L_2\cap M_1)|,\\ &|M_1\cap M_2|))\end{aligned}$$

$$\int_F \left(y_2^2 z_1 + (z_1-z_2)^2 + y_1^2 z_2 - y_1 y_2(z_1+z_2)\right)$$
$$y_1^{j_1}(y_1+z_1+1)^{k_1} y_2^{j_2}(y_2+z_2+1)^{k_2} dy_1 dz_1 dy_2 dz_2 =$$

$$\frac{2^{4d+(d-3)(d-4)/2-17}}{(d-4)!}\sum_{j_1=0}^{d-4}\sum_{k_1=0}^{d-j_1-4}\sum_{j_2=0}^{d-4}\sum_{k_2=0}^{d-j_2-4}\frac{(-1)^{k_1+k_2}}{2^{j_1+j_2+2k_1+2k_2}}$$

$$\sum_{l_{22}=\max(0,-d+j_1+j_2+4)}^{\min(j_1,j_2)}$$
$$\sum_{m_{22}=\max(0,d-j_1-k_1-j_2-k_2+l_{22}-4)}^{\min(d-j_1-k_1-4,d-j_2-k_2-4,d-j_1-j_2+l_{22}-4)}$$
$$\sum_{l_{11}=\max(0,-d+j_1+j_2+k_2-l_{22}+m_{22}+4)}^{\min(j_1-l_{22},k_2,-d+j_1+j_2+k_1+k_2-l_{22}+m_{22}+4)}$$
$$\sum_{l_{21}=\max(0,-d+j_1+j_2+k_1-l_{22}+m_{22}+4)}^{\min(j_2-l_{22},d-l_{22}-m_{22}-l_{11}-4,-d+j_1+j_2+k_1+k_2-l_{22}+m_{22}-l_{11}+4)}$$
$$\sum_{l_{14}=0}^{l_{22}}(-1)^{l_{22}-l_{14}}$$

$$\binom{d-4}{\begin{array}{c}m_{22},l_{11},l_{21},l_{14},l_{22}-l_{14},j_1-l_{11}-l_{22},j_2-l_{21}-l_{22},\\ d-j_1-j_2-k_2+l_{11}+l_{22}-m_{22}-4,\\ d-j_1-j_2-k_1+l_{21}+l_{22}-m_{22}-4,\\ -d+j_1+j_2+k_1+k_2-l_{11}-l_{21}-l_{22}+m_{22}+4\end{array}}$$

$$\begin{aligned}I_4(\mathcal{P},(&l_{11}+l_{21}+l_{14},\\ &2d-2j_1-2j_2-k_1-k_2+3l_{22}-2m_{22}+l_{11}+l_{21}-l_{14}-8,\\ &j_1+j_2-2l_{22}-l_{11}-l_{21},\\ &m_{22}))\end{aligned}$$

$$\int_F \left(y_2^2 z_1 + (z_1-z_2)^2 + y_1^2 z_2 - y_1 y_2(z_1+z_2)\right)$$
$$y_1^{j_1}(y_1+z_1+1)^{k_1} y_2^{j_2}(y_2+z_2+1)^{k_2} dy_1 dz_1 dy_2 dz_2 =$$

$$\frac{2^{4d+(d-3)(d-4)/2-17}}{(d-4)!}\sum_{j_1=0}^{d-4}\sum_{k_1=0}^{d-j_1-4}\sum_{j_2=0}^{d-4}\sum_{k_2=0}^{d-j_2-4}\frac{(-1)^{k_1+k_2}}{2^{j_1+j_2+2k_1+2k_2}}$$
$$\sum\nolimits_{m_{22}=\max(0,d-j_1-k_1-j_2-k_2-4)}^{\min(d-j_1-k_1-4,d-j_2-k_2-4)}$$
$$\sum\nolimits_{l_{22}=\max(0,-d+j_1+j_2+m_{22}+4)}^{\min(j_1,j_2,-d+j_1+j_2+k_1+k_2+m_{22}+4)}$$
$$\sum\nolimits_{m_{11}=\max(0,d-j_1-j_2-k_1-m_{22}+l_{22}-4)}^{\min(d-j_1-k_1-m_{22}-4,k_2,d-j_1-j_2-m_{22}+l_{22}-4)}$$
$$\sum\nolimits_{m_{21}=\max(0,d-j_1-j_2-k_2-m_{22}+l_{22}-4}^{\min(d-j_2-k_2-m_{22}-4,d-m_{22}-l_{22}-m_{11}-4,d-j_1-j_2-m_{22}+l_{22}-m_{11}-4)}$$
$$\sum\nolimits_{l_{14}=0}^{l_{22}}(-1)^{l_{22}-l_{14}}$$
$$\begin{pmatrix} d-4 \\ m_{22}, m_{11}, m_{21}, l_{14}, l_{22}-l_{14}, \\ d-j_1-k_1-m_{22}-m_{11}-4, d-j_2-k_2-m_{22}-m_{21}-4, \\ -d+j_1+j_2+k_1+m_{22}-l_{22}+m_{11}+4, \\ -d+j_1+j_2+k_2+m_{22}-l_{22}+m_{21}+4, \\ d-j_1-j_2-m_{22}+l_{22}-m_{11}-m_{21}-4 \end{pmatrix}$$
$$\begin{aligned} I_4(\mathcal{P}, (&-2d+2j_1+2j_2+k_1+k_2-2l_{22}+2m_{22}+m_{11}+m_{21}+l_{14}+8, \\ &m_{11}+m_{21}+l_{22}-l_{14}, \\ &2d-j_1-j_2-k_1-k_2-2m_{22}-m_{11}-m_{21}-8, \\ &m_{22})) \end{aligned}$$
$$\int_F \left(y_2^2 z_1+(z_1-z_2)^2+y_1^2 z_2-y_1y_2(z_1+z_2)\right)$$
$$y_1^{j_1}(y_1+z_1+1)^{k_1}y_2^{j_2}(y_2+z_2+1)^{k_2}dy_1dz_1dy_2dz_2.$$

□

CHAPTER 3

Shift Radix Systems and the finiteness property

3.1. Introduction and definitions

In the following chapter we will define and discuss Shift Radix Systems, which were first introduced by Akiyama, Borbély, Brunotte, Pethő, and Thuswaldner in [**Akiyama et al., 2005**] (compare also [**Akiyama et al., 2006b, Akiyama et al., 2008b, Akiyama et al., 2008c, Akiyama et al., 2006a, Kirschenhofer and Thuswaldner, 2014**]). Shift Radix Systems are dynamical systems which are closely related to two important number systems known as β-expansions and Canonical Number Systems. It is these relations which led to their introduction in the first place. Indeed, they form a generalization of both and we will discuss the relations among them in detail in the upcoming sections. Furthermore, Shift Radix Systems are almost linear mappings and for one well-studied dynamical property it suffices to consider the linear part only and to ignore the error term, which, in the case of Shift Radix Systems, is introduced by application of the floor function at one point. It will turn out that the behavior of the linear part is closely related to the Schur-Cohn region introduced in Chapter 2. Another important dynamical property - the so called finiteness property - is highly dependent on the error term though and its study gave rise to the introduction of so-called critical points at which the Shift Radix Systems behave highly chaotic. In order to get a grip on this dynamical property even in the vicinity of critical points, so-called cut-out polyhedra and sets of witnesses were invented. The detailed discussion of these two notions will close the chapter at which point we will - at least in theory - have enough tools to characterize the set of those Shift Radix Systems which have the finiteness property as precisely as we wish. In practice however, the tools reviewed in this chapter will face intrinsic obstacles quite soon. The discussion of techniques and algorithms which will allow to push back these obstacles to a point where topologically surprising things do happen will be the subject of Chapter 4.

In this chapter only "real" Shift Radix Systems will be treated, that is Shift Radix Systems with a real vector as its parameter operating on integer vectors. Generalizations to complex numbers and Gaussian integers as well as to irrational quadratic fields and Euclidean integer rings will be considered in Chapter 5 and Chapter 6.

Definition 3.1.1. [**Akiyama et al., 2005**] *For $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ the mapping*

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$$
$$\mathbf{a} = (a_1, \ldots, a_d) \mapsto (a_2, \ldots, a_d, -\lfloor \mathbf{r}\mathbf{a} \rfloor)$$

where $\mathbf{r}\mathbf{a} = \sum_{i=1}^{d} r_i a_i$ is the scalar product of $\mathbf{r}$ and $\mathbf{a}$, is called the d-dimensional Shift Radix System (SRS for short) associated with $\mathbf{r}$ and $\mathbf{r}$ is called the parameter of $\tau_{\mathbf{r}}$. Furthermore we define

$$\mathcal{D}_d := \left\{\mathbf{r} \in \mathbb{R}^d \mid \forall\, \mathbf{a} \in \mathbb{Z}^d : \exists\, i, j \in \mathbb{N} : \tau_{\mathbf{r}}^i(\mathbf{a}) = \tau_{\mathbf{r}}^{i+j}(\mathbf{a})\right\}$$
$$\mathcal{D}_d^{(0)} := \left\{\mathbf{r} \in \mathbb{R}^d \mid \forall\, \mathbf{a} \in \mathbb{Z}^d : \exists\, i \in \mathbb{N} : \tau_{\mathbf{r}}^i(\mathbf{a}) = \mathbf{0}\right\}$$

where $\tau_{\mathbf{r}}^i(\mathbf{a})$ means i-fold application of $\tau_{\mathbf{r}}$ to $\mathbf{a}$. Elements of $\mathcal{D}_d^{(0)}$ are said to have the finiteness property.

It is a trivial but important observation that $\mathcal{D}_d^{(0)} \subset \mathcal{D}_d$ for all $d \in \mathbb{N}$. Whereas it is easily seen that $\mathcal{D}_1 = [-1, 1]$ and $\mathcal{D}_1^{(0)} = [0, 1)$, it is well-known that $\mathcal{D}_d^{(0)}$ has a very complicated structure even for $d = 2$ (cf. e.g. [**Surer, 2007**]).

3.2. A relation to β-expansions

Shift Radix Systems form a generalization [**Akiyama et al., 2005**] of so-called β-expansions (cf. e.g. [**Rényi, 1957, Parry, 1960, Frougny and Solomyak, 1992**]) which in turn are a natural generalization of positional notation systems. In positional notation systems an arbitrary integer b greater than 1 is used as a basis to represent any real number x in the form

$$x = \operatorname{sgn}(x)\left(a_m b^m + a_{m-1} b^{m-1} + \ldots\right)$$

where m is some integer and a_i is an element of the set of digits $[\![0, b-1]\!]$ for $i \in \mathbb{Z}_{\leq m}$. This representation is "almost" unique. Indeed, if one either forbids the infinite period 0 or the infinite period $b-1$ in the digit expansion, it is unique. Forbidding the period $b-1$ is equivalent to the following restriction: Whenever there are multiple digit representations of a given number (in the case of positional notation systems there can be at most 2) take the one that is largest with respect to lexicographical order. So out of the two base 10 representations "0.999..." and "1.000..." of the natural number 1, the latter one is preferred.

The way in which β-expansions generalize positional notation systems is that instead of taking an integer b as a basis one simply takes any real number β greater than 1. It is clear that in many regards number systems defined in such a way behave quite differently than positional notation systems. One thing that both systems have in common though is that you can force the representations to be unique if you demand them to be largest possible with respect to lexicographical order. Doing so will lead to the so-called greedy expansion with respect to β.

Definition 3.2.1. [**Rényi, 1957**] *Let $\beta > 1$ be a non-integral real number and $\gamma \in [0, \infty)$. The set $\mathcal{A}_\beta := [\![0, \lfloor \beta \rfloor]\!]$ is called the set of digits (for β) and the representation*

$$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \ldots$$

with $m \in \mathbb{Z}$ and $a_i \in \mathcal{A}_\beta$ for all $i \in \mathbb{Z}_{\leq m}$, satisfying

$$0 \leq \gamma - \sum_{i=k}^{m} a_i \beta^i < \beta^k$$

for all $m \geq k \in \mathbb{Z}$ is called the greedy expansion of γ with respect to β.

As stated above, the greedy expansion of a number is always unique. Indeed, for $\gamma \in [0, 1)$ it can be computed using the so-called β transformation:

Definition 3.2.2. [**Rényi, 1957**] *For a non-integral real number $\beta > 1$ the mapping*

$$\begin{aligned} T_\beta : [0,1) &\to [0,1) \\ \gamma &\mapsto \beta\gamma - \lfloor \beta\gamma \rfloor = \{\beta\gamma\} \end{aligned}$$

is called β-transformation.

Lemma 3.2.3. [**Rényi, 1957**] *For a non-integral real number $\beta > 1$ and a $\gamma \in [0, 1)$ the greedy expansion of γ with respect to β is given by*

$$\gamma = \sum_{i=1}^{\infty} \left\lfloor \beta T_\beta^{i-1}(\gamma) \right\rfloor \beta^{-i}$$

The uniqueness is something β-expansions have in common with usual positional notation systems. One aspect in which they differ is that with positional notation systems it doesn't really matter which number one takes as a basis - all of them are equally suitable but with β-expansions there is a certain quality of the basis which not all choices do fulfill. This quality is related to the question on how large the set of those numbers is which do have a finite greedy expansion with respect to the given β. It turns out that for some but not all choices of β this set is largest possible.

Definition 3.2.4. [**Frougny and Solomyak, 1992**] *For a non-integral real number $\beta > 1$ let $\operatorname{Fin}(\beta)$ denote the set of all $\gamma \in [0, \infty)$ which have finite greedy expansion with respect to β. β is said to have property (F) iff $\operatorname{Fin}(\beta) = \mathbb{Z}[\frac{1}{\beta}] \cap [0, \infty)$.*

It is clear that the inclusion $\mathrm{Fin}(\beta) \subseteq \mathbb{Z}[\frac{1}{\beta}] \cap [0, \infty)$ always holds, so β having property (F) is also equivalent to $\mathbb{Z}[\frac{1}{\beta}] \cap [0, \infty) \subseteq \mathrm{Fin}(\beta)$. Characterizing all β which have property (F) turns out to be a very difficult problem. A lemma from [**Frougny and Solomyak, 1992**] provides a first hint on where to look for such numbers. But before we can state the Lemma we need to give another

Definition 3.2.5. *A real algebraic integer greater than* 1 *is called Pisot number* [**Thue, 1912, Hardy, 1919, Pisot, 1919**] *iff all of its Galois conjugates are less than* 1 *in absolute value.*

A real algebraic integer greater than 1 *is called Salem number* [**Salem, 1963**] *iff all of its Galois conjugates are less than or equal to* 1 *in absolute value and at least one of its Galois conjugates has an absolute value of exactly* 1.

Lemma 3.2.6. [**Frougny and Solomyak, 1992**] *Let $\beta > 1$ be a non-integral real number. Then*

(i) $\mathbb{N} \subseteq \mathrm{Fin}(\beta) \Rightarrow \beta$ *is a Pisot or Salem number*

(ii) β *has property (F)* $\Rightarrow \beta$ *is a Pisot number.*

From the previous lemma it follows in particular that every β which has property (F) is an algebraic integer. From this one can derive the main relation between property (F) of β-expansion on the one hand and the finiteness property of Shift Radix Systems on the other hand. The following theorem was first proven in [**Hollander, 1996**] and was adapted to fit the notion of Shift Radix Systems.

Theorem 3.2.7. [**Akiyama et al., 2005**] *Let $\beta > 1$ be a non-integral algebraic integer and $(x - \beta)(x^{d-1} + r_{d-2}x^{d-2} + \ldots + r_0)$ its minimal polynomial. Then*

$$\beta \text{ has property (F)} \Leftrightarrow (r_0, \ldots, r_{d-2}) \in \mathcal{D}_{d-1}^{(0)}.$$

3.3. A relation to Canonical Number Systems

Canonical Number Systems were already studied in [**Knuth, 1960**] (cf. also [**Knuth, 1998**]) and [**Penney, 1965**]. Just like β-expansions, Canonical Number Systems (or CNS for short) generalize usual positional notation systems but in a different way. The initial observation which led to their discovery was that every non-zero Gaussian integer γ can be represented uniquely in the form

$$\gamma = c_0 + c_1 b + \ldots + c_m b^m$$

where $m \in \mathbb{N}$, $c_i \in \{0, 1\}$ for $i \in [\![0, m-1]\!]$, $c_m = 1$, and b is a very specific Gaussian integer itself, that is $-1 + \mathrm{i}$. Since then the original notion of Canonical Number Systems has been generalized first to arbitrary quadratic number fields [**Gilbert, 1981, Kátai and Kovács, 1980, Kátai and Szabó, 1975, Kátai and Kovács, 1981**] and later also to arbitrary number fields [**Kovács, 1981, Kovács and Pethő, 1991**]. Finally an even more general concept of Canonical Number Systems was established in [**Pethő, 1991**] which is the one we shall use here.

Definition 3.3.1. [**Pethő, 1991**] *For $d \in \mathbb{N}$ and $P(X) := X^d + p_{d-1}X^{d-1} + \ldots + p_0 \in \mathbb{Z}[X]$ let $\mathcal{A}_P := [\![0, |p_0| - 1]\!]$ denote the set of digits (for P). Furthermore let $R_P := \mathbb{Z}[X]/P(X)Z[X]$ and $x := X + P(X)Z[X]$ - the image of X under the canonical epimorphism from $Z[X]$ to R_P. Then P is said to be a CNS-polynomial iff every non-zero $A \in R_P$ has a unique representation (which shall be called the CNS-representation of A) in the form*

$$A = a_0 + a_1 x + \ldots + a_m x^m$$

where $m \in \mathbb{N}_0$, $a_i \in \mathcal{A}_P$ for all $i \in [\![0, m]\!]$, and $a_m \neq 0$.

Since P in the previous definition is normed it is clear that any given coset $A \in R_P$ has a unique element of degree at most $d - 1$. If this element is given by

$$A = A_0 + A_1 x + \ldots + A_{d-1} x^{d-1}$$

where $A_i \in \mathbb{Z}$ for $i \in [\![0, d-1]\!]$, then the CNS-representation of A can, just as β-expansions, be computed by a rather simple dynamical system defined by backward division.

DEFINITION 3.3.2. [**Akiyama et al., 2005**] *For* $d \in \mathbb{N}$*,* $P(X) := X^d + p_{d-1}X^{d-1} + \ldots + p_0 \in \mathbb{Z}[X]$*, and* $A_d := 0$ *let*

$$T_P : \{A \in \mathbb{Z}[X] \mid \deg A < d\} \to \{A \in \mathbb{Z}[X] \mid \deg A < d\}$$
$$A = A_0 + A_1X + \ldots + A_{d-1}X^{d-1} \mapsto \sum_{i=0}^{d-1}(A_{i+1} - \lfloor A_0/p_0 \rfloor p_{i+1})X^i$$

It follows directly from the definition that for every $A \in \{A \in \mathbb{Z}[X] \mid \deg A < d\}$ we get

$$A = (A_0 - \lfloor A_0/p_0 \rfloor p_0) + XT_P(A)$$

where $(A_0 - \lfloor A_0/p_0 \rfloor p_0) \in \mathcal{A}_P$. If the orbit of A under T_P ends up in 0, iterative application of the transformation above will give the CNS-representation of A. Indeed, A does admit a CNS-representation iff there is a $k \in \mathbb{N}$ such that $T_P^k(A) = 0$ which proves the following lemma.

LEMMA 3.3.3. [**Akiyama et al., 2005**] *Let* $d \in \mathbb{N}$ *and* $P(X) := X^d + p_{d-1}X^{d-1} + \ldots + p_0 \in \mathbb{Z}[X]$*. Then* P *is a CNS-polynomial iff all orbits of* T_P *end up in* 0*.*

Due to the following lemma it is convenient to consider the conjugate mapping $\tilde{T_P}$ instead of T_P. Independently in [**Brunotte, 2001**] and [**Scheicher and Thuswaldner, 2004**] it was observed that the basis transformation

$$\left\{1, X, \ldots, X^{d-1}\right\} \to \{\omega_1, \ldots, \omega_d\}$$

where

$$\omega_j = \sum_{i=d-j+1}^{d} p_i X^{-d+j+i-1}, \text{ for } j \in [\![0, d]\!] \text{ and } p_d := 1$$

allows for a very nice and easy to apply representation of the $\tilde{T_P}$:

LEMMA 3.3.4. [**Brunotte, 2001**] *Let* $d \in \mathbb{N}$*,* $P(X) := X^d + p_{d-1}X^{d-1} + \ldots + p_0 \in \mathbb{Z}[X]$*,* $p_d := 1$*, and* $A = \sum_{j=1}^{d} a_j\omega_j \in R_P$ *where* $a_i \in \mathbb{Z}$ *for* $i \in [\![1, d]\!]$*. Then*

$$\tilde{T_P}(A) = -\left\lfloor \frac{p_1a_d + \ldots + p_da_1}{p_0} \right\rfloor \omega_d + \sum_{j=1}^{d-1} a_{j+1}\omega_j$$

This observation gave rise to the definition of the following function which is known as Brunotte's mapping.

DEFINITION 3.3.5. [**Brunotte, 2001**] *For* $d \in \mathbb{N}$*,* $P(X) := X^d + p_{d-1}X^{d-1} + \ldots + p_0 \in \mathbb{Z}[X]$*, and* $p_d := 1$ *let*

$$\tau_P : \mathbb{Z}^d \to \mathbb{Z}^d$$
$$(a_1, \ldots, a_d) \mapsto \left(a_2, \ldots, a_d, -\left\lfloor \frac{p_1a_d + \ldots + p_da_1}{p_0} \right\rfloor\right)$$

It is clear that P is a CNS-polynomial iff all orbits of τ_P end up in $\mathbf{0}$. But for $\mathbf{r} := \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right)$ we get that $\tau_P(\mathbf{a}) = \tau_{\mathbf{r}}(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{Z}^d$ which finally reveals the relation between Canonical Number Systems and Shift Radix Systems as it proves the following theorem.

THEOREM 3.3.6. [**Akiyama et al., 2005**] *Let* $d \in \mathbb{N}$ *and* $P(X) := X^d + p_{d-1}X^{d-1} + \ldots + p_0 \in \mathbb{Z}[X]$*. Then*

$$P \text{ is a CNS-polynomial} \Leftrightarrow \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right) \in \mathcal{D}_d^{(0)}.$$

3.4. The Schur-Cohn region and ultimately periodic orbits

In this section we will discuss the relation between Shift Radix Systems and the Schur-Cohn region treated in Chapter 2. As pointed out in the introduction of the present chapter we are mostly interested in two dynamical properties of Shift Radix Systems which, for a given parameter $\mathbf{r} \in \mathbb{R}^d$, boil down to the following two questions:

- Are all orbits of $\tau_{\mathbf{r}}$ eventually periodic? (Is $\mathbf{r} \in \mathcal{D}_d$?)
- Do all orbits of $\tau_{\mathbf{r}}$ end up in $\mathbf{0}$? (Is $\mathbf{r} \in \mathcal{D}_d^{(0)}$?)

The reason why we are interested in answering these questions was pointed out in Section 3.2 and Section 3.3. The present section will deal with the first of the two questions. Indeed, with the help of the Schur-Cohn region the set $\mathcal{D}_d$ can be characterized almost everywhere (with respect to the Lebesgue-measure) or, more precisely, everywhere but on its boundary. The following theorem can be found in [**Akiyama et al., 2005**] and is the SRS analogue of a theorem on CNS given in [**Gilbert, 1981**]. We repeat the proof as certain parts (especially the norm $\|\cdot\|_\rho$) are needed later.

Definition 3.4.1. *For $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ let*

$$R(\mathbf{r}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & \cdots & \cdots & \cdots & -r_d \end{pmatrix} \in \mathbb{R}^{d \times d}.$$

Theorem 3.4.2. [**Akiyama et al., 2005**] *Let $d \in \mathbb{N}$. Then*

$$\mathcal{E}_d^{(\mathbb{R})} \subseteq \mathcal{D}_d \subseteq \overline{\mathcal{E}_d^{(\mathbb{R})}}.$$

Proof. Let $\mathbf{r} = (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d$ and $P(x) := = x^d + x^{d-1}r_{d-1} + \ldots + r_0 \in \mathbb{R}[x]$. Then it is clear (cf. Section 2.1) that $R(\mathbf{r}) = C(P)$ and

$$\mathcal{E}_d^{(\mathbb{R})} = \left\{\mathbf{r} \in \mathbb{R}^d \mid \rho(R(\mathbf{r}) < 1\right\}$$
$$\overline{\mathcal{E}_d^{(\mathbb{R})}} = \left\{\mathbf{r} \in \mathbb{R}^d \mid \rho(R(\mathbf{r}) = 1\right\}$$

(the second statement follows from the fact that the coefficients of a polynomial depend continuously on its roots [**Naulin and Pabst, 1994**]). On the other hand we have

$$\tau_{\mathbf{r}}(\mathbf{a}) = R(\mathbf{r})\mathbf{a} + (0, \ldots, 0, \{\mathbf{ra}\})$$

for all $\mathbf{a} \in \mathbb{Z}^d$. Assume that $0 < \rho(R(\mathbf{r})) < 1$ and let $\rho \in (\rho(R(\mathbf{r})), 1)$. Then there exists a norm $\|\cdot\|_\rho$ on $\mathbb{R}^d$ such that

$$\|R(\mathbf{r})\mathbf{a}\|_\rho \leq \rho \|\mathbf{a}\|_\rho$$

for all $\mathbf{a} \in \mathbb{R}^d$ (cf. [**Lagarias and Wang, 1996**, Formula (3.2)]). Thus we get for all $\mathbf{a} \in \mathbb{Z}^d$ with $\|\mathbf{a}\|_\rho > \frac{\|(0,\ldots,0,1)\|_\rho}{1-\rho} \Leftrightarrow \rho \|\mathbf{a}\|_\rho < \|\mathbf{a}\|_\rho - \|(0, \ldots, 0, 1)\|_\rho$ that

$$\|\tau_{\mathbf{r}}(\mathbf{a})\|_\rho \leq \|R(\mathbf{r})\mathbf{a}\|_\rho + \|(0, \ldots, 0, \{\mathbf{ra}\}\|_\rho < \rho \|\mathbf{a}\|_\rho + \|(0, \ldots, 0, 1)\|_\rho < \|\mathbf{a}\|_\rho.$$

So whenever $\rho(R(\mathbf{r})) < 1$ the associated Shift Radix System is contractive outside the ball $\left\{\mathbf{a} \in \mathbb{Z}^d \mid \|\mathbf{a}\|_\rho \leq \frac{\|(0,\ldots,0,1)\|_\rho}{1-\rho}\right\}$ which means that all orbits of $\tau_{\mathbf{r}}$ eventually end up inside the ball and must therefore get periodic. In conclusion we have

$$\mathcal{E}_d^{(\mathbb{R})} = \left\{\mathbf{r} \in \mathbb{R}^d \mid \rho(R(\mathbf{r})) < 1\right\} \subseteq \mathcal{D}_d.$$

For the proof of the other inclusion assume to the contrary that there is a $\mathbf{r} \in \mathcal{D}_d$ with $\rho(R(\mathbf{r})) > 1$. First we conclude by induction that

$$\tau_{\mathbf{r}}^n(\mathbf{a}) = R(\mathbf{r})^n(\mathbf{a}) + \sum_{i=1}^{n} R(\mathbf{r})^{n-i} \left(0, \ldots, 0, \left\{\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{a})\right\}\right)^\intercal$$

for all $n \in \mathbb{N}_0$ and $\mathbf{a} \in \mathbb{Z}^d$. Let λ be an eigenvalue of $R(\mathbf{r})$ with $|\lambda| > 1$, $\mathbf{v}$ a corresponding left eigenvector, $c \in \mathbb{R}$ such that

$$\left|\mathbf{v}\left(0,\ldots,0,\left\{\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{a})\right\}\right)\right| \le c$$

for all $i \in \mathbb{N}$ (c exists since $\left|\left(0,\ldots,0,\left\{\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{a})\right\}\right)\right| < 1$ for all $i \in \mathbb{N}$) and let $\mathbf{a} \in \mathbb{Z}^d$ such that

$$|\mathbf{va}| > \frac{c+1}{|\lambda|-1}.$$

Then

$$\begin{aligned}\left|\sum_{i=1}^{n}\lambda^{n-i}\mathbf{v}\left(0,\ldots,0,\left\{\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{a})\right\}\right)\right| &\le \sum_{i=1}^{n}|\lambda|^{n-i}\left|\mathbf{v}\left(0,\ldots,0,\left\{\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{a})\right)\right\}\right| \\ &\le c\sum_{i=1}^{n}|\lambda|^{n-i} = c\sum_{i=0}^{n-1}|\lambda|^{i} = c\frac{|\lambda|^n-1}{|\lambda|-1} \le \frac{c\,|\lambda|^n+|\lambda|^n}{|\lambda|-1} \\ &< |\lambda|^n\,|\mathbf{va}|\end{aligned}$$

and therefore

$$\begin{aligned}|\mathbf{v}\tau_{\mathbf{r}}^{n}(\mathbf{a})| &= \left|\lambda^n\mathbf{va}+\sum_{i=1}^{n}\lambda^{n-i}\mathbf{v}\left(0,\ldots,0,\left\{\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{a})\right\}\right)\right| \\ &\ge \left||\lambda|^n\,|\mathbf{va}| - \left|\sum_{i=1}^{n}\lambda^{n-i}\mathbf{v}\left(0,\ldots,0,\left\{\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{a})\right\}\right)\right|\right| \\ &= |\lambda|^n\,|\mathbf{va}| - \left|\sum_{i=1}^{n}\lambda^{n-i}\mathbf{v}\left(0,\ldots,0,\left\{\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{a})\right\}\right)\right| \\ &\ge |\lambda|^n\,|\mathbf{va}| - \sum_{i=1}^{n}|\lambda|^{n-i}\left|\mathbf{v}\left(0,\ldots,0,\left\{\mathbf{r}\tau_{\mathbf{r}}^{i-1}(\mathbf{a})\right\}\right)\right| \\ &> |\lambda|^n\,\frac{c+1}{|\lambda|-1} - c\sum_{i=1}^{n}|\lambda|^{n-i} = |\lambda|^n\,\frac{c+1}{|\lambda|-1} - c\frac{|\lambda|^n-1}{|\lambda|-1} = \frac{|\lambda|^n+c}{|\lambda|-1} > |\lambda|^{n-1}\end{aligned}$$

for all $n \in \mathbb{N}_0$. Thus the orbit of $\mathbf{a}$ under $\tau_{\mathbf{r}}$ is not eventually periodic which contradicts $\mathbf{r} \in \mathcal{D}_d$. □

COROLLARY 3.4.3. [**Akiyama et al., 2005**] *Let $d \in \mathbb{N}$. Then*

(i) $\mathcal{D}_d \subseteq \left\{\mathbf{r} \in \mathbb{R}^d \mid \rho(R(\mathbf{r})) \le 1\right\}$

(ii) $\mathcal{D}_d \supseteq \left\{\mathbf{r} \in \mathbb{R}^d \mid \rho(R(\mathbf{r})) < 1\right\}$

(iii) $\partial\mathcal{D}_d = \left\{\mathbf{r} \in \mathbb{R}^d \mid \rho(R(\mathbf{r})) = 1\right\}$.

The previous theorem characterizes $\mathcal{D}_d$ everywhere but on its boundary. It turns out that on the boundary it is much harder to decide whether or not a point belongs to $\mathcal{D}_d$. Even for the case $d = 2$ only partial results could be achieved by now. To close this section we shall repeat previous results and a very important open conjecture on the boundary of $\mathcal{D}_2$.

THEOREM 3.4.4. [**Akiyama et al., 2006b**]

(i) $\{(x, x+1) \mid x \in [-1, 1)\} \cup \{(x, -x-1) \mid x \in [-1, 0] \cup \{(1, -1), (1, 0), (1, 1)\}\} \subseteq \mathcal{D}_2$

(ii) $(\{(x, -x-1) \mid x \in (0, 1]\} \cup \{(1, 2)\}) \cap \mathcal{D}_2 = \emptyset$.

The next theorem has a very complicated proof and can be found in [**Akiyama et al., 2008a**] (cf. also [**Pethő, 2009, Kirschenhofer et al., 2008**]).

THEOREM 3.4.5. $\left\{\left(1, \frac{\pm 1 \pm \sqrt{5}}{2}\right), \left(1, \pm\sqrt{2}\right), \left(1, \pm\sqrt{3}\right)\right\} \subseteq \mathcal{D}_2$.

The previous theorem settles special cases of a conjecture which has been formulated in several different contexts and many publications such as [**Akiyama et al., 2006b, Bruin et al., 2003, Lowenstein et al., 1997, Vivaldi, 1994**].

CONJECTURE 3.4.6. $\{(1, y) \mid y \in (-2, 2)\} \subseteq \mathcal{D}_2$.

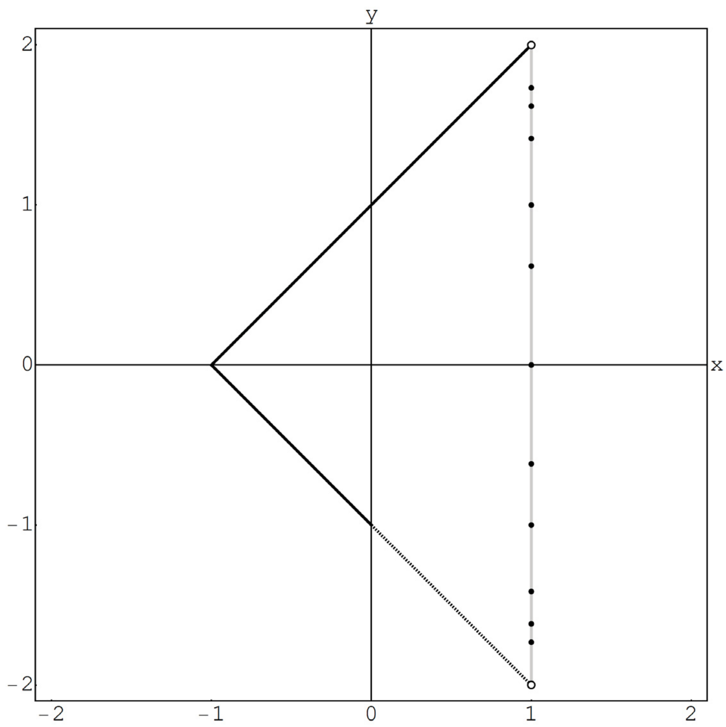


FIGURE 1. Overview of settled parts on the boundary of $\mathcal{D}_2$.

3.5. Cycles and polyhedra

Referring to the two questions asked at the beginning of the previous section where we presented several results on the first, we now turn the second, which will be partially answered in the upcoming Theorem 3.5.10. Our first trivial, yet decisive observation at the task of characterizing $\mathcal{D}_d^{(0)}$ is that it is a subset of $\mathcal{D}_d$ which is identical to $\mathcal{E}_d^{(\mathbb{R})}$ up to its boundary. We recall the difference between $\mathcal{D}_d$ and $\mathcal{D}_d^{(0)}$: While the orbits of Shift Radix Systems associated with parameters from $\mathcal{D}_d$ end up in any period, in case of $\mathcal{D}_d^{(0)}$ they end up in a special period, that is the trivial $(\mathbf{0})$. The following lemma is therefore a direct consequence of the definitions.

DEFINITION 3.5.1. [**Akiyama et al., 2005**] *For $d \in \mathbb{N}$ let $\mathcal{C}_d^{(\mathbb{Z})} := \bigcup_{n \in \mathbb{N}_0} (\mathbb{Z}^d)^n$ denote the set of (d-dimensional, real) cycles.*

For a cycle $\pi = (\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathcal{C}_d^{(\mathbb{Z})}$ let $P_{\mathbb{R}}(\pi) := \left\{ \mathbf{r} \in \mathbb{R}^d \mid \forall\, i \in [\![1, k]\!] : \tau_{\mathbf{r}}(\mathbf{a}_i) = \mathbf{a}_{i\%k+1} \right\}$, i.e. the set of those parameters $\mathbf{r}$ for which π is a cycle of the associated Shift Radix System. $P_{\mathbb{R}}(\pi)$ shall be referred to as the cutout polyhedron of π.

LEMMA 3.5.2. [**Akiyama et al., 2005**] *Let $d \in \mathbb{N}$. Then*

$$\mathcal{D}_d^{(0)} = \mathcal{D}_d \setminus \bigcup_{\pi \in \mathcal{C}_d^{(\mathbb{Z})} \setminus \{(\mathbf{0})\}} P_{\mathbb{R}}(\pi).$$

The lemma above explains the first half of the notion "cutout polyhedron". Lemma 3.5.4 will explain the second.

DEFINITION 3.5.3. *A set $P \subseteq \mathbb{R}^d$ ($d \in \mathbb{N}$) is called (real, convex, d-dimensional) polyhedron iff it is the intersection of finitely many half-spaces or $\mathbb{R}^d$ itself. A polyhedron is considered non-degenerate iff it has positive and finite Lebesgue measure and degenerate otherwise. The set of all real d-dimensional polyhedra shall be denoted by $\mathcal{P}_d^{(\mathbb{R})}$.*

LEMMA 3.5.4. [**Akiyama et al., 2005**] *Let $d \in \mathbb{N}$ and $\mathbf{a} = (a_1, \ldots, a_d), \mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{Z}^d$. Then*

$$\left\{\mathbf{r} \in \mathbb{R}^d \mid \tau_\mathbf{r}(\mathbf{a}) = \mathbf{b}\right\} = \{\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d \mid \forall\, i \in [\![1, d-1]\!] : b_i = a_{i+1} \\ 0 \le r_1 a_1 + \ldots + r_d a_d + b_d < 1\}.$$

In particular: If $\pi \in \mathcal{C}_d^{(\mathbb{Z})}$ then $P_\mathbb{R}(\pi)$ is a (possibly degenerate) convex polyhedron.

PROOF. Let $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$. Then

$$\begin{aligned}\tau_\mathbf{r}(\mathbf{a}) = \mathbf{b} &\Leftrightarrow (b_1, \ldots, b_d) = (a_2, \ldots, a_d, -\lfloor r_1 a_1 + \ldots + r_d a_d \rfloor) \\ &\Leftrightarrow \forall\, i \in [\![1, d-1]\!] : b_i = a_{i+1} \quad \wedge \quad b_d = -\lfloor r_1 a_1 + \ldots + r_d a_d \rfloor \\ &\Leftrightarrow \forall\, i \in [\![1, d-1]\!] : b_i = a_{i+1} \quad \wedge \quad \lfloor r_1 a_1 + \ldots + r_d a_d + b_d \rfloor = 0 \\ &\Leftrightarrow \forall\, i \in [\![1, d-1]\!] : b_i = a_{i+1} \quad \wedge \quad 0 \le r_1 a_1 + \ldots + r_d a_d + b_d < 1.\end{aligned}$$

The proof of the "In particular" part obviously follows by induction. □

We now know how to compute the cutout polyhedron of a given cycle. But how do we even find a cycle which corresponds to a non-empty cutout polyhedron? A first hint is given by the following lemma.

LEMMA 3.5.5. [**Akiyama et al., 2005**] *Let $d \in \mathbb{N}$, $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$, $\rho \in (\rho(R(\mathbf{r})), 1)$, $\|\cdot\|_\rho$ norm on $\mathbb{R}^d$ with $\|R(\mathbf{r})\mathbf{a}\|_\rho \le \rho \|\mathbf{a}\|_\rho$ for all $\mathbf{a} \in \mathbb{R}^d$ (cf. proof of Theorem 3.4.2), and $\mathbf{a} \in \mathbb{Z}^d$ such that $\tau_\mathbf{r}^k(\mathbf{a}) = \mathbf{a}$ for some $k \in \mathbb{N}$. Then*

$$\|\mathbf{a}\|_\rho \le \frac{\|(0, \ldots, 0, 1)\|_\rho}{1 - \rho}.$$

In particular: $\left\{\pi \in \mathcal{C}_d^{(\mathbb{Z})} \mid \mathbf{r} \in P_\mathbb{R}(\pi)\right\}$ is a finite set.

PROOF. From the proof of Theorem 3.4.2 we have that

$$\|\mathbf{a}\|_\rho > \frac{\|(0, \ldots, 0, 1)\|_\rho}{1 - \rho} \Rightarrow \|\tau_\mathbf{r}(\mathbf{a})\|_\rho < \|\mathbf{a}\|_\rho$$

which already proves the lemma. □

The following theorem is essentially the same as Theorem 3.2 in [**Weitzer, 2015a**] (cf. also Lemma 7.2 in [**Akiyama et al., 2005**]) and improves the "In particular" part of the previous lemma.

THEOREM 3.5.6. [**Weitzer, 2015a**] *Let $d \in \mathbb{N}$ and $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$. Then there is an open neighborhood B of $\mathbf{r}$ for which*

$$\left\{\pi \in \mathcal{C}_d^{(\mathbb{Z})} \mid B \cap P_\mathbb{R}(\pi) \ne \emptyset\right\}$$

is a finite set.
In particular: If $M \subseteq \operatorname{int}(\mathcal{D}_d)$ with $\operatorname{dist}(M, \partial\mathcal{D}_d) > 0$ then $\left\{\pi \in \mathcal{C}_d^{(\mathbb{Z})} \mid M \cap P_\mathbb{R}(\pi) \ne \emptyset\right\}$ is a finite set.

PROOF. Let $\rho \in (\rho(R(\mathbf{r})), 1)$ and $\|\cdot\|_\rho$ norm on $\mathbb{R}^d$ with $\|R(\mathbf{r})\mathbf{a}\|_\rho \le \rho \|\mathbf{a}\|_\rho$ for all $\mathbf{a} \in \mathbb{R}^d$ (cf. proof of Theorem 3.4.2). As the function which maps $\mathbf{s} \in \mathbb{R}^d$ to $\max\left\{\frac{\|R(\mathbf{s})\mathbf{a}\|_\rho}{\|\mathbf{a}\|_\rho} \mid \mathbf{a} \in \mathbb{R}^d \wedge \|\mathbf{a}\|_\rho = 1\right\}$ is continuous, there is an open neighborhood B of $\mathbf{r}$ such that $\|R(\mathbf{s})\mathbf{a}\|_\rho \le \rho \|\mathbf{a}\|_\rho$ for every $\mathbf{s} \in B$ and $\mathbf{a} \in \mathbb{R}^d$. As in the proof of Theorem 3.4.2 we deduce that

$$\|\mathbf{a}\|_\rho > \frac{\|(0, \ldots, 0, 1)\|_\rho}{1 - \rho} \Rightarrow \|\tau_\mathbf{s}(\mathbf{a})\|_\rho < \|\mathbf{a}\|_\rho$$

for all $\mathbf{s} \in B$. Therefore the set of all cycles of d-dimensional Shift Radix Systems associated with parameters in B is contained in the ball $\left\{\mathbf{a} \in \mathbb{Z}^d \mid \|\mathbf{a}\|_\rho \le \frac{\|(0,\ldots,0,1)\|_\rho}{1-\rho}\right\}$ and is thus finite as claimed.

For the in "In particular" part let C be a compact set with $M \subseteq C \subseteq \operatorname{int}(\mathcal{D}_d)$ (note that $\mathcal{D}_d$ is bounded according to Theorem 3.4.2) and for every $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$ let $B_{\mathbf{r}}$ be an open neighborhood of $\mathbf{r}$ for which $\left\{\pi \in \mathcal{C}_d^{(\mathbb{Z})} \mid B_{\mathbf{r}} \cap P_{\mathbb{R}}(\pi) \neq \emptyset\right\}$ is a finite set. Then $\{B_{\mathbf{r}} \mid \mathbf{r} \in C\}$ is an open cover of C and therefore there exists a finite subcover $\{B_{\mathbf{r}_1}, \ldots, B_{\mathbf{r}_k}\}$, $k \in \mathbb{N}$. The set $\left\{\pi \in \mathcal{C}_d^{(\mathbb{Z})} \mid M \cap P_{\mathbb{R}}(\pi) \neq \emptyset\right\}$ is then a subset of the finite union of finite sets $\bigcup_{i=1}^k \left\{\pi \in \mathcal{C}_d^{(\mathbb{Z})} \mid B_{\mathbf{r}_i} \cap P_{\mathbb{R}}(\pi) \neq \emptyset\right\}$ and therefore finite itself. □

According to Lemma 3.5.2 and Lemma 3.5.4, $\mathcal{D}_d^{(0)}$ is the set difference of $\mathcal{D}_d$ and countably many convex polyhedra. Furthermore Theorem 3.5.6 implies that any subset of the interior of $\mathcal{D}_d$ which has a positive distance from the boundary of $\mathcal{D}_d$ intersects with only finitely many cutout polyhedra. Altogether we get that the boundary of $\mathcal{D}_d^{(0)}$ cannot have any curved parts in the interior of $\mathcal{D}_d$ but is characterized by finitely many hyperplanes in any subset of $\mathcal{D}_d$ that has a positive distance from the boundary of $\mathcal{D}_d$. At this point one might hope that $\mathcal{D}_d$ itself intersects with only finitely many cutout polyhedra or at least that only finitely many are non-redundant. For $d = 1$ this is in fact true but even for $d = 2$ infinitely many cutout polyhedra are necessary to gain $\mathcal{D}_2^{(0)}$ from $\mathcal{D}_2$ which gives rise to the definition of so-called critical points which we will introduce in Section 3.7.

As mentioned before, even in the case of $d = 2$ infinitely many cutout polyhedra and thus infinitely many cycles are needed to describe $\mathcal{D}_d^{(0)}$. One way to deal with infinitely many cycles is to combine them to infinite parameterized families. To compute the corresponding cutout polyhedra of such a family one can use the following lemma from [**Weitzer, 2015a**]. But before we need a few definitions (an adapted version of [**Grünbaum and Shephard, 1967**, Chapter 3] is used to cover degenerate polyhedra).

DEFINITION 3.5.7. *A face of a polyhedron $P \in \mathcal{P}_d^{(\mathbb{R})}$ ($d \in \mathbb{N}$) is any intersection of $\overline{P}$ with a closed half-space such that the interior of P (with respect to the smallest affine subspace of $\mathbb{R}^d$ containing P) and the boundary of the half-space are disjoint. In addition $\emptyset$ and $\mathbb{R}^d$ shall be considered faces if $P = \mathbb{R}^d$. The set of faces of P shall be denoted by $\mathcal{F}(P)$.*

The face lattice of P is the set of faces $\mathcal{F}(P)$ of P together with the partial order given by set inclusion.

For a face F of P let F° denote the set difference of F and the union of all faces of P that are less than F (in the face lattice of P). Any F° where $F \in \mathcal{F}(P)$ shall be referred to as open face of P and consequently $\mathcal{F}^\circ(P) := \{F^\circ \mid F \in \mathcal{F}(P)\}$ as the set of open faces of P.

LEMMA 3.5.8. [**Weitzer, 2015a**] *Let $\mathcal{H}$ denote a finite set of half-spaces in $\mathbb{R}^d$ and $P \in \mathcal{P}_d^{(\mathbb{R})}$ be bounded. Furthermore let $\mathcal{H}_o := \{H \in \mathcal{H} \mid H \text{ open}\}$, $\mathcal{H}_c := \{H \in \mathcal{H} \mid H \text{ closed}\}$, $\mathcal{F}_o(P) := \mathcal{F}^\circ(P) \setminus \mathcal{P}(P)$, $\mathcal{F}_c(P) := \mathcal{F}^\circ(P) \cap \mathcal{P}(P)$ and A_M the smallest affine subspace of $\mathbb{R}^d$ containing M for all $M \subseteq \mathbb{R}^d$. Then $P = \bigcap \mathcal{H}$ iff the following holds:*

(i) $\forall\, F \in \mathcal{F}^\circ(P) : F \text{ singleton} \Rightarrow \forall\, H \in \mathcal{H} : F \subseteq \overline{H}$
(ii) $A_P = \mathbb{R}^d \,\vee\, \exists\, \mathcal{H}' \subseteq \mathcal{H} : A_P = \bigcap \mathcal{H}'$
(iii) $\forall\, F \in \mathcal{F}_o(P) : \exists\, H \in \mathcal{H}_o : F \subseteq \partial H$
(iv) $\forall\, F \in \mathcal{F}_c(P) : \overline{F} \neq \overline{P} \Rightarrow \exists\, H \in \mathcal{H}_c : F \subseteq \partial H \wedge P \nsubseteq \partial H$
(v) $\forall\, F \in \mathcal{F}_c(P) : \nexists\, H \in \mathcal{H}_o : F \subseteq \partial H$.

PROOF. It is obvious that $P = \bigcap \mathcal{H} \Rightarrow$ (i)$\wedge \ldots \wedge$(v). For the other direction let $Q := \bigcap \mathcal{H}$. Then $Q \in \mathcal{P}_d^{(\mathbb{R})}$ and (i) implies that all vertices of P are contained in $\overline{Q}$ and thus $\overline{P} \subseteq \overline{Q}$ since P is bounded. Furthermore (v) guarantees that no face belonging to P is being cut off and therefore $P \subseteq Q$. We are left to show the other inclusion. By (ii) we get $Q \subseteq A_P$ and thus $A_P = A_Q$. Therefore we can assume w.l.o.g. that P is nondegenerate (i.e. $A_P = \mathbb{R}^d$). But then (iii) and (iv) (and $P \subseteq Q$) imply that every face of P is not only a face of Q but also the types of the faces (contained or not contained in P resp. Q) coincide and thus $P = Q$. Note that by our assumption $A_P = \mathbb{R}^d$ statement (iv) simplifies to $\forall\, F \in \mathcal{F}_c(P) : \overline{F} \neq \overline{P} \Rightarrow \exists\, H \in \mathcal{H}_c : F \subseteq \partial H$. The $P \nsubseteq \partial H$ part is only needed to guarantee a cut also in the degenerate case. □

Definition 3.5.9. *For two finite tuples S and T let $S \sqcup T$ denote the tuple obtained by concatenation of S and T.*

For a tuple $(T_1, \dots, T_n)$ of $n \in \mathbb{N}$ finite tuples let $\operatorname{shuffle}(T_1, \dots, T_n)$ *denote the tuple obtained by successively stringing together the first entries of the tuples (in the given order) followed by the second entries and so forth, with tuples having too little entries being skipped (e.g.* $\operatorname{shuffle}((1,2),(3),(4,5,6)) = (1,3,4,2,5,6)$*).*

Using the notions of the previous definition we define the six infinite families of cycles given in the following theorem. Note that the families C_2 and C_6 were already found in [**Surer, 2007**] and are added for completeness.

Theorem 3.5.10. [**Weitzer, 2015a**] *Let*

$$
\begin{aligned}
C_0(1) &:= ((-3,3),(3,-2),(-2,1),(1,1),(1,-2),(-2,3),(3,-3))\\
C_0(2) &:= ((-5,1),(1,5),(5,-3),(-3,-3),(-3,5),(5,1),(1,-5),(-5,2),(2,4),\\
&\quad\ (4,-4),(-4,-1),(-1,5),(5,-1),(-1,-4),(-4,4),(4,2),(2,-5))
\end{aligned}
$$

$$
\begin{aligned}
C_1^{(1)}(n) &:= ((-2n,2k))_{k=1}^{n} \sqcup ((-2n+2k,2n))_{k=1}^{n-1} \sqcup ((2k-1,2n-2k))_{k=1}^{n-1} \sqcup\\
&\quad\ ((2n-1,-2k+1))_{k=1}^{n} \sqcup ((2n-2k-1,-2n+1))_{k=1}^{n-1} \sqcup\\
&\quad\ ((-2k,-2n+2k+1))_{k=1}^{n-1}\\
C_1^{(2)}(n) &:= ((2k,2n-2k))_{k=1}^{n-1} \sqcup ((2n,-2k+1))_{k=1}^{n} \sqcup ((2n-2k,-2n+1))_{k=1}^{n-1} \sqcup\\
&\quad\ ((-2k+1,-2n+2k+1))_{k=1}^{n-1} \sqcup ((-2n+1,2k))_{k=1}^{n} \sqcup\\
&\quad\ ((-2n+2k+1,2n))_{k=1}^{n-1}\\
C_1^{(3)}(n) &:= ((2n-2k,-2n))_{k=1}^{n-1} \sqcup ((-2k+1,-2n+2k))_{k=1}^{n-1} \sqcup\\
&\quad\ ((-2n+1,2k-1))_{k=1}^{n} \sqcup ((-2n+2k+1,2n-1))_{k=1}^{n-1} \sqcup\\
&\quad\ ((2k,2n-2k-1))_{k=1}^{n-1} \sqcup ((2n,-2k))_{k=1}^{n}\\
C_1(n) &:= \operatorname{shuffle}(C_1^{(1)}(n), C_1^{(2)}(n), C_1^{(3)}(n)),\ n \geq 2
\end{aligned}
$$

$$
\begin{aligned}
C_2^{(1)}(n) &:= ((-2n,2k-1))_{k=1}^{n+1} \sqcup ((-2n+2k,2n+1))_{k=1}^{n-1}\\
C_2^{(2)}(n) &:= ((2k-1,2n-2k+1))_{k=1}^{n} \sqcup ((2n+1,-2k))_{k=1}^{n}\\
C_2^{(3)}(n) &:= ((2n-2k+1,-2n))_{k=1}^{n} \sqcup ((-2k,-2n+2k))_{k=1}^{n-1}\\
C_2(n) &:= \operatorname{shuffle}(C_2^{(1)}(n), C_2^{(2)}(n), C_2^{(3)}(n)),\ n \geq 1
\end{aligned}
$$

$$
\begin{aligned}
C_3^{(1)}(n) &:= ((-2n-1,1)) \sqcup ((-2n+2k-2,-2k))_{k=1}^{n} \sqcup ((2k-1,-2n-1))_{k=1}^{n}\\
C_3^{(2)}(n) &:= ((1,2n+1)) \sqcup ((-2k,2n+2))_{k=1}^{n-1} \sqcup ((-2n,2n+1)) \sqcup\\
&\quad\ ((-2n-1,2n-2k+1))_{k=1}^{n-1}\\
C_3^{(3)}(n) &:= ((2n+1,-2n)) \sqcup ((2n+2,-2n+2k))_{k=1}^{n-1} \sqcup ((2n-2k+3,2k-1))_{k=1}^{n}\\
C_3(n) &:= \operatorname{shuffle}(C_3^{(1)}(n), C_3^{(2)}(n), C_3^{(3)}(n)),\ n \geq 2
\end{aligned}
$$

$$
\begin{aligned}
C_4^{(1)}(n) &:= ((-2n-1,2)) \sqcup ((-2n+2k-2,-2k+1))_{k=1}^{n} \sqcup ((2k-1,-2n))_{k=1}^{n-1} \sqcup\\
&\quad\ ((2n-1,-2n+1)) \sqcup ((2n,-2n+2k+1))_{k=1}^{n-1} \sqcup ((2n-2k+1,2k))_{k=1}^{n} \sqcup\\
&\quad\ ((-2k,2n+1))_{k=1}^{n-1} \sqcup ((-2n,2n)) \sqcup ((-2n-1,2n-2k))_{k=1}^{n-2}\\
C_4^{(2)}(n) &:= ((2,2n)) \sqcup ((-2k+1,2n+1))_{k=1}^{n-1} \sqcup ((-2n+1,2n)) \sqcup\\
&\quad\ ((-2n,2n-2k))_{k=1}^{n-1} \sqcup ((-2n+2k-1,-2k+1))_{k=1}^{n} \sqcup ((2k,-2n))_{k=1}^{n-1} \sqcup\\
&\quad\ ((2n,-2n+1)) \sqcup ((2n+1,-2n+2k+1))_{k=1}^{n-1} \sqcup ((2n-2k+2,2k))_{k=1}^{n-1}\\
C_4^{(3)}(n) &:= ((2n,-2n)) \sqcup ((2n+1,-2n+2k))_{k=1}^{n-1} \sqcup ((2n-2k+2,2k-1))_{k=1}^{n} \sqcup\\
&\quad\ ((-2k+1,2n))_{k=1}^{n-1} \sqcup ((-2n+1,2n-1)) \sqcup ((-2n,2n-2k-1))_{k=1}^{n-1} \sqcup\\
&\quad\ ((-2n+2k-1,-2k))_{k=1}^{n} \sqcup ((2k,-2n-1))_{k=1}^{n-1}\\
C_4(n) &:= \operatorname{shuffle}(C_4^{(1)}(n), C_4^{(2)}(n), C_4^{(3)}(n)),\ n \geq 2
\end{aligned}
$$

$$
\begin{aligned}
C_5^{(1)}(n) &:= ((-n-1,1)) \sqcup ((-n+k-1,k+2))_{k=1}^{n-1}\\
C_5^{(2)}(n) &:= ((1,n+1)) \sqcup ((k+2,n-k+1))_{k=1}^{n-2} \sqcup ((n+1,1))\\
C_5^{(3)}(n) &:= ((n-k+2,-k-1))_{k=1}^{n-1} \sqcup ((1,-n-1))\\
C_5^{(4)}(n) &:= ((-k-1,-n+k-1))_{k=1}^{n-1}\\
C_5(n) &:= \text{shuffle}(C_5^{(1)}(n), C_5^{(2)}(n), C_5^{(3)}(n), C_5^{(4)}(n)),\ n \geq 2
\end{aligned}
$$

$$
\begin{aligned}
C_6^{(1)}(n) &:= ((-n+k-1,-k))_{k=1}^{n} \sqcup ((1,-n))\\
C_6^{(2)}(n) &:= ((-k,n-k+1))_{k=1}^{n} \sqcup ((n-k+1,k+1))_{k=1}^{n}\\
C_6^{(3)}(n) &:= ((k+1,-n+k))_{k=1}^{n-1} \sqcup ((n+1,1))\\
C_6(n) &:= \text{shuffle}(C_6^{(1)}(n), C_6^{(2)}(n), C_6^{(3)}(n)),\ n \geq 1
\end{aligned}
$$

Then we have the following corresponding infinite families of cutout polygons, where every polygon is given by a list of its vertices in counterclockwise order, a vertex is overlined iff it belongs to the respective polygon, and belonging of an edge is indicated by a solid (contained) or dotted (not contained) line between the endpoints:

$$C_0(n),\ n=1\text{:}\ \left(\tfrac{3}{4},\tfrac{3}{2}\right) — \left(1,\tfrac{5}{3}\right) \text{- -} \left(\tfrac{7}{6},\tfrac{11}{6}\right) \text{- -} \left(1,2\right) \text{- -}$$

$$n=2\text{:}\ \left(\tfrac{25}{26},\tfrac{15}{26}\right) — \left(1,\tfrac{1}{2}\right) — \left(\tfrac{28}{27},\tfrac{16}{27}\right) \text{- -} \left(1,\tfrac{3}{5}\right) \text{- -}$$

$$C_1(n),\ n\geq 2\text{:}\ \left(1-\tfrac{1}{4n^2-4n+2}, 1+\tfrac{2n-1}{4n^2-4n+2}\right) \text{- -} \left(1, 1+\tfrac{1}{2n-1}\right) \text{- -}$$
$$\left(1+\tfrac{1}{4n^2-2}, 1+\tfrac{2n+2}{4n^2-2}\right) \text{- -} \left(1, 1+\tfrac{1}{2n-2}\right) \text{- -}$$

$$C_2(n),\ n=1\text{:}\ \left(\tfrac{2}{3},\tfrac{4}{3}\right) \text{- -} \left(1,\tfrac{3}{2}\right) \text{- -} \left(\tfrac{6}{5},\tfrac{9}{5}\right) \text{- -} \left(\tfrac{3}{4},\tfrac{3}{2}\right) \text{- -}$$

$$n\geq 2\text{:}\ \left(1-\tfrac{1}{4n^2-2n+1}, 1+\tfrac{2n-1}{4n^2-2n+1}\right) \text{- -} \left(1, 1+\tfrac{1}{2n}\right) \text{- -}$$
$$\left(1+\tfrac{1}{4n^2+2n-1}, 1+\tfrac{2n+2}{4n^2+2n-1}\right) \text{- -} \left(1, 1+\tfrac{1}{2n-1}\right) \text{- -}$$

$$C_3(n),\ n\geq 2\text{:}\ \overline{\left(1-\tfrac{1}{4n^2+6n-1}, 1-\tfrac{2n+4}{4n^2+6n-1}\right)} — \overline{\left(1, 1-\tfrac{1}{2n-1}\right)} —$$
$$\overline{\left(1+\tfrac{1}{4n^2+6n-2}, 1-\tfrac{2n+3}{4n^2+6n-2}\right)} — \overline{\left(1, 1-\tfrac{1}{2n}\right)} —$$

$$C_4(n),\ n=2\text{:}\ \overline{\left(\tfrac{19}{20},\tfrac{3}{5}\right)} — \overline{\left(\tfrac{21}{22},\tfrac{13}{22}\right)} — \left(1,\tfrac{3}{5}\right) \text{- -} \left(\tfrac{22}{21},\tfrac{13}{21}\right) \text{- -} \left(\tfrac{20}{19},\tfrac{12}{19}\right) \text{- -} \left(1,\tfrac{2}{3}\right) —$$

$$n\geq 3\text{:}\ \overline{\left(1-\tfrac{1}{4n^2+4n-4}, 1-\tfrac{2n+4}{4n^2+4n-4}\right)} — \left(1, 1-\tfrac{1}{2n-2}\right) \text{- -}$$
$$\left(1+\tfrac{1}{4n^2+4n-5}, 1-\tfrac{2n+3}{4n^2+4n-5}\right) \text{- -} \left(1, 1-\tfrac{1}{2n-1}\right) —$$

$$C_5(n),\ n=2\text{:}\ \overline{\left(\tfrac{10}{11},\tfrac{4}{11}\right)} — \left(1,\tfrac{1}{3}\right) — \overline{\left(\tfrac{11}{10},\tfrac{2}{5}\right)} — \overline{\left(1,\tfrac{1}{2}\right)} —$$

$$n=3\text{:}\ \left(\tfrac{14}{15},\tfrac{4}{15}\right) — \left(1,\tfrac{1}{4}\right) — \left(\tfrac{19}{18},\tfrac{5}{18}\right) \text{- -} \left(1,\tfrac{1}{3}\right) \text{- -}$$

$$n=4\text{:}\ \left(\tfrac{22}{23},\tfrac{5}{23}\right) — \overline{\left(\tfrac{23}{24},\tfrac{5}{24}\right)} — \left(1,\tfrac{1}{5}\right) — \left(\tfrac{24}{23},\tfrac{5}{23}\right) \text{- -} \left(1,\tfrac{1}{4}\right) \text{- -}$$

$$n\geq 5\text{:}\ \left(1-\tfrac{1}{n^2+n+3}, \tfrac{n+1}{n^2+n+3}\right) — \overline{\left(1-\tfrac{1}{n^2+2n}, \tfrac{n+1}{n^2+2n}\right)} — \left(1, \tfrac{1}{n+1}\right) —$$
$$\overline{\left(1+\tfrac{1}{n^2+2n-1}, \tfrac{n+1}{n^2+2n-1}\right)} — \left(1+\tfrac{1}{n^2+n+3}, \tfrac{n+1}{n^2+n+3}\right) \text{- -} \left(1, \tfrac{1}{n}\right) \text{- -}$$

$$C_6(n),\ n=1\ \ \left(\tfrac{2}{3},-\tfrac{1}{3}\right) — \left(1,-1\right) \text{- -} \left(\tfrac{4}{3},-\tfrac{2}{3}\right) \text{- -}$$

$$n\geq 2\text{:}\ \left(1-\tfrac{1}{n^2+2}, -\tfrac{n}{n^2+2}\right) — \left(1, -\tfrac{1}{n}\right) \text{- -} \left(1+\tfrac{1}{n^2+n+1}, -\tfrac{n+1}{n^2+n+1}\right) \text{- -}$$
$$\left(1+\tfrac{1}{n^2+2n}, -\tfrac{n+1}{n^2+2n}\right) \text{- -} \left(1-\tfrac{1}{n^2+n+1}, -\tfrac{n}{n^2+n+1}\right) \text{- -}$$

PROOF. Using Lemma 3.5.4 one can compute the systems of linear inequalities that correspond to the given cycles. Computer-aided computations then show that these systems of inequalities together with the claimed shapes of the cutout polygons meet the conditions of Lemma 3.5.8 which proves the theorem. □

Figure 2 shows a compilation of all families. The polygons are threaded on the line segment between $(1,-1)$ and $(1,2)$ and they tend to either of the two points $(1,0)$ and $(1,1)$. These two points will turn out to be so-called critical points which we will define in Section 3.7.

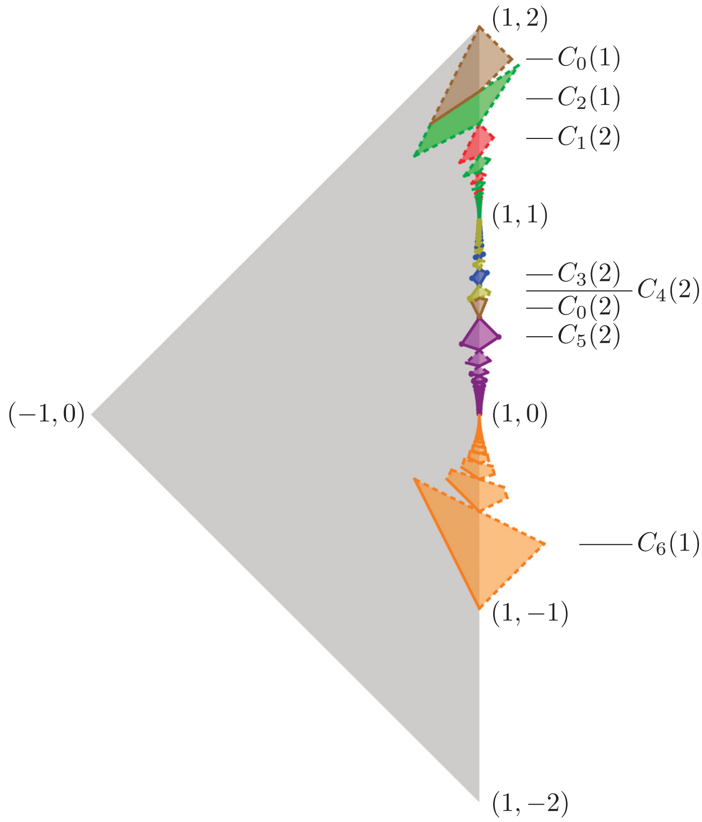


FIGURE 2. Six families of cutout polygons for $d = 2$.

3.6. Brunotte's algorithm: Sets of witnesses

In the previous section we provided a tool which allows to prove that a certain parameter $\mathbf{r} \in \mathbb{R}^d$ does not belong to $\mathcal{D}_d^{(0)}$. In the present section we will recall another basic tool which will allow to decide whether or not a given parameter $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$ belongs to $\mathcal{D}_d^{(0)}$. It is the so-called sets of witnesses which form the basis of what is known as Brunotte's algorithm [**Brunotte, 2001**]. Furthermore the concept of sets of witnesses can be generalized such that they not only settle the finiteness property for single parameters but for whole convex regions of parameters contained in the interior of $\mathcal{D}_d$.

DEFINITION 3.6.1. Cf. [**Brunotte, 2001**] *A set $V \subseteq \mathbb{Z}^d$ is called a set of witnesses for $\mathbf{r} \in \mathbb{R}^d$ ($d \in \mathbb{N}$) iff it is stable under $\tau_{\mathbf{r}}$ and $\tau_{\mathbf{r}}^{\star} := -\tau_{\mathbf{r}} \circ (-\operatorname{id}_{\mathbb{Z}^d})$ and contains a generating set of the group $(\mathbb{Z}^d, +)$ which is closed under taking inverses.*

All sets of witnesses have the following decisive property:

LEMMA 3.6.2. [**Akiyama et al., 2005**] *Let $d \in \mathbb{N}$, $\mathbf{r} \in \mathbb{R}^d$, and $V \subseteq \mathbb{Z}^d$ a set of witnesses for $\mathbf{r}$. Then*

$$\mathbf{r} \in \mathcal{D}_d^{(0)} \Leftrightarrow \forall\, \mathbf{a} \in V : \exists\, n \in \mathbb{N} : \tau_{\mathbf{r}}^n(\mathbf{a}) = \mathbf{0}.$$

If one could find a finite set of witnesses for a given parameter $\mathbf{r}$ this would of course provide a method to decide whether or not the parameter belongs to $\mathcal{D}_d^{(0)}$. This is exactly what Brunotte's algorithm does for any $\mathbf{r}$ in the interior of $\mathcal{D}_d$ - it computes a finite set of witnesses.

DEFINITION 3.6.3. [**Akiyama et al., 2005**] *For* $d \in \mathbb{N}$ *and* $\mathbf{r} \in \mathbb{R}^d$ *let*

$$\begin{aligned} V_{\mathbf{r},0} &:= \{(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\} \\ \forall\, n \in \mathbb{N} : V_{\mathbf{r},n} &:= V_{\mathbf{r},n-1} \cup \tau_{\mathbf{r}}(V_{\mathbf{r},n-1}) \cup \tau_{\mathbf{r}}^{\star}(V_{\mathbf{r},n-1}) \\ V_{\mathbf{r}} &:= \bigcup_{n \in \mathbb{N}_0} V_{\mathbf{r},n} \end{aligned}$$

$V_{\mathbf{r}}$ *shall be referred to as the set of witnesses associated with* $\mathbf{r}$.

THEOREM 3.6.4. [**Akiyama et al., 2005**] *Let* $d \in \mathbb{N}$ *and* $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$. *Then* $V_{\mathbf{r}}$ *is a finite set of witnesses for* $\mathbf{r}$.

As indicated in the beginning of the section the concept of sets of witnesses can be adapted to work for whole regions of parameters.

DEFINITION 3.6.5. *For* $d \in \mathbb{N}$ *and* $M \subseteq \mathbb{R}^d$ *let*

$$\begin{aligned} \overline{\tau}_M : \mathcal{P}(\mathbb{Z}^d) &\to \mathcal{P}(\mathbb{Z}^d). \\ V &\mapsto \{\tau_{\mathbf{r}}(\mathbf{a}) \mid \mathbf{r} \in M \wedge \mathbf{a} \in V\} \end{aligned}$$

With this notation one can define sets of witnesses for regions of parameters in complete analogy to Definition 3.6.1. The concepts and theorems for single parameters translate as expected.

DEFINITION 3.6.6. Cf. [**Akiyama et al., 2005**] *A set* $V \subseteq \mathbb{Z}^d$ *is called a set of witnesses for* $M \subseteq \mathbb{R}^d$ $(d \in \mathbb{N})$ *iff it is stable under* $\overline{\tau}_M$ *and* $\overline{\tau}_M^{\star} := -\overline{\tau}_M \circ (-\operatorname{id}_{\mathbb{Z}^d})$ *and contains a generating set of the group* $(\mathbb{Z}^d, +)$ *which is closed under taking inverses.*

LEMMA 3.6.7. [**Akiyama et al., 2005**] *Let* $d \in \mathbb{N}$, $M \subseteq \mathbb{R}^d$, *and* $V \subseteq \mathbb{Z}^d$ *a set of witnesses for* M. *Then*

$$M \cap \mathcal{D}_d^{(0)} = M \setminus \bigcup_{\substack{\pi = (\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathcal{C}_d^{(\mathbb{Z})} \\ \{\mathbf{a}_1, \ldots, \mathbf{a}_k\} \subseteq V}} P_{\mathbb{R}}(\pi)$$

DEFINITION 3.6.8. Cf. [**Akiyama et al., 2005**] *For* $d \in \mathbb{N}$ *and* $M \subseteq \mathbb{R}^d$ *let*

$$\begin{aligned} V_{M,0} &:= \{(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\} \\ \forall\, n \in \mathbb{N} : V_{M,n} &:= V_{M,n-1} \cup \overline{\tau}_M(V_{M,n-1}) \cup \overline{\tau}_M^{\star}(V_{M,n-1}) \\ V_M &:= \bigcup_{n \in \mathbb{N}_0} V_{M,n} \end{aligned}$$

V_M *shall be referred to as the set of witnesses associated with* M.

THEOREM 3.6.9. [**Weitzer, 2015a**] *Let* $d \in \mathbb{N}$ *and* $M \subseteq \operatorname{int}(\mathcal{D}_d)$ *with* $\operatorname{dist}(M, \partial\mathcal{D}_d) > 0$. *Then there is a* $k \in \mathbb{N}$ *and there are* $B_1, \ldots, B_k \subseteq \mathbb{R}^d$ *such that* $M = \bigcup_{i=1}^k B_i$ *and* V_{B_i} *is a finite set of witnesses for* B_i *for all* $i \in [\![1, k]\!]$.

PROOF. The proof of Theorem 3.5.6 can easily be adapted. □

We now have almost everything we need to define an algorithm which computes the intersection of $\mathcal{D}_d^{(0)}$ and any given subset M of the interior of $\mathcal{D}_d$ which has a positive distance from the boundary of $\mathcal{D}_d$.

The first step is to subdivide M into finitely many, sufficiently small sets for which the iteration in Definition 3.6.8 becomes stationary. Theorem 3.6.9 implies that such a subdivision always exists even though it does not provide a constructive method on how to compute it. It turns out that this is only of theoretical relevance as it is perfectly practicable to just try a certain subdivision and refine it if the iteration does not seem to hold. This can be decided heuristically by introducing a sufficient upper bound for the size (e.g. its diameter) of the set of witnesses. What makes a sufficient upper bound can be estimated by the actual upper bound (as given in the proof of Theorem 3.4.2) for the size of the set of witnesses of that parameter $\mathbf{r} \in M$ the associated matrix $R(\mathbf{r})$ of which has the largest spectral radius. Though not of practical importance we shall nevertheless overcome this heuristic guessing in Section 4.2 where we will introduce a "real" algorithm which is guaranteed to terminate for all inputs M.

The only question which remains open at this point is how to compute $\overline{\tau}_B$ and $\overline{\tau}^\star_B$ that are needed to actually compute the finite set of witnesses of an element B of the subdivision. In [**Akiyama et al., 2005**] a method is provided which we will not repeat here as we will introduce a more efficient method in Section 4.3 (cf. Remark 4.3.4).

When finite sets of witnesses for all elements of the subdivision are found we are left with the very time-consuming task of subtracting from M all the cutout polyhedra coming from cycles in the sets of witnesses. We shall provide a much more efficient alternative to this approach in Section 4.4.

3.7. Critical points

At this point we have enough tools at hand to characterize $\mathcal{D}_d^{(0)}$ in an arbitrarily large region in finite time. We know that $\mathcal{D}_d^{(0)}$ is contained in $\mathcal{D}_d$, which is a bounded set. In the interior of $\mathcal{D}_d$ we can always characterize $\mathcal{D}_d^{(0)}$ as long as we keep a positive distance from the boundary of $\mathcal{D}_d$. The only points we cannot handle in general yet are those on the boundary itself. Furthermore we have a full characterization of $\mathcal{D}_1^{(0)}$ which is known to be equal to $[0, 1)$. In this particular case two cutout polyhedra are sufficient to describe $\mathcal{D}_d^{(0)}$ in terms of $\mathcal{D}_d$. Unfortunately already for $d = 2$ this is not the case anymore. We cannot hope to represent $\mathcal{D}_2^{(0)}$ as the set difference of $\mathcal{D}_2$ and finitely many cutout polyhedra and the reason for that is the existence of so-called critical points.

Definition 3.7.1. Cf. [**Akiyama et al., 2005**] *Let* $d \in \mathbb{N}$ *and* $\mathbf{r} \in \mathbb{R}^d$.

- $\mathbf{r}$ *is called a regular point (for* $\mathcal{D}_d^{(0)}$*) iff there exists an open neighborhood of* $\mathbf{r}$ *which intersects with only finitely many cutout polyhedra.*
- $\mathbf{r}$ *is called a weakly critical point (for* $\mathcal{D}_d^{(0)}$*) iff any open neighborhood of* $\mathbf{r}$ *intersects with infinitely many cutout polyhedra.*
- $\mathbf{r}$ *is called a critical point (for* $\mathcal{D}_d^{(0)}$*) iff for every open neighborhood* B *of* $\mathbf{r}$ *the set* $B \setminus \mathcal{D}_d^{(0)}$ *cannot be covered by finitely many cutout polyhedra.*
- $\mathbf{r}$ *is called a strongly critical point (for* $\mathcal{D}_d^{(0)}$*) iff for every open neighborhood* B *of* $\mathbf{r}$ *the set* $B \setminus \mathcal{D}_d^{(0)}$ *cannot be covered by finitely many polyhedra.*

In [**Akiyama et al., 2005**] only the first three types of points are considered and it is argued that the difference between those two notions is that in the neighborhood of a critical point, $\mathcal{D}_d^{(0)}$ cannot be characterized by finitely many polyhedra. This is not completely correct as it cannot be ruled out that the infinitely many cutout polyhedra in the neighborhood of a true critical point (i.e. that is not a weakly critical point) can be covered by finitely many polyhedra that might not be cutout polyhedra though. The existence of critical points alone therefore does not necessarily imply that $\mathcal{D}_d^{(0)}$ has a complicated structure in the sense that it cannot be given as the union of finitely many polyhedra. Only the existence of strongly critical points could, but up to now there is no point for $d \in \mathbb{N}$ that is proven to be strongly critical. However, all previous observations do suggest that every critical point is in fact strongly critical.

The following theorem is proven in [**Akiyama et al., 2005**]. We will not repeat the proof at this point but we shall show the existence of a critical point in the complex setting in Section 5.4 applying similar methods.

Theorem 3.7.2. [**Akiyama et al., 2005**] *Let* $d \in \mathbb{N}_{\geq 2}$. *Then* $(0, \ldots, 0, 1, 0) \in \mathbb{R}^d$ *is a critical point for* $\mathcal{D}_d^{(0)}$.

We close the chapter with an image of $\mathcal{D}_2^{(0)}$ which gives a first idea of its true shape. It can be seen that there are most likely two critical points which are $(1, 0)$ (which is already proven to be critical) and $(1, 1)$. In Chapter 4 we will take a closer look on the surroundings of these critical points and settle two topological questions: Is $\mathcal{D}_2^{(0)}$ connected and do its connected components have trivial fundamental group? In both cases the answer will be: No.

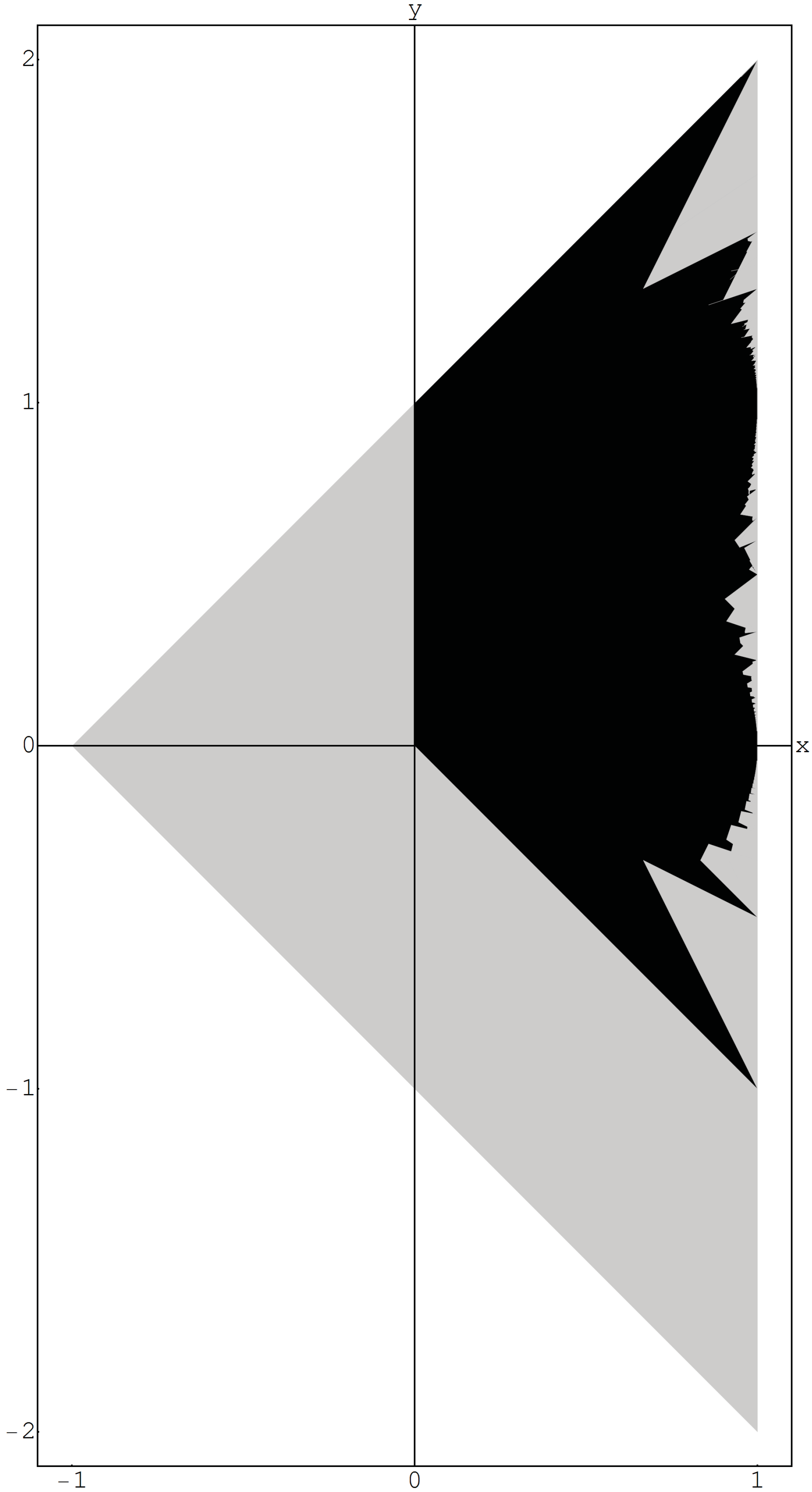


FIGURE 3. $\mathcal{D}_2$ and $\mathcal{D}_2^{(0)}$.

CHAPTER 4

New algorithms and topological results

In this chapter we will present two algorithms which allow the characterization of the intersection of $\mathcal{D}_d^{(0)}$ and any closed convex hull of finitely many interior points of $\mathcal{D}_d$ which is completely contained in the interior of $\mathcal{D}_d$. These algorithms form an alternative to Brunotte's algorithm for regions introduced in Section 3.6. Both have individual advantages which will be discussed in the respective sections. The algorithms have been applied to settle two previously open questions on the topology of $\mathcal{D}_2^{(0)}$: Is $\mathcal{D}_2^{(0)}$ connected, and do its connected components have trivial fundamental group? The answers to both questions (which is "no" in both cases) will be discussed in detail in the last section. Most of the material in this chapter has been published in [**Weitzer, 2015a**].

4.1. Graphs of witnesses

In Section 3.5 we defined cutout polyhedra, which consist of exactly those parameters, the corresponding Shift Radix Systems of which admit a given common cycle. In Section 3.6 we introduced the concept of sets of witnesses and, by analogy, we shall now define the set of those parameters which share a given common set of witnesses. To be more specific, we don't only demand the sets of witnesses to be identical but also the way in which τ and $\tau^\star$ act on them. For that purpose we first have to extend our definition to "graphs of witnesses" the vertex set of which is the set of witnesses and whose edges are given by the actions of τ and $\tau^\star$.

DEFINITION 4.1.1. [**Weitzer, 2015a**] *For $d \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{R}^d$, $\Pi_\mathbf{r}$ - the graph of witnesses associated with $\mathbf{r}$ - denotes the edge-colored multidigraph with vertex set $V_\mathbf{r}$ and an edge of color 1 from a vertex $\mathbf{a}$ to a vertex $\mathbf{b}$ iff $\tau_\mathbf{r}(\mathbf{a}) = \mathbf{b}$ and an edge of color 2 from $\mathbf{a}$ to $\mathbf{b}$ iff $\tau^\star_\mathbf{r}(\mathbf{a}) = \mathbf{b}$.*

If E_1 is the set of all edges (ordered pairs) of color 1 and E_2 the set of all edges of color 2, then the graph $\Pi_\mathbf{r}$ is completely characterized by the pair $(E_1, E_2) \in \mathcal{P}((\mathbb{Z}^d)^2)^2$ (as there are no isolated vertices) and thus the graph and the pair can be identified. For any such graph $\Pi = (E_1, E_2) \in \mathcal{P}((\mathbb{Z}^d)^2)^2$ let $P_\mathbb{R}(\Pi) := \left\{\mathbf{r} \in \mathbb{R}^d \mid \forall\, (\mathbf{a},\mathbf{b}) \in E_1 : \tau_\mathbf{r}(\mathbf{a}) = \mathbf{b} \wedge \forall\, (\mathbf{a},\mathbf{b}) \in E_2 : \tau^\star_\mathbf{r}(\mathbf{a}) = \mathbf{b}\right\}$.

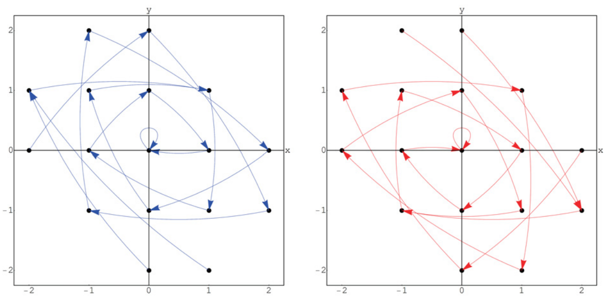


FIGURE 1. E_1 (left) and E_2 of $\Pi_\mathbf{r}$ for $\mathbf{r} = \left(\frac{3}{4}, \frac{1}{2}\right)$.

DEFINITION 4.1.2. *A strip $S \subseteq \mathbb{R}^d$ is the intersection of two parallel oppositely oriented half-spaces, or $\mathbb{R}^d$ itself. The empty set and the whole space $\mathbb{R}^d$ are considered degenerate and all others nondegenerate strips. For nondegenerate strips the attributes open, half-open, and closed shall indicate belonging of the hyperplanes bounding the strip. The width of a strip is the normal distance of these hyperplanes if it is nondegenerate or $-\infty$ or ∞ if the strip is the empty set or the whole space respectively.*

Every strip having positive width can be represented in one of the four ways

$$\left\{\mathbf{r}\in\mathbb{R}^d \mid 0\left\{\begin{matrix}<\\ \leq\end{matrix}\right\}\mathbf{ar}+b\left\{\begin{matrix}<\\ \leq\end{matrix}\right\}1\right\} \qquad (\mathbf{a}\in\mathbb{R}^d,\, b\in\mathbb{R}),$$

where $\mathbf{a}$ is normal to the strip's bounding hyperplanes and $\frac{1}{\|\mathbf{a}\|}$ is the strip's width.

Lemma 4.1.3. [**Weitzer, 2015a**] *Let* $\mathbf{a}=(a_1,\dots,a_d),\mathbf{b}=(b_1,\dots,b_d)\in\mathbb{Z}^d$, *and* $\mathbf{r}\in\mathbb{R}^d$. *Then*

(i) $\left\{\mathbf{r}\in\mathbb{R}^d \mid \tau_{\mathbf{r}}(\mathbf{a})=\mathbf{b}\right\}=\left\{\mathbf{r}\in\mathbb{R}^d \mid \forall\, i\in[\![1,d-1]\!]: b_i=a_{i+1}\ \wedge\ 0\leq \mathbf{ra}+b_d<1\right\}$

(ii) $\left\{\mathbf{r}\in\mathbb{R}^d \mid \tau^\star_{\mathbf{r}}(\mathbf{a})=\mathbf{b}\right\}=\left\{\mathbf{r}\in\mathbb{R}^d \mid \forall\, i\in[\![1,d-1]\!]: b_i=a_{i+1}\ \wedge\ 0\leq -\mathbf{ra}-b_d<1\right\}$

(iii) $\left\{\mathbf{s}\in\mathbb{R}^d \mid \tau_{\mathbf{s}}(\mathbf{a})=\tau_{\mathbf{r}}(\mathbf{a})\ \wedge\ \tau^\star_{\mathbf{s}}(\mathbf{a})=\tau^\star_{\mathbf{r}}(\mathbf{a})\right\}=\begin{cases}\left\{\mathbf{s}\in\mathbb{R}^d \mid \mathbf{sa}-\mathbf{ra}=0\right\} & \textit{if } \mathbf{ra}\in\mathbb{Z}\\ \left\{\mathbf{s}\in\mathbb{R}^d \mid 0<\mathbf{sa}-\lfloor\mathbf{ra}\rfloor<1\right\} & \textit{if } \mathbf{ra}\notin\mathbb{Z}\end{cases}$

Proof. (i) and (ii) can be proven in the same way as Lemma 3.5.4 (indeed, (i) is even identical to Lemma 3.5.4 but is repeated here for convenience).

For the proof of (iii) let $M := \left\{\mathbf{s}\in\mathbb{R}^d \mid \tau_{\mathbf{s}}(\mathbf{a})=\tau_{\mathbf{r}}(\mathbf{a})\ \wedge\ \tau^\star_{\mathbf{s}}(\mathbf{a})=\tau^\star_{\mathbf{r}}(\mathbf{a})\right\}$. Then (i) and (ii) imply that $M=\left\{\mathbf{s}\in\mathbb{R}^d \mid 0\leq\mathbf{sa}-\lfloor\mathbf{ra}\rfloor<1\ \wedge\ 0\leq-\mathbf{sa}-\lfloor-\mathbf{ra}\rfloor<1\right\}$. If $\mathbf{ra}\in\mathbb{Z}$ then $-\lfloor-\mathbf{ra}\rfloor=\mathbf{ra}$ and therefore $M=\left\{\mathbf{s}\in\mathbb{R}^d \mid \mathbf{sa}-\mathbf{ra}=0\right\}$. If $\mathbf{ra}\notin\mathbb{Z}$ then $-\lfloor-\mathbf{ra}\rfloor=\lfloor\mathbf{ra}\rfloor+1$ and therefore $M=\left\{\mathbf{s}\in\mathbb{R}^d \mid 0<\mathbf{sa}-\lfloor\mathbf{ra}\rfloor<1\right\}$. □

Lemma 4.1.4. [**Weitzer, 2015a**] *Let* $\mathbf{r}\in\operatorname{int}(\mathcal{D}_d)$. *Then* $P_{\mathbb{R}}(\Pi_{\mathbf{r}})$ *is the intersection of a nondegenerate, open polyhedron and an affine subspace of* $\mathbb{R}^d$.

Proof. Lemma 4.1.3 (iii) implies that $P_{\mathbb{R}}(\Pi_{\mathbf{r}})$ is the intersection of finitely many hyperplanes and finitely many open strips and thus is the intersection of an open polyhedron and an affine subspace of $\mathbb{R}^d$. Furthermore it is non-empty as $\mathbf{r}\in P_{\mathbb{R}}(\Pi_{\mathbf{r}})$ and it is bounded as $\{(\pm1,0,\dots,0),\dots,(0,\dots,0,\pm1)\}\subseteq V_{\mathbf{r}}$ and therefore $P_{\mathbb{R}}(\Pi_{\mathbf{r}})\subseteq\lfloor\mathbf{r}\rfloor+[0,1]^d$. □

Lemma 4.1.5. [**Weitzer, 2015a**] *Let* $d\in\mathbb{N}$ *and* $\mathbf{r}\in\mathbb{R}^d$. *Then*

(i) $\mathbf{r}\in P_{\mathbb{R}}(\Pi_{\mathbf{r}})$, *and* $\mathbf{r}\in\mathcal{D}_d^{(0)}\Leftrightarrow P_{\mathbb{R}}(\Pi_{\mathbf{r}})\subseteq\mathcal{D}_d^{(0)}$

(ii) $\mathcal{D}_d^{(0)}=\bigcup\left\{P_{\mathbb{R}}(\Pi_{\mathbf{r}}) \mid \mathbf{r}\in\mathcal{D}_d^{(0)}\right\}$ *and this union is disjoint.*

Proof. Follows directly from the definition of $P_{\mathbb{R}}(\Pi_{\mathbf{r}})$. □

4.2. A "real" algorithm

The following algorithm is a straightforward application of Lemma 4.1.5.

Algorithm 1 [**Weitzer, 2015a**] Determination of $\operatorname{conv}(\{\mathbf{r}_1,\dots,\mathbf{r}_k\})\cap\mathcal{D}_d^{(0)}$

Input: $(\mathbf{r}_1,\dots,\mathbf{r}_k)\in\operatorname{int}(\mathcal{D}_d)^k$ such that $\operatorname{conv}(\{\mathbf{r}_1,\dots,\mathbf{r}_k\})\subset\operatorname{int}(\mathcal{D}_d)$.
Output: $\mathcal{P}\subseteq\mathcal{P}_d^{(\mathbb{R})}$ with $\operatorname{conv}(\{\mathbf{r}_1,\dots,\mathbf{r}_k\})\cap\mathcal{D}_d^{(0)}=\bigcup\mathcal{P}$ disjoint.

$H\leftarrow\operatorname{conv}(\{\mathbf{r}_1,\dots,\mathbf{r}_k\})$
$\mathcal{P}\leftarrow\emptyset$
while $H\setminus\bigcup\mathcal{P}\neq\emptyset$ **do**
  select $\mathbf{r}\in H\setminus\bigcup\mathcal{P}$
  $\mathcal{P}\leftarrow\mathcal{P}\cup\{H\cap P_{\mathbb{R}}(\Pi_{\mathbf{r}})\}$
  if $\mathbf{r}\in\mathcal{D}_d^{(0)}$ **then** {use Brunotte's algorithm}
    $\mathrm{fin}_{H\cap P_{\mathbb{R}}(\Pi_{\mathbf{r}})}\leftarrow$**true**
  else
    $\mathrm{fin}_{H\cap P_{\mathbb{R}}(\Pi_{\mathbf{r}})}\leftarrow$**false**
  end if
end while
return $\{P\in\mathcal{P} \mid \mathrm{fin}_P=\textbf{true}\}$

Of course the question arises whether the while loop actually terminates, which is equivalent to the possibility of exhausting H by finitely many $P_{\mathbb{R}}(\Pi_{\mathbf{r}})$.

Theorem 4.2.1. [**Weitzer, 2015a**] *Algorithm 1 terminates for all inputs.*

Proof. By Theorem 3.6.9 the set $\bigcup_{\mathbf{r}\in H} V_{\mathbf{r}}$ is finite and since only finitely many graphs can be defined on a finite number of vertices the set $\{P_{\mathbb{R}}(\Pi_{\mathbf{r}}) \mid \mathbf{r} \in H\}$ is also finite. □

How does Algorithm 1 compare to Brunotte's algorithm for regions introduced in Section 3.6? The first difference is that Algorithm 1 is a "real" algorithm in the sense that it is guaranteed to hold for all inputs by Theorem 4.2.1 whereas with Brunotte's algorithm for regions this is not the case. But as discussed at the end of Section 3.6 this is only of theoretic relevance. Practically both algorithms are perfectly applicable. The second difference lies in the running times of the algorithms. It might not be obvious but in many situations Algorithm 1 is much faster than Brunotte's algorithm for regions. To get an idea why this is the case let us recapitulate what the two algorithms do and what the different approaches are. In both cases we start with a suitable (which has a different meaning for each of the two algorithms) convex hull H and the goal is to characterize $H \cap \mathcal{D}_d^{(0)}$. The output of Algorithm 1 is a list of pairwise disjoint polyhedra the union of which is just $H \cap \mathcal{D}_d^{(0)}$, while the output of Brunotte's algorithm for regions is a list of cutout polyhedra which, if subtracted from H, again give $H \cap \mathcal{D}_d^{(0)}$. Both algorithms use sets of witnesses but while Algorithm 1 uses all sets of witnesses occurring for parameters in H one by one and computes the largest possible subset of H which can be settled by a given set of witnesses, Brunotte's algorithm computes a set of witnesses for the whole set H, which contains as a subset (in relevant cases even as a proper subset) an overlapping of all sets of witnesses considered in Algorithm 1. After computing the set of witnesses for H, Brunotte's algorithm for regions considers the overlapping of all graphs that can be defined on the set of witnesses by parameters from H simultaneously to search for cycles the corresponding cutout polyhedra of which are then subtracted from H. But since an overlapping of many graphs is considered, many cycles will be found which lead to empty cutout polyhedra. In Algorithm 1 the different layers of the overlapped graph are treated separately and all cycles found in a layer are guaranteed to be relevant.

So what are the time-consuming steps in both algorithms? For Brunotte's algorithm for regions it is the computation of all cycles in the overlapped graph and, even more so, the computation of the corresponding cutout polyhedra. The latter is done by solving a system of linear inequalities which come in pairs for every step in the respective cycle according to Lemma 3.5.4. But this system of linear inequalities needs to be solved even if the solution is the empty set which might not be obvious right from the beginning.

For Algorithm 1 the time-consuming step is the computation of the polyhedra which correspond to the sets of witnesses that are found. At first sight it might seem that this will be much slower than computing cutout polyhedra for cycles even if many of them are empty hence useless. After all, the systems of linear inequalities coming from sets of witnesses are much bigger as there are four inequalities for every element in the set of witnesses by Lemma 4.1.3. But there is one decisive advantage one has when solving these systems, which is not available when solving the systems coming from cycles in Brunotte's algorithm for regions: One solution of the system is already known. It is the parameter $\mathbf{r}$ which has been used to find the set of witnesses in the first place and which is trivially contained in the corresponding polyhedron. Using this known solution the solution set of the whole system can be found very efficiently. We shall give a short summary of the approach for $d = 2$ which can be adapted to higher dimensions as well. Figure 2 below shows a possible situation and will be referred to in the following description.

The first step is to compute the normal distances of the parameter $\mathbf{r}$ (black point) and any of the lines bounding the half-planes defined by the linear inequalities. Those that are closest to $\mathbf{r}$ are certainly non-redundant (thick black line). Starting from such a line one can intersect it with all the other lines and select among those the one the intersection point of which is closest to the foot of the perpendicular from $\mathbf{r}$ to the first line in a given direction (clockwise or counter-clockwise). If several lines have the same minimal distance then one of those has to be selected which encloses the smallest angle with the first line with respect to the chosen direction. This line is again guaranteed to be non-redundant (one of the two thick orange lines adjacent to the thick black line). One then continues in the same fashion to go around $\mathbf{r}$ and find all non-redundant lines till the first line is reached again (the other thick orange lines). If n is the number of linear inequalities and m is the number of edges of the resulting polygon then nm intersections have to be computed. Among several millions of polygons found that way there was not a single one where m was larger than 5 (and this was the case in only a single situation, in all others m was less than 5) which makes the seemingly costly task of solving the systems of linear inequalities coming from sets of witnesses not so time-consuming after all. Note that some details (which cause no additional time exposure) of the whole procedure have been skipped. These are mostly related to the fact that there are actually two types (strict and non-strict) of linear inequalities that have to be considered.

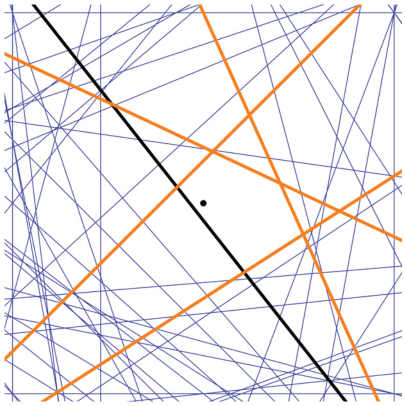


FIGURE 2. Going around the known solution $\mathbf{r}$ to find the whole solution set $P_{\mathbb{R}}(\Pi_{\mathbf{r}})$.

4.3. Finer classes

In all our applications Algorithm 1 presented in the previous section performed faster than Brunotte's algorithm for regions and can therefore be used to practically settle larger regions of $\mathcal{D}_d^{(0)}$. In the next section an algorithm will be introduced which again performed much faster than Algorithm 1 and the present section will serve to discuss necessary preliminaries.

From now on let $d \in \mathbb{N}$ and $(\mathbf{r}_1, \ldots, \mathbf{r}_k) \in \operatorname{int}(\mathcal{D}_d)^k$ such that $H = \operatorname{conv}(\{\mathbf{r}_1, \ldots, \mathbf{r}_k\}) \subset \operatorname{int}(\mathcal{D}_d)$. Algorithm 1 computes a decomposition of H into finitely many disjoint polyhedra (from which it selects those which are contained in $\mathcal{D}_d^{(0)}$ in the final step). Algorithm 2 from the next section uses any finite superset V (which has to be fixed initially) of $\tilde{V}_H := \bigcup_{\mathbf{r} \in H} V_{\mathbf{r}}$ to compute a refinement of this decomposition (cf. Figure 3). $\tilde{V}_H$ itself is a finite set according to Theorem 3.6.9 and can be computed by Algorithm 1. Though $\tilde{V}_H$ would be the optimal choice for V, its determination by Algorithm 1 would of course be pointless. But at least for some choices of H another finite superset of $\tilde{V}_H$ can be calculated efficiently using Brunotte's algorithm for regions. It calculates a common set of witnesses for all $\mathbf{r} \in H$. Unfortunately the set V_H found in this way need not always be finite even if H is contained in the interior of $\mathcal{D}_d$. However in practice this causes no troubles as discussed at the end of Section 3.6.

From now on let $V \subseteq \mathbb{Z}^d$ fix any finite superset of $\tilde{V}_H$ and consider the equivalence relation

DEFINITION 4.3.1. [**Weitzer, 2015a**]

$$\sim := \left\{(\mathbf{r}_1, \mathbf{r}_2) \in (\mathbb{R}^d)^2 \mid \forall\, \mathbf{a} \in V : \tau_{\mathbf{r}_1}(\mathbf{a}) = \tau_{\mathbf{r}_2}(\mathbf{a}) \wedge \tau^\star_{\mathbf{r}_1}(\mathbf{a}) = \tau^\star_{\mathbf{r}_2}(\mathbf{a})\right\}.$$

Then the set $H/_\sim = \left\{[\mathbf{r}]_\sim \cap H \mid \mathbf{r} \in \mathbb{R}^d\right\}$ is a refinement of the decomposition of H calculated by Algorithm 1. If $\mathcal{R} \subseteq H$ is any system of representatives of $H/_\sim$ then the intersection of $\mathcal{D}_d^{(0)}$ and H is given by the finite disjoint union $\bigcup \left\{[\mathbf{r}]_\sim \cap H \mid \mathbf{r} \in \mathcal{R} \cap \mathcal{D}_d^{(0)}\right\}$. In order to determine the complete list of equivalence classes (Theorem 4.3.3 below) the notion of face lattices of (convex) polyhedra proves useful again. To proceed we need the following technical definition.

Definition 4.3.2. [**Weitzer, 2015a**] *For* $(\mathbf{a}, b) \in \mathbb{R}^d \times \mathbb{R}$ *let* $\mathrm{P}(\mathbf{a}, b) := \left\{\mathbf{r} \in \mathbb{R}^d \mid \mathbf{ar} + b = 0\right\}$. *A hyperplane* $P \subseteq \mathbb{R}^d$ *is called integer if there is a tuple* $(\mathbf{a}, b) \in \mathbb{Z}^d \times \mathbb{Z}$ *such that* $P = \mathrm{P}(\mathbf{a}, b)$. *Any such tuple shall then be denoted as generator of* P. *The unique generator which satisfies that the first nonzero entry of* $\mathbf{a}$ *is positive and that the greatest common divisor of the entries of* $\mathbf{a}$ *and* b *is* 1 *is the canonical generator of* P *and shall be denoted by* $\mathrm{CG}(P) = (\mathrm{CG}_1(P), \mathrm{CG}_2(P))$. *The first entry* $\mathrm{CG}_1(P)$ *of the canonical generator is the canonical normal vector of* P.

Theorem 4.3.3. [**Weitzer, 2015a**] *For all* $\mathbf{a} \in V$ *let* $B_{\mathbf{a}} := \{-\mathbf{ar}_i \mid i \in [\![1, k]\!]\}$ *and*

$$\mathcal{G} := \{\mathrm{CG}(\mathrm{P}(\mathbf{a}, b)) \mid \mathbf{a} \in V \setminus \{\mathbf{0}\} \wedge b \in \{\lfloor \min(B_{\mathbf{a}}) \rfloor, \ldots, \lceil \max(B_{\mathbf{a}}) \rceil\}\}.$$

Furthermore let $\phi : \mathbb{R}^d \to \{-1,0,1\}^{|\mathcal{G}|}$ *where* $\phi(\mathbf{r}) = (\mathrm{sgn}(\mathbf{ar} + b))_{(\mathbf{a},b)\in\mathcal{G}}$ *and let* $\mathcal{P}$ *denote the set of all minimal nondegenerate polyhedra having non-empty intersection with* H *which are the intersection of some selection of half-spaces from the set* $\left\{\left\{\mathbf{r} \in \mathbb{R}^d \mid \mathbf{ar} + b \geq 0\right\} \mid (\mathbf{a}, b) \in \mathcal{G}\right\} \cup \left\{\left\{\mathbf{r} \in \mathbb{R}^d \mid -\mathbf{ar} - b \geq 0\right\} \mid (\mathbf{a}, b) \in \mathcal{G}\right\}$. *Then*

$$H/_{\sim} = \left\{\phi^{-1}(S) \cap H \mid S \in \{-1,0,1\}^{|\mathcal{G}|}\right\} \setminus \{\emptyset\} = \left\{F \cap H \mid F \in \bigcup_{P \in \mathcal{P}} \mathcal{F}^{\circ}(P)\right\} \setminus \{\emptyset\}.$$

Proof. Lemma 4.1.3 implies that $\Phi_1 : H/_{\sim} \to \{-1,0,1\}^{|\mathcal{G}|}$, where $\Phi_1([\mathbf{r}]_{\sim}) = \phi(\mathbf{r})$ is well-defined and injective. On the other hand it follows from the definitions of open faces and $\mathcal{P}$ that $\Phi_2 : \left\{F \cap H \mid F \in \bigcup_{P\in\mathcal{P}} \mathcal{F}^{\circ}(P)\right\} \setminus \{\emptyset\} \to \{-1,0,1\}^{|\mathcal{G}|}$ given by $\Phi_2(F) = (\mathrm{sgn}(\mathbf{av} + b))_{(\mathbf{a},b)\in\mathcal{G}}$ with $\mathbf{v} \in F$ is also well-defined and injective and $\Phi_1^{-1}(S) = \Phi_2^{-1}(S)$ for all $S \in \{-1,0,1\}^{|\mathcal{G}|}$ which proves the statement. □

Theorem 4.3.3 gives a geometric interpretation of the equivalence classes of $\sim$. The hyperplanes which are generated by the elements of $\mathcal{G}$ cut $\mathbb{R}^d$ into pieces of polyhedral shape and the set of all (nonempty) open faces of these polyhedra is exactly the set of equivalence classes of $\sim$. The use of canonical generators eliminates redundant hyperplanes, which is not needed in the proof but will speed up the process of actually finding the set of all open faces. If $d = 2$ this is not too difficult but one would probably approach the problem in reverse order than what Theorem 4.3.3 suggests. Instead of calculating the set $\mathcal{P}$ of polygons directly and the set of open faces (singletons (vertices), open line segments (edges), and nondegenerate open polygons (interiors)) afterwards, one can first find all vertices by pairwise intersection of the given lines, then the edges (pair of distinct vertices that lie on a common line with no other vertex lying in between), and at last the interiors (use any algorithm to find the graph theoretic faces of a planar embedding of a graph).

In higher dimensions one could use the cylindrical algebraic decomposition algorithm which, for a given set of polynomials in $\mathbb{R}[x_1, \ldots, x_d]$, finds a decomposition of $\mathbb{R}^d$ into regions on which each polynomial has constant sign [**Collins, 1975**].

Remark 4.3.4. [**Weitzer, 2015a**] *The set* $H/_{\sim}$ *of equivalence classes is also useful when calculating* $\overline{\tau}_H(V)$ *for some finite* $V \subseteq \mathbb{Z}^d$. *It follows from the definition of* $\overline{\tau}_H$ *that* $\overline{\tau}_H(V) = \bigcup_{\mathbf{a}\in V} \overline{\tau}_H(\{\mathbf{a}\})$ *and for any* $\mathbf{a} \in \mathbb{Z}^d$ *one gets that*

$$\overline{\tau}_H(\{\mathbf{a}\}) = \{\tau_{\mathbf{r}}(\mathbf{a}) \mid [\mathbf{r}]_{\sim} \in H/_{\sim}\},$$

where $\sim := \left\{(\mathbf{r}_1, \mathbf{r}_2) \in \mathbb{R}^2 \mid \tau_{\mathbf{r}_1}(\mathbf{a}) = \tau_{\mathbf{r}_2}(\mathbf{a}) \wedge \tau^{\star}_{\mathbf{r}_1}(\mathbf{a}) = \tau^{\star}_{\mathbf{r}_2}(\mathbf{a})\right\}$.

If the set $H/_{\sim}$ of equivalence classes is known one could use Brunotte's algorithm to decide whether or not a given class belongs to $\mathcal{D}_d^{(0)}$. The definition of $\sim$ guarantees that the result will be the same for all parameters in the class. But instead of treating all classes independently two decisive optimizations can be made to speed up the process considerably.

If $[\mathbf{r}]_{\sim} \in H/_{\sim}$ is any class then $V_{\mathbf{r}} \subseteq V$ and in all situations of practical relevance (where V has been found with Brunotte's algorithm for regions) V probably will not be much larger than $V_{\mathbf{r}}$. So instead of calculating $V_{\mathbf{r}}$ and checking it for nontrivial cycles one can just check the similar superset V.

The second optimization relies on the fact that the graph defined by $\tau_{\mathbf{r}}$ on V can only change at specific vertices if the parameter is changed to $\mathbf{s}$ where $[\mathbf{s}]_{\sim} \in H/_{\sim}$ is any class that is adjacent to $[\mathbf{r}]_\sim$ and two classes are considered *adjacent* if they are distinct and their topological boundaries intersect. The following theorem describes on which nodes the graph on V needs to be updated (at most) in this situation. If both classes have a positive distance from the boundary of H, the set M of these nodes consists of those elements of V which are integer multiples of the canonical normal vectors of any hyperplane containing the intersection of the boundaries of $[\mathbf{r}]_\sim$ and $[\mathbf{s}]_\sim$ and any (d-1)-dimensional class in the intersection of the "closed neighborhoods" of $[\mathbf{r}]_\sim$ and $[\mathbf{s}]_\sim$.

THEOREM 4.3.5. [**Weitzer, 2015a**] *Let $[\mathbf{r}]_\sim \in H/_\sim$ and $[\mathbf{s}]_\sim \in H/_\sim$ be adjacent and for any class $C \in H/_\sim$ let $\overline{N(C)} := \{D \in H/_\sim \mid D$ adjacent to $C\} \cup \{C\}$ (closed neighborhood of C). Furthermore let $M := \{\mathbf{a} \in V \mid \exists\, b \in \mathbb{Z} : \partial[\mathbf{r}]_\sim \cap \partial[\mathbf{s}]_\sim \subseteq \mathrm{P}(\mathbf{a}, b)\}$. Then*

(i) $\{\mathbf{a} \in V \mid \tau_{\mathbf{r}}(\mathbf{a}) \neq \tau_{\mathbf{s}}(\mathbf{a})\} \subseteq M$

(ii) $\partial[\mathbf{r}]_\sim \cap \partial H = \emptyset \wedge \partial[\mathbf{s}]_\sim \cap \partial H = \emptyset \Rightarrow$

$$M = V \cap \{\lambda\, \mathrm{CG}_1(\mathrm{span}_{\mathbb{R}}([\mathbf{t}]_\sim - \mathbf{t}) + \mathbf{t}) \mid \lambda \in \mathbb{Z} \wedge [\mathbf{t}]_\sim \in \overline{N([\mathbf{r}]_\sim)} \cap \overline{N([\mathbf{s}]_\sim)} \wedge \dim_{\mathbb{R}}(\mathrm{span}_{\mathbb{R}}([\mathbf{t}]_\sim - \mathbf{t})) = d-1 \wedge \partial[\mathbf{r}]_\sim \cap \partial[\mathbf{s}]_\sim \subseteq \mathrm{span}_{\mathbb{R}}([\mathbf{t}]_\sim - \mathbf{t}) + \mathbf{t}\}$$

PROOF. We say that a hyperplane separates two classes from $H/_\sim$ iff either one class is contained in the hyperplane while the other has empty intersection with it or each class has empty intersection with exactly one of the two open half-spaces $\mathbb{R}^d$ is divided into by the hyperplane. For every $\mathbf{a} \in V$ with $\tau_{\mathbf{r}}(\mathbf{a}) \neq \tau_{\mathbf{s}}(\mathbf{a})$ there is a $b \in \mathbb{Z}$ such that $\mathrm{P}(\mathbf{a}, b)$ separates $[\mathbf{r}]_\sim$ and $[\mathbf{s}]_\sim$ according to Theorem 4.3.3. And every hyperplane separating the two distinct but "touching" classes $[\mathbf{r}]_\sim$ and $[\mathbf{s}]_\sim$ has to contain the intersection of their topological boundaries which shows (i).

For the proof of (ii) assume that $[\mathbf{r}]_\sim$ and $[\mathbf{s}]_\sim$ both have a positive distance from the boundary of H and let N be the set on the right-hand side of the claimed equation. It is easy to see that $N \subseteq M$. For the other inclusion let $\mathbf{a} \in M$ and $b \in \mathbb{Z}$ such that $\partial[\mathbf{r}]_\sim \cap \partial[\mathbf{s}]_\sim \subseteq \mathrm{P}(\mathbf{a}, b)$. Then it follows from Theorem 4.3.3 that there is a class $[\mathbf{t}]_\sim \in \overline{N([\mathbf{r}]_\sim)} \cap \overline{N([\mathbf{s}]_\sim)}$ with $\mathrm{P}(\mathbf{a}, b) = \mathrm{span}_{\mathbb{R}}([\mathbf{t}]_\sim - \mathbf{t}) + \mathbf{t}$ and thus $\dim_{\mathbb{R}}(\mathrm{span}_{\mathbb{R}}([\mathbf{t}]_\sim - \mathbf{t})) = \dim_{\mathbb{R}}(\mathrm{P}(\mathbf{a}, b) - \mathbf{t}) = d - 1$. Furthermore it follows from the definition of CG_1 that $\mathbf{a} = \lambda\, \mathrm{CG}_1(\mathrm{P}(\mathbf{a}, b))$ for some $\lambda \in \mathbb{Z}$ which proves $M \subseteq N$. □

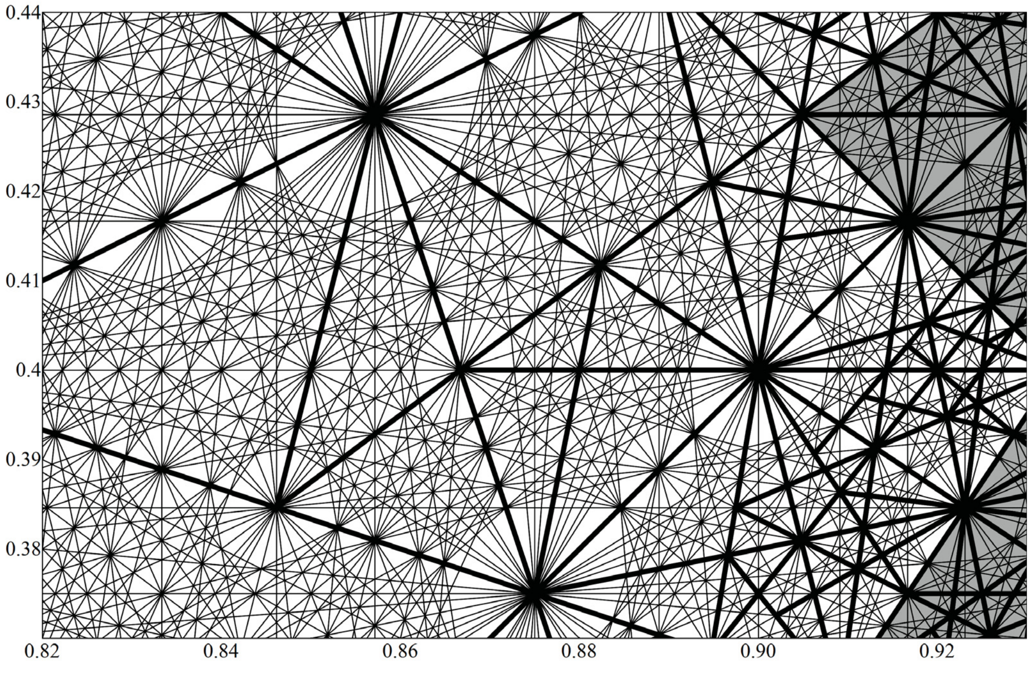


FIGURE 3. Comparison of the decompositions obtained by Algorithm 1 (bold) and Algorithm 2 of $H = \mathrm{conv}\left(\left\{\left(\frac{41}{50}, \frac{37}{100}\right), \left(\frac{93}{100}, \frac{37}{100}\right), \left(\frac{93}{100}, \frac{11}{25}\right), \left(\frac{41}{50}, \frac{11}{25}\right)\right\}\right)$. Dark regions are contained in cutout polygons.

4.4. A fast algorithm

Algorithm 2 [**Weitzer, 2015a**] Determination of $\operatorname{conv}(\{\mathbf{r}_1, \dots, \mathbf{r}_k\}) \cap \mathcal{D}_d^{(0)}$

Input: $(\mathbf{r}_1, \dots, \mathbf{r}_k) \in \operatorname{int}(\mathcal{D}_d)^k$ such that $\operatorname{conv}(\{\mathbf{r}_1, \dots, \mathbf{r}_k\}) \subset \operatorname{int}(\mathcal{D}_d)$,
$\tilde{V}_{\operatorname{conv}(\{\mathbf{r}_1,\dots,\mathbf{r}_k\})} \subseteq V \subseteq \mathbb{Z}^d$ finite.
Output: $\mathcal{C} \subseteq \mathcal{C}_d^{(\mathbb{Z})}$ with $\operatorname{conv}(\{\mathbf{r}_1, \dots, \mathbf{r}_k\}) \cap \mathcal{D}_d^{(0)} = \operatorname{conv}(\{\mathbf{r}_1, \dots, \mathbf{r}_k\}) \setminus \bigcup_{\pi \in \mathcal{C}} P(\pi)$.

1: $H \leftarrow \operatorname{conv}(\{\mathbf{r}_1, \dots, \mathbf{r}_k\})$
2: $\mathcal{C} \leftarrow \emptyset$
3: $G = (V(G), E(G)) \leftarrow (V, \emptyset)$ {edgeless digraph with vertex set V}
4: calculate $H/_\sim$ according to Theorem 4.3.3
5: **for all** $C \in H/_\sim$ **do**
6:     $N_C \leftarrow \{D \in H/_\sim \mid D \text{ adjacent to } C\}$
7:     $B_C \leftarrow$**false**
8: **end for**
9: **for all** $[\mathbf{r}]_\sim \in H/_\sim$ with $B_{[\mathbf{r}]_\sim} =$ **false** and $\partial[\mathbf{r}]_\sim \cap \partial H \neq \emptyset$ **do**
10:     **if** $\mathbf{r} \in \mathcal{D}_d^{(0)}$ **then** {search for cycles of $\mathbf{r}$ on V}
11:         $B_{[\mathbf{r}]_\sim} \leftarrow$**true**
12:     **else**
13:         select π nontrivial cycle of $\mathbf{r}$ on V
14:         $\mathcal{C} \leftarrow \mathcal{C} \cup \{\pi\}$
15:         **for all** $[\mathbf{s}]_\sim \in H/_\sim$ with $B_{[\mathbf{s}]_\sim} =$ **false** and $\mathbf{s} \in P(\pi)$ **do**
16:             $B_{[\mathbf{s}]_\sim} \leftarrow$**true**
17:         **end for**
18:     **end if**
19: **end for**
20: **while** $\exists\, C \in H/_\sim : B_C =$ **false do**
21:     select $[\mathbf{r}]_\sim \in H/_\sim$ with $B_{[\mathbf{r}]_\sim} =$ **false**
22:     $E(G) \leftarrow \{(\mathbf{a}, \tau_{\mathbf{r}}(\mathbf{a})) \mid \mathbf{a} \in V\}$
23:     $W \leftarrow V$
24:     **loop**
25:         **if** $\mathbf{r} \in \mathcal{D}_d^{(0)}$ **then** {search for cycles of G starting at the vertices in W}
26:             $B_{[\mathbf{r}]_\sim} \leftarrow$**true**
27:             **if** $\exists\, C \in N_{[\mathbf{r}]_\sim} : B_C =$ **false then**
28:                 select $C \in N_{[\mathbf{r}]_\sim}$ with $B_C =$ **false**
29:                 update $E(G)$ according to Theorem 4.3.5
30:                 save the tails of the changed edges in W
31:                 $[\mathbf{r}]_\sim \leftarrow C$
32:             **else**
33:                 **break**
34:             **end if**
35:         **else**
36:             select π nontrivial cycle of G
37:             $\mathcal{C} \leftarrow \mathcal{C} \cup \{\pi\}$
38:             **for all** $[\mathbf{s}]_\sim \in H/_\sim$ with $B_{[\mathbf{s}]_\sim} =$ **false** and $\mathbf{s} \in P(\pi)$ **do**
39:                 $B_{[\mathbf{s}]_\sim} \leftarrow$**true**
40:             **end for**
41:             **break**
42:         **end if**
43:     **end loop**
44: **end while**
45: **return** $\mathcal{C}$

Algorithm 2 computes a minimal set (with respect to set inclusion but not necessarily cardinality) of cutout polyhedra which characterizes $\mathcal{D}_d^{(0)}$ inside of H. After initialization of required variables (steps 1-8) the classes "touching" the boundary of H are treated directly (steps 9-19). If a nontrivial cycle is found all classes contained in the corresponding cutout polyhedron are considered "done" (the associated B-flag is set to **true**) and the cycle is added to the output set $\mathcal{C}$.

After that the main part of the algorithm follows (steps 20-44). The classes are treated along walks in the graph defined on $H/_\sim$ by the adjacency relation. When a nontrivial cycle is found it is handled as before and a new walk begins, as it does when the walk reaches a dead end (i.e. if there are no neighbors yet to be treated). Any time a new walk starts all edges of G have to be updated and checked for cycles, which consumes much more time than updating and checking only those edges which are changed when going from one class to an adjacent one. In order to minimize the number of restarts it is crucial to make a good choice when selecting the next node (step 28). A Hamiltonian path would of course be an optimal but also costly choice. Instead the following heuristic turns out to be adequate: Of all possible neighbors of least dimension take the one (or one of those) which has the highest number of neighbors that are already treated. This way the walks tend to stay "compact" and will not cut the graph into too many pieces of pending vertices.

Figure 3 illustrates the relation between the resulting decompositions of Algorithm 1 and Algorithm 2 and cutout polygons. Even though Algorithm 2 considers more equivalence classes than Algorithm 1 it is still much faster as only those orbits are considered which change when going from one class to an adjacent one. Another advantage of Algorithm 2 over both Algorithm 1 and Brunotte's algorithm for regions is that it has a very compact output. None of the cutout polyhedra in $\mathcal{C}$ are redundant.

4.5. Topological results on Shift Radix Systems

The following theorem characterizes large parts of $\mathcal{D}_2^{(0)}$ and the subsequent corollary summarizes topological features of the characterized region.

THEOREM 4.5.1. [**Weitzer, 2015a**] *Let* $K = \frac{1}{20}$, $L = \frac{1}{512}$, *and* $C := C_1 \setminus C_2$ *where*

$$C_1 := \{(x, y) \in \mathbb{R}^2 \mid x \le 1 - L\}$$

$$C_2 := \operatorname{int}\left(\left(\operatorname{conv}\left(\left\{(1-K, 2-K), (1-K+\sqrt{2}L, 2-K), (1-\sqrt{2}L, 2-2\sqrt{2}L), (1,2)\right\}\right)\right)\right).$$

Furthermore let every 5-tuple (n, x, y, a, b) *in the list of Table 1 represent a cutout polygon* P *in the following way:* $\mathbf{r} := (\frac{x}{n}, \frac{y}{n})$, $\mathbf{a} := (a, b)$, $m := \min\{k \in \mathbb{N} \mid \tau_{\mathbf{r}}^k(\mathbf{a}) = \mathbf{a}\}$, $\pi := (\tau_{\mathbf{r}}(\mathbf{a}), \ldots, \tau_{\mathbf{r}}^m(\mathbf{a}))$, *and* $P := P(\pi)$. *If* $\{P_1, \ldots, P_{598}\}$ *is the set of the* 598 *cutout polygons then*

$$\mathcal{D}_2^{(0)} \cap C = \{(x, y) \in \mathbb{R}^2 \mid x \le 1 \wedge |y| \le x + 1\} \cap C \setminus \bigcup_{k=1}^{598} P_k$$

and none of the 598 *cutout polygons are redundant.*

PROOF. The list of Table 1 has been found by Algorithm 2 of the previous section. An annotated version of the C++ program which computed these results can be found on the CD coming with this thesis or at:

`http://institute.unileoben.ac.at/mathstat/personal/weitzer.htm`

The convex sets used as inputs for the algorithm were the closed squares $\left[\frac{x}{n}, \frac{x+1}{n}\right] \times \left[\frac{y}{n}, \frac{y+1}{n}\right]$, where $n = 8192$, $(x, y) \in \mathbb{Z}^2$ with $\left\lfloor \frac{2n}{3} \right\rfloor \le x \le L - \frac{1}{n}$ and $-\frac{n}{2} \le y \le \frac{3n}{2} - 1$. The remaining regions have already been characterized in [**Akiyama et al., 2005**] (especially by Theorem 4.8 there, which covers the region $\{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1 \wedge 0 < y < x + 1 \wedge 4x < y^2 \wedge y > \frac{x}{\gamma_6} + \gamma_6\}$ where γ_q is the positive root of $qt^3 + qt^2 - qt - q + 1$, $q \in \mathbb{N}$, and therefore reaches the boundary of $\mathcal{D}_2$) or were also treated by Algorithm 2. Any of the given parameters is contained solely in the corresponding cutout polygon which shows that none of the cutout polygons are redundant. □

Note that the region characterized by Theorem 4.5.1 is considerably larger than what has been achieved with Brunotte's algorithm for regions so far ($L = \frac{1}{100}$, [**Surer, 2007**]). Also note that $\{(x, y) \in \mathbb{R}^2 \mid x \le 1 \wedge |y| \le x + 1\}$ is the topological closure of $\mathcal{D}_2$ (Theorem 3.4.2) and that C_2 is a small open quadrangle of width L touching the boundary of $\mathcal{D}_2$ left of $(1, 2)$.

The analysis of the list of cutout polygons leads to the following corollary.

COROLLARY 4.5.2. [**Weitzer, 2015a**]

(i) $\mathcal{D}_2^{(0)}$ *has at least 22 connected components*

(ii) *The largest connected component of* $\mathcal{D}_2^{(0)}$ *has at least 3 holes.*

PROOF. The parameters

$(\frac{1}{2},\frac{1}{2})$, $(\frac{152}{157},\frac{193}{157})$, $(\frac{313}{315},\frac{239}{210})$, $(\frac{167}{168},\frac{255}{224})$, $(\frac{314}{317},\frac{359}{317})$, $(\frac{453}{455},\frac{496}{455})$, $(\frac{305}{306},\frac{37}{34})$, $(\frac{362}{363},\frac{259}{242})$, $(\frac{356}{357},\frac{382}{357})$, $(\frac{358}{359},\frac{384}{359})$, $(\frac{1121}{1124},\frac{601}{562})$, $(\frac{1375}{1378},\frac{640}{689})$, $(\frac{2061}{2066},\frac{959}{1033})$, $(\frac{309}{310},\frac{141}{155})$, $(\frac{1533}{1538},\frac{699}{769})$, $(\frac{989}{992},\frac{901}{992})$, $(\frac{1127}{1133},\frac{1009}{1133})$, $(\frac{1607}{1612},\frac{691}{806})$, $(\frac{694}{697},\frac{521}{697})$, $(\frac{92}{93},\frac{16}{31})$, $(\frac{537}{539},\frac{67}{539})$, $(\frac{304}{305},\frac{38}{305})$

are contained in 22 distinct connected components of $\mathcal{D}_2^{(0)}$.

The parameters

$(\frac{911}{914},\frac{391}{457})$, $(\frac{2455}{2463},\frac{2108}{2463})$, $(\frac{265}{266},\frac{1}{4})$

are contained in 3 distinct holes of the largest connected component of $\mathcal{D}_2^{(0)}$. □

The figures below show the calculated cutout polygons and the resulting shape of $\mathcal{D}_2^{(0)}$ in the corresponding region. Figure 4 gives an overview of $\mathcal{D}_2^{(0)}$ and shows the regions which lie above and below the point $(1,1)$, Figure 5 shows several connected components, and Figure 6 gives an example of holes.

The cutout polygons are represented in the following way: If an edge belongs to the polygon it is plotted solid, and dotted otherwise. Belonging of a vertex is indicated by a prominent dot at the respective position. In the images which show the resulting shape of $\mathcal{D}_2^{(0)}$ black regions do belong to $\mathcal{D}_2^{(0)}$, white regions do not and gray regions are not settled by now.

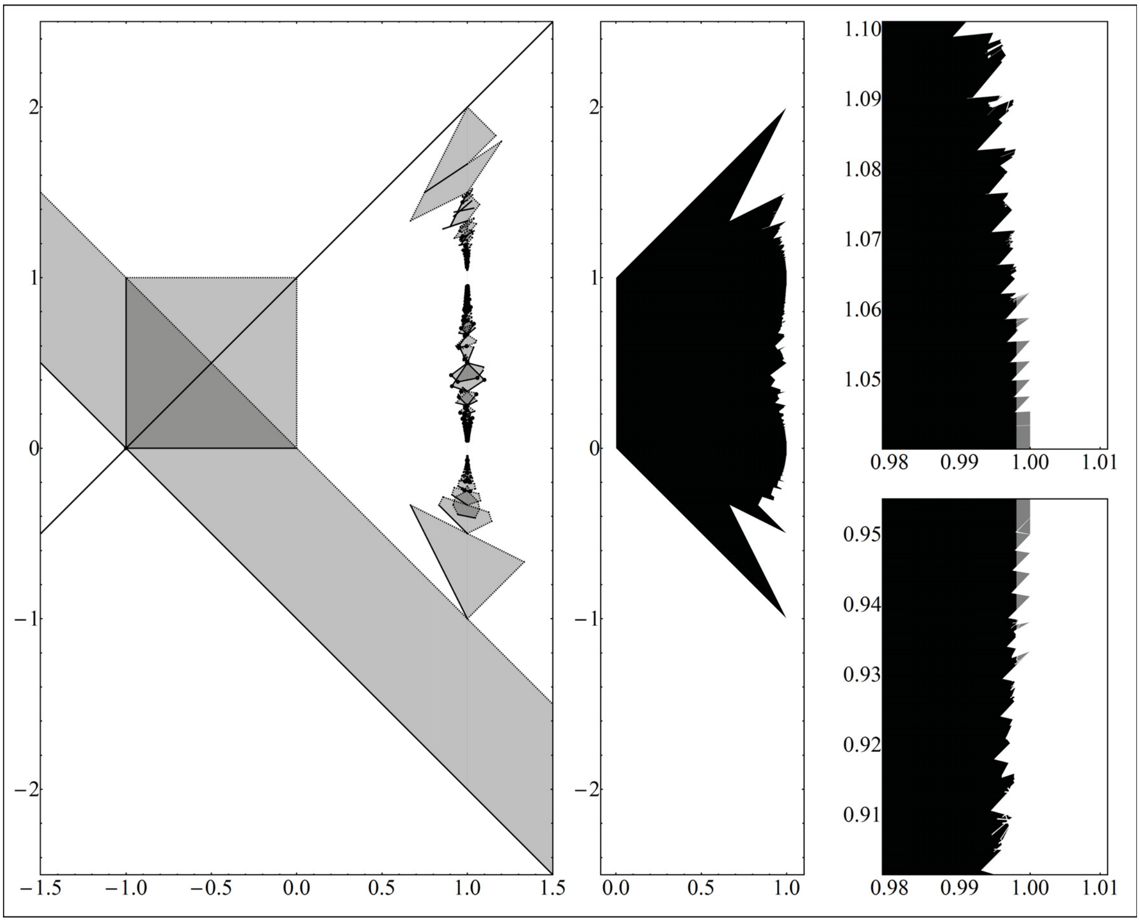


FIGURE 4. Overview of $\mathcal{D}_2^{(0)}$ and the regions above and below $(1,1)$.

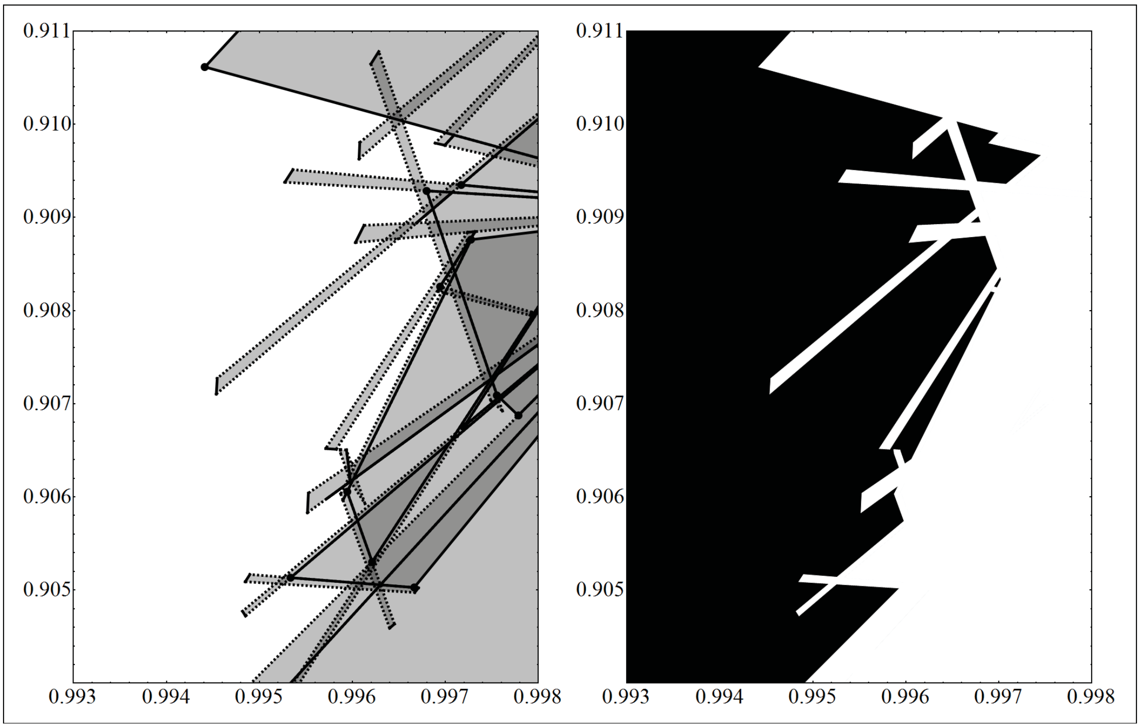


FIGURE 5. Four connected components of $\mathcal{D}_2^{(0)}$.

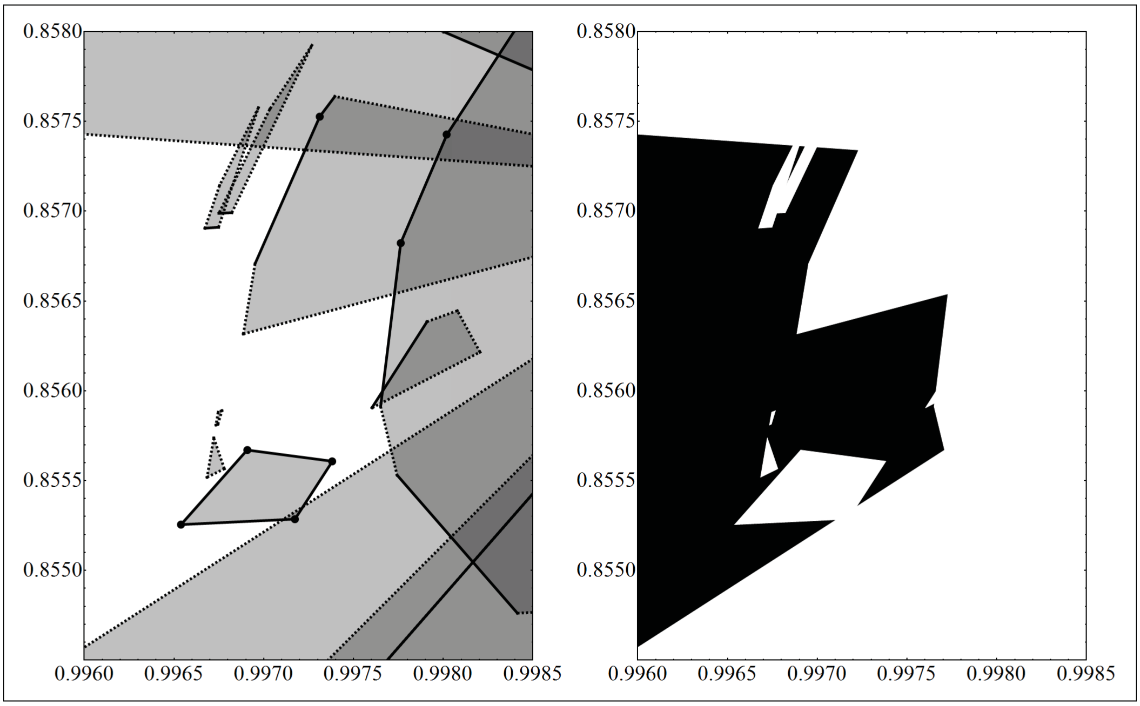


FIGURE 6. Two holes of $\mathcal{D}_2^{(0)}$.

(1, 0, -1, 1, 1)	(1, 0, 1, -1, 1)	(3, -1, 1, 0, 1)	(3, 3, -2, 2, 1)	(3, 3, -1, 5, 1)	(3, 3, 2, 5, 1)	(4, 4, 1, 6, 1)	(4, 4, 7, 1, 1)
(5, 5, 1, 11, 4)	(5, 5, 8, 1, 1)	(6, 6, -1, 9, 1)	(6, 6, 1, 10, 1)	(6, 6, 5, 7, 1)	(7, 7, -3, 3, 1)	(7, 7, -1, 16, 4)	(7, 7, 4, 5, 1)
(7, 7, 9, 3, 1)	(8, 8, -1, 21, 4)	(8, 8, 3, 3, 1)	(8, 8, 5, 4, 1)	(8, 8, 7, 9, 1)	(8, 8, 9, 25, 2)	(8, 8, 11, 2, 1)	(9, 9, -2, 5, 1)
(9, 9, -1, 13, 1)	(9, 9, 1, 14, 1)	(9, 9, 7, 6, 1)	(9, 9, 8, 11, 1)	(9, 9, 10, 39, 4)	(9, 9, 11, 4, 1)	(10, 9, 13, 2, 1)	(10, 10, -1, 29, 4)
(10, 10, 11, 15, 2)	(11, 11, 10, 13, 1)	(11, 11, 12, 27, 2)	(11, 11, 13, 5, 1)	(12, 11, 5, 3, 3)	(12, 11, 16, 2, 1)		
(12, 12, -1, 17, 1)	(12, 12, 1, 18, 1)	(13, 13, -1, 35, 4)	(13, 13, 3, 5, 1)	(13, 13, 11, 8, 1)	(13, 13, 15, 6, 1)		
(14, 13, -4, 4, 1)	(14, 14, -1, 54, 2)	(14, 14, 15, 30, 2)	(15, 15, -2, 8, 1)	(15, 15, -1, 21, 1)	(15, 15, 1, 22, 1)		
(15, 15, 2, 8, 1)	(15, 15, 14, 17, 1)	(15, 15, 16, 35, 2)	(15, 15, 17, 7, 1)	(16, 16, 17, 25, 2)	(17, 17, 1, 26, 1)		
(17, 17, 16, 19, 1)	(17, 17, 19, 8, 1)	(18, 18, -1, 25, 1)	(19, 18, 5, 4, 1)	(19, 19, 18, 21, 1)	(19, 19, 21, 9, 1)		
(21, 21, 19, 12, 1)	(21, 21, 23, 10, 1)	(22, 21, 28, 2, 4)	(23, 23, 25, 11, 1)	(25, 24, 17, 6, 1)	(25, 25, 27, 12, 1)		
(26, 25, 31, 4, 1)	(27, 26, 9, 9, 1)	(27, 26, 35, 8, 2)	(27, 27, -2, 14, 1)	(27, 27, 25, 15, 1)	(27, 27, 29, 13, 1)		
(28, 27, 34, 7, 2)	(29, 29, 27, 16, 1)	(29, 29, 31, 14, 1)	(30, 29, 36, 4, 1)	(30, 29, 37, 7, 1)	(31, 29, 43, 4, 1)		
(31, 31, 33, 15, 1)	(32, 30, 19, 3, 2)	(32, 31, -6, 6, 1)	(32, 32, -3, 11, 1)	(33, 33, -2, 17, 1)	(33, 33, 2, 17, 1)		
(33, 33, 31, 35, 2)	(33, 33, 35, 16, 1)	(34, 33, 6, 6, 1)	(34, 33, 42, 7, 2)	(35, 34, 43, 7, 2)	(35, 35, 37, 17, 1)		
(37, 35, 22, 4, 2)	(37, 37, 35, 20, 1)	(37, 37, 39, 18, 1)	(38, 37, 56, 15, 3)	(39, 39, -2, 20, 1)	(39, 39, 2, 20, 1)		
(39, 39, 41, 19, 1)	(40, 39, 29, 8, 1)	(40, 39, 30, 10, 2)	(41, 38, 56, 4, 2)	(41, 41, -2, 21, 1)	(41, 41, 2, 21, 1)		
(41, 41, 43, 20, 1)	(43, 43, -2, 22, 1)	(43, 43, 2, 22, 1)	(43, 43, 45, 21, 1)	(44, 43, -7, 7, 1)	(45, 45, -2, 23, 1)		
(45, 45, 2, 23, 1)	(46, 44, 27, 4, 2)	(46, 45, 7, 7, 1)	(49, 47, 60, 5, 2)	(50, 49, 57, 6, 1)	(50, 49, 58, 15, 2)		
(52, 51, 27, 15, 1)	(54, 53, 41, 9, 2)	(55, 54, 11, 11, 1)	(55, 54, 42, 10, 4)	(55, 54, 64, 15, 2)	(56, 55, 64, 6, 1)		
(56, 55, 65, 15, 2)	(61, 60, -15, 16, 2)	(61, 60, 71, 15, 3)	(61, 60, 72, 7, 4)	(63, 62, 75, 9, 6)	(67, 66, 55, 18, 3)		
(68, 67, 45, 21, 2)	(68, 67, 54, 13, 2)	(68, 67, 56, 20, 4)	(68, 67, 79, 14, 3)	(68, 68, 5, 14, 1)	(71, 69, -17, 9, 2)		
(71, 70, 59, 11, 1)	(72, 71, 83, 9, 3)	(74, 72, 57, 5, 2)	(74, 73, -9, 9, 1)	(74, 73, 18, 17, 1)	(74, 73, 59, 14, 2)		
(74, 73, 86, 14, 2)	(76, 74, 57, 9, 3)	(76, 75, 9, 9, 1)	(79, 76, 97, 6, 1)	(79, 77, 59, 10, 3)	(80, 78, 59, 8, 3)		
(81, 80, 92, 10, 2)	(82, 81, 66, 11, 3)	(82, 81, 91, 8, 1)	(83, 81, 64, 6, 2)	(86, 85, 69, 13, 4)	(89, 87, 48, 9, 2)		
(90, 89, 100, 8, 1)	(92, 91, -10, 10, 1)	(94, 93, 10, 10, 1)	(98, 97, 14, 14, 1)	(98, 97, 111, 24, 2)	(99, 97, 75, 8, 3)		
(100, 99, 85, 13, 1)	(114, 113, 11, 11, 1)	(116, 115, 102, 10, 1)	(120, 119, 136, 22, 3)	(121, 120, 96, 21, 3)			
(122, 121, 133, 10, 1)	(122, 121, 137, 14, 1)	(124, 123, 141, 16, 2)	(127, 123, 183, 7, 1)	(127, 126, 109, 14, 2)			
(128, 127, 16, 15, 3)	(129, 127, 106, 17, 4)	(132, 130, 111, 7, 2)	(132, 131, 144, 10, 1)	(134, 133, -12, 12, 1)			
(134, 133, 152, 26, 4)	(136, 135, 12, 12, 1)	(136, 135, 152, 13, 3)	(137, 134, 74, 9, 2)	(137, 136, 154, 14, 2)			
(139, 138, 121, 16, 2)	(140, 139, 155, 38, 3)	(141, 139, 34, 19, 2)	(141, 140, 20, 21, 2)	(141, 140, 160, 26, 4)			
(142, 141, 157, 14, 3)	(144, 142, 115, 14, 2)	(145, 143, 122, 8, 2)	(147, 146, 163, 14, 3)	(148, 146, -29, 15, 2)			
(148, 147, 172, 22, 4)	(151, 150, 134, 16, 1)	(153, 151, -30, 15, 4)	(154, 152, 37, 18, 2)	(155, 153, 127, 19, 5)			
(156, 155, 22, 28, 2)	(158, 155, 183, 13, 3)	(158, 157, -13, 13, 1)	(159, 157, 127, 14, 2)	(160, 159, 13, 13, 1)			
(168, 167, 183, 28, 2)	(170, 169, 183, 12, 1)	(172, 170, 195, 11, 2)	(173, 169, 129, 10, 3)	(177, 176, 194, 16, 2)			
(179, 177, 147, 19, 4)	(179, 178, 195, 28, 2)	(180, 178, 93, 22, 1)	(180, 179, 196, 27, 2)	(182, 181, 136, 44, 3)			
(182, 181, 196, 12, 1)	(183, 182, 201, 16, 3)	(185, 183, 152, 19, 4)	(185, 183, 211, 10, 4)	(189, 187, 214, 23, 2)			
(191, 189, 157, 19, 3)	(191, 190, 208, 27, 2)	(191, 190, 212, 19, 1)	(192, 191, 209, 29, 3)	(193, 192, 173, 18, 1)			
(195, 192, 160, 18, 4)	(200, 199, 222, 19, 1)	(201, 200, 219, 27, 3)	(202, 199, 106, 15, 1)	(202, 201, 179, 21, 3)			
(203, 202, 25, 24, 4)	(203, 202, 178, 36, 4)	(206, 204, 181, 9, 2)	(212, 211, -15, 15, 1)	(214, 213, 15, 15, 1)			
(218, 215, 114, 16, 4)	(220, 216, 255, 14, 2)	(220, 217, 115, 16, 4)	(220, 219, 202, 14, 1)	(221, 218, 181, 18, 4)			
(222, 221, 242, 54, 4)	(223, 221, 196, 10, 2)	(226, 225, 241, 14, 1)	(230, 229, 207, 19, 2)	(231, 230, 259, 23, 7)			
(232, 231, 210, 62, 6)	(233, 232, 174, 53, 3)	(238, 237, 267, 35, 7)	(240, 239, 256, 14, 1)	(241, 240, 262, 18, 3)			
(242, 239, 275, 11, 1)	(242, 241, -16, 16, 1)	(242, 241, 263, 20, 2)	(244, 243, 16, 16, 1)	(244, 243, 146, 48, 2)			
(244, 243, 268, 24, 5)	(245, 244, 222, 20, 2)	(249, 248, 269, 19, 3)	(250, 249, 25, 25, 1)	(254, 251, 288, 12, 1)			
(255, 254, 191, 63, 3)	(255, 254, 231, 60, 7)	(256, 255, 23, 22, 3)	(258, 257, 32, 68, 3)	(258, 257, 283, 24, 1)			
(260, 259, 238, 21, 1)	(265, 264, 33, 67, 3)	(265, 264, 66, 66, 1)	(265, 264, 238, 26, 2)	(265, 264, 285, 25, 2)			
(268, 267, 294, 24, 1)	(270, 268, 235, 15, 3)	(272, 270, 301, 38, 4)	(273, 271, 302, 39, 4)	(273, 272, 34, 68, 3)			
(275, 273, 239, 15, 2)	(276, 275, 248, 27, 2)	(277, 275, 241, 16, 3)	(277, 276, 298, 25, 2)	(278, 276, 243, 15, 3)			
(278, 277, 250, 27, 2)	(278, 277, 299, 25, 2)	(278, 277, 301, 20, 3)	(279, 277, 242, 29, 6)	(280, 279, 301, 20, 2)			
(281, 280, -28, 29, 2)	(286, 285, -57, 56, 4)	(290, 288, 321, 39, 3)	(290, 289, 307, 16, 1)	(293, 289, 240, 18, 4)			
(295, 294, 321, 30, 2)	(296, 294, 267, 11, 2)	(299, 298, 262, 27, 3)	(301, 300, 322, 95, 7)	(304, 303, 260, 79, 5)			
(306, 305, 324, 16, 1)	(308, 306, 341, 39, 5)	(308, 307, -18, 18, 1)	(309, 308, 278, 26, 4)	(310, 309, 18, 18, 1)			
(317, 315, 286, 12, 2)	(318, 317, 293, 28, 2)	(327, 325, 284, 16, 3)	(327, 326, 297, 31, 4)	(328, 327, 302, 23, 1)			
(329, 328, 299, 33, 3)	(331, 330, 354, 96, 6)	(335, 333, 291, 17, 3)	(335, 334, 287, 45, 1)	(337, 336, 303, 31, 5)			
(338, 337, 386, 46, 1)	(341, 336, 281, 20, 4)	(343, 341, 298, 17, 2)	(344, 343, -19, 19, 1)	(346, 345, 19, 19, 1)			
(351, 349, 389, 19, 1)	(352, 351, 32, 64, 2)	(352, 351, 402, 47, 1)	(353, 352, 321, 33, 2)	(353, 352, 411, 54, 4)			
(354, 353, 379, 24, 1)	(356, 355, 378, 41, 2)	(356, 355, 381, 95, 4)	(360, 359, 329, 48, 2)	(361, 360, 386, 23, 3)			
(362, 361, 381, 18, 1)	(363, 361, 403, 35, 3)	(363, 362, 335, 24, 2)	(363, 362, 388, 49, 3)	(365, 362, 317, 15, 2)			
(365, 364, 331, 61, 6)	(367, 365, 411, 21, 1)	(368, 367, 394, 24, 2)	(369, 368, 395, 30, 2)	(370, 369, 396, 30, 2)			
(372, 370, 413, 34, 5)	(372, 371, 398, 95, 6)	(373, 372, 396, 40, 2)	(374, 372, 333, 44, 7)	(374, 373, 347, 102, 9)			
(377, 376, 403, 49, 3)	(380, 379, 400, 18, 1)	(381, 380, 406, 24, 3)	(381, 380, 435, 53, 1)	(385, 384, 410, 24, 3)			
(386, 384, 55, 27, 5)	(389, 387, 353, 32, 3)	(389, 388, 413, 40, 2)	(390, 388, 353, 62, 7)	(390, 389, 414, 42, 3)			
(391, 390, 423, 69, 4)	(393, 392, 49, 67, 4)	(395, 393, 350, 20, 4)	(400, 399, 373, 26, 1)	(401, 400, 50, 67, 4)			
(402, 400, 369, 13, 2)	(403, 401, 357, 21, 4)	(403, 402, 31, 30, 4)	(403, 402, 439, 34, 5)	(404, 403, 429, 40, 3)			
(404, 403, 437, 69, 3)	(405, 404, 376, 104, 8)	(406, 405, 431, 39, 3)	(407, 404, 353, 29, 5)	(408, 407, 378, 26, 2)			
(409, 408, 29, 29, 1)	(409, 408, 436, 25, 3)	(411, 410, 381, 33, 2)	(416, 415, 445, 95, 6)	(416, 415, 450, 69, 3)			
(417, 409, 483, 14, 2)	(423, 422, 387, 33, 1)	(423, 422, 461, 38, 1)	(427, 425, 392, 14, 2)	(428, 425, 380, 43, 6)			
(431, 430, 369, 60, 3)	(433, 432, 379, 43, 5)	(434, 433, 473, 38, 1)	(435, 434, 466, 45, 3)	(437, 436, 400, 34, 5)			
(442, 441, 463, 20, 1)	(442, 441, 469, 26, 2)	(450, 447, 394, 16, 2)	(450, 449, 487, 37, 2)	(454, 453, 425, 28, 1)			
(462, 459, 236, 34, 3)	(462, 461, 484, 20, 1)	(462, 461, 500, 37, 2)	(463, 462, 496, 90, 4)	(464, 463, 431, 105, 9)			
(465, 463, 507, 55, 3)	(471, 468, 536, 22, 2)	(473, 471, 354, 57, 6)	(479, 476, 426, 44, 6)	(479, 478, 445, 56, 3)			
(481, 476, 545, 24, 2)	(482, 480, 437, 31, 3)	(485, 482, 552, 22, 2)	(487, 484, 423, 16, 3)	(487, 486, 519, 35, 2)			
(489, 487, 529, 19, 3)	(492, 489, 560, 23, 1)	(493, 491, 541, 24, 1)	(494, 493, 522, 72, 4)	(495, 494, 249, 115, 17)			
(496, 493, 544, 16, 1)	(497, 496, 465, 131, 9)	(499, 497, 452, 60, 8)	(500, 499, 533, 35, 2)	(503, 501, 551, 20, 1)			
(503, 501, 552, 25, 1)	(509, 508, 477, 29, 2)	(510, 507, 559, 17, 1)	(511, 510, 540, 72, 5)	(512, 511, 32, 32, 1)			
(513, 511, 465, 59, 8)	(513, 512, 545, 26, 3)	(514, 511, 560, 28, 3)	(516, 513, 572, 36, 3)	(519, 518, 37, 46, 2)			
(521, 518, 462, 42, 6)	(521, 519, 474, 22, 2)	(523, 522, 553, 28, 3)	(524, 522, 487, 15, 2)	(528, 525, 575, 27, 2)			
(530, 529, 553, 22, 1)	(532, 530, 133, 66, 1)	(536, 532, 461, 30, 5)	(536, 532, 593, 39, 4)	(538, 536, 67, 67, 3)			
(553, 551, 514, 16, 2)	(555, 553, 499, 27, 2)	(563, 560, 613, 26, 3)	(565, 563, 516, 23, 2)	(576, 574, 619, 21, 2)			
(577, 575, 525, 33, 4)	(581, 579, 535, 27, 3)	(586, 583, 520, 45, 6)	(586, 584, 73, 65, 3)	(594, 592, 547, 28, 3)			
(599, 597, 652, 30, 2)	(608, 606, 521, 39, 3)	(610, 607, 153, 48, 6)	(627, 625, 573, 23, 3)	(637, 635, 546, 40, 3)			
(641, 639, 690, 21, 3)	(654, 652, 595, 23, 4)	(662, 660, 621, 17, 2)	(668, 663, 739, 39, 4)	(670, 667, 607, 20, 2)			
(677, 674, 738, 55, 3)	(681, 677, 742, 28, 2)	(681, 678, 509, 53, 8)	(687, 685, 735, 95, 5)	(692, 690, -53, 27, 2)			
(694, 692, 63, 63, 2)	(695, 693, 652, 18, 2)	(697, 686, 572, 18, 4)	(697, 695, 746, 95, 5)	(706, 704, 755, 48, 2)			
(707, 705, 756, 47, 3)	(719, 717, 652, 60, 7)	(729, 726, 545, 56, 10)	(730, 728, 781, 95, 5)	(737, 734, 668, 62, 8)			
(737, 735, 683, 25, 2)	(740, 738, 687, 103, 10)	(741, 738, 808, 54, 4)	(742, 740, 695, 45, 3)	(750, 747, 823, 24, 2)			
(752, 749, 820, 54, 4)	(752, 749, 844, 31, 2)	(756, 754, 809, 95, 4)	(758, 755, 851, 31, 1)	(764, 762, 709, 101, 10)			
(768, 765, 698, 21, 2)	(777, 773, 703, 61, 7)	(794, 789, 904, 22, 1)	(795, 793, 737, 32, 3)	(801, 796, 912, 22, 1)			
(804, 802, 747, 27, 2)	(808, 806, 749, 33, 3)	(811, 808, 203, 66, 1)	(816, 814, 771, 19, 2)	(822, 817, 936, 23, 1)			
(833, 830, 899, 73, 4)	(839, 836, 902, 21, 1)	(847, 845, 725, 81, 5)	(850, 846, 773, 33, 3)	(852, 849, 920, 71, 4)			
(853, 851, 806, 20, 2)	(855, 853, 794, 103, 9)	(857, 854, 921, 22, 1)	(884, 882, 821, 103, 9)	(913, 911, 978, 90, 4)			
(914, 911, 782, 45, 1)	(921, 919, 858, 29, 2)	(923, 920, 839, 32, 3)	(931, 927, 696, 55, 3)	(933, 930, 856, 39, 4)			
(933, 931, 866, 105, 10)	(943, 939, 1057, 58, 12)	(943, 941, 876, 104, 10)	(944, 942, 877, 56, 4)	(978, 976, 911, 29, 2)			
(986, 981, 875, 20, 4)	(986, 984, 937, 21, 2)	(993, 989, 900, 62, 8)	(995, 991, 1085, 54, 4)	(996, 992, 1093, 24, 1)			
(1005, 1003, 1062, 72, 4)	(1005, 1003, 1171, 81, 1)	(1006, 1002, 1097, 54, 5)	(1008, 1006, 943, 129, 11)				
(1009, 1006, 918, 24, 4)	(1009, 1007, 944, 131, 10)	(1014, 1012, 1073, 29, 1)	(1017, 1015, 948, 37, 3)				
(1022, 1020, 957, 44, 3)	(1027, 1025, 976, 22, 2)	(1031, 1029, 1091, 29, 2)	(1039, 1036, 1112, 47, 2)				
(1039, 1037, 972, 127, 11)	(1046, 1044, 975, 38, 3)	(1064, 1061, 986, 25, 2)	(1069, 1066, 1135, 41, 3)				
(1072, 1069, 1146, 50, 2)	(1076, 1073, 1164, 29, 2)	(1088, 1085, 1164, 94, 6)	(1089, 1086, 1166, 25, 1)				
(1094, 1091, 1015, 102, 10)	(1095, 1092, 1171, 48, 2)	(1102, 1099, 1170, 40, 2)	(1104, 1101, 1181, 97, 6)				
(1109, 1105, 278, 67, 1)	(1110, 1107, 1187, 48, 3)	(1113, 1109, 279, 67, 1)	(1136, 1133, 1054, 102, 10)				
(1151, 1147, 1035, 27, 2)	(1155, 1152, 1072, 26, 2)	(1170, 1167, 1087, 52, 5)	(1260, 1255, 1374, 55, 4)				
(1273, 1270, 1351, 26, 1)	(1282, 1279, 1373, 93, 4)	(1293, 1290, 1372, 27, 1)	(1319, 1316, 1225, 51, 5)				
(1349, 1345, 1470, 30, 1)	(1353, 1349, 1448, 46, 3)	(1370, 1366, 1465, 97, 5)	(1377, 1372, 1486, 36, 3)				
(1382, 1378, 1479, 95, 5)	(1389, 1384, 1499, 72, 5)	(1389, 1386, 1289, 108, 9)	(1393, 1389, 1489, 49, 4)				
(1428, 1425, 1336, 130, 10)	(1460, 1457, 1360, 29, 2)	(1471, 1468, 1376, 128, 11)	(1472, 1469, 1594, 37, 2)				
(1506, 1502, 1599, 41, 2)	(1514, 1508, 1651, 55, 4)	(1514, 1511, 1764, 81, 1)	(1613, 1609, 1512, 46, 3)				
(1645, 1640, 1494, 59, 7)	(1647, 1643, 1529, 103, 10)	(1654, 1648, 1785, 72, 4)	(1667, 1661, 1799, 72, 5)				
(1683, 1679, 1821, 69, 2)	(1690, 1685, 421, 83, 5)	(1694, 1689, 422, 84, 5)	(1695, 1691, 1834, 70, 2)				
(1741, 1736, 1861, 49, 3)	(1760, 1755, 1882, 47, 2)	(1773, 1768, 1897, 96, 5)	(1812, 1808, 1941, 90, 4)				
(1817, 1813, 1688, 104, 10)	(1865, 1861, 1633, 44, 5)	(1885, 1878, 235, 67, 3)	(1921, 1914, 2073, 72, 4)				
(1932, 1921, 2142, 37, 2)	(1945, 1938, 2099, 72, 5)	(1975, 1971, 2087, 72, 4)	(1982, 1977, 1840, 103, 10)				
(2173, 2166, 2346, 75, 5)	(2199, 2191, 2373, 72, 4)	(2248, 2242, 2405, 94, 6)	(2259, 2251, 2438, 72, 6)				
(2290, 2281, 1715, 63, 3)	(2456, 2445, 1837, 53, 8)	(2457, 2451, 2281, 51, 5)	(2463, 2455, 2108, 44, 1)				
(2487, 2479, 2685, 75, 5)	(2519, 2514, 2935, 81, 1)	(2560, 2555, 2709, 30, 1)	(2894, 2884, 3125, 71, 4)				
(2911, 2903, 2701, 101, 10)	(2952, 2945, 3194, 70, 3)	(3067, 3059, 3318, 69, 3)	(3514, 3505, 3760, 96, 6)				
(3602, 3593, 3344, 104, 9)	(3616, 3607, 3357, 104, 9)	(4148, 4139, 3107, 114, 5)	(4457, 4441, 4810, 73, 4)				
(4644, 4631, 4966, 48, 2)	(10499, 10471, 11235, 94, 5)	(12774, 12742, 11859, 104, 9)					

TABLE 1. List of cutout polygons.

CHAPTER 5

Gaussian Shift Radix Systems and Pethő's Loudspeaker

5.1. Introduction and definitions

Gaussian Shift Radix Systems (or GSRS for short) define Shift Radix Systems in the sense of Chapter 3 on Gaussian integers and were first introduced in [**Brunotte et al., 2011**]. Many definitions and properties carry over from Shift Radix Systems with no or only slight modifications and, as a start, we shall give a list of them. The very similar proofs will not be given and can easily be derived from the corresponding ones in Chapter 3. In the upcoming sections we will formulate a conjecture (Conjecture 5.2.2) on the full characterization of $\mathcal{G}_1^{(0)}$ (the analogue to $\mathcal{D}_1^{(0)}$) known as Pethő's Loudspeaker and prove it in parts (Theorem 5.3.1)). Furthermore we will present an even stronger conjecture which might provide a way to prove the original conjecture in the future. We will also derive some consequences under the assumption that the Loudspeaker has the conjectured shape like its perimeter and area. The proven parts of the conjecture are sufficient to identify all weakly critical and critical points of $\mathcal{G}_1^{(0)}$, which will be the subject of another section. Finally we will explain a kind of "self-similarity" in a hidden pattern of the Loudspeaker revealed by the GSRS analogue of Algorithm 1 presented in Section 4.2. Parts of the material presented in this chapter have been published in [**Weitzer, 2015b**].

Throughout the chapter we will identify $\mathbb{C}^d \simeq \mathbb{R}^{2d}$ and $\mathbb{Z}[\mathrm{i}]^d \simeq \mathbb{Z}^{2d}$ for $d \in \mathbb{N}$ if convenient.

Definition 5.1.1. [**Brunotte et al., 2011**] *For $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{C}^d$ the mapping*

$$\gamma_{\mathbf{r}} : \mathbb{Z}[\mathrm{i}]^d \to \mathbb{Z}[\mathrm{i}]^d$$
$$\mathbf{a} = (a_1, \dots, a_d) \mapsto (a_2, \dots, a_d, -\lfloor \mathbf{ra} \rfloor)$$

where $\mathbf{ra} = \sum_{i=1}^d r_i a_i$ (note that this definition slightly differs from the scalar product of complex vectors which is given by $\sum_{i=1}^d r_i \overline{a_i}$), is called the d-dimensional Gaussian Shift Radix System (GSRS for short) associated with $\mathbf{r}$ and $\mathbf{r}$ is called the parameter of $\gamma_{\mathbf{r}}$. Furthermore we define

$$\mathcal{G}_d := \left\{ \mathbf{r} \in \mathbb{C}^d \mid \forall\, \mathbf{a} \in \mathbb{Z}[\mathrm{i}]^d : \exists\, i, j \in \mathbb{N} : \gamma_{\mathbf{r}}^i(\mathbf{a}) = \gamma_{\mathbf{r}}^{i+j}(\mathbf{a}) \right\}$$
$$\mathcal{G}_d^{(0)} := \left\{ \mathbf{r} \in \mathbb{C}^d \mid \forall\, \mathbf{a} \in \mathbb{Z}[\mathrm{i}]^d : \exists\, i \in \mathbb{N} : \gamma_{\mathbf{r}}^i(\mathbf{a}) = \mathbf{0} \right\}$$

where $\gamma_{\mathbf{r}}^i(\mathbf{a})$ means i-fold application of $\gamma_{\mathbf{r}}$ to $\mathbf{a}$. Elements of $\mathcal{G}_d^{(0)}$ are said to have the finiteness property.

As with real Shift Radix Systems we can characterize $\mathcal{G}_d$ almost everywhere by the Schur-Cohn region:

Definition 5.1.2. *For $d \in \mathbb{N}$ and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{C}^d$ let*

$$R(\mathbf{r}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & \cdots & \cdots & \cdots & -r_d \end{pmatrix} \in \mathbb{C}^{d \times d}$$

Theorem 5.1.3. [**Brunotte et al., 2011**] *Let $d \in \mathbb{N}$. Then*

$$\mathcal{E}_d^{(\mathbb{C})} \subseteq \mathcal{G}_d \subseteq \overline{\mathcal{E}_d^{(\mathbb{C})}}.$$

COROLLARY 5.1.4. [**Brunotte et al., 2011**] *Let* $d \in \mathbb{N}$. *Then*

(i) $\mathcal{G}_d \subseteq \left\{\mathbf{r} \in \mathbb{C}^d \mid \rho(R(\mathbf{r})) \leq 1\right\}$

(ii) $\mathcal{G}_d \supseteq \left\{\mathbf{r} \in \mathbb{C}^d \mid \rho(R(\mathbf{r})) < 1\right\}$

(iii) $\partial\mathcal{G}_d = \left\{\mathbf{r} \in \mathbb{C}^d \mid \rho(R(\mathbf{r})) = 1\right\}$.

All notions and theorems for cycles translate to the complex case as expected:

DEFINITION 5.1.5. [**Brunotte et al., 2011**] *For* $d \in \mathbb{N}$ *let* $\mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])} := \bigcup_{n\in\mathbb{N}_0} (\mathbb{Z}[\mathrm{i}]^d)^n$ *denote the set of (d-dimensional, complex) cycles.*

For a cycle $\pi = (\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])}$ *let* $P_{\mathbb{C}}(\pi) := \left\{\mathbf{r} \in \mathbb{C}^d \mid \forall\, i \in [\![1,k]\!] : \gamma_{\mathbf{r}}(\mathbf{a}_i) = \mathbf{a}_{i\%k+1}\right\}$, *i.e. the set of those parameters* $\mathbf{r}$ *for which* π *is a cycle of the associated Gaussian Shift Radix System.* $P_{\mathbb{C}}(\pi)$ *shall be referred to as the cutout polyhedron of* π.

LEMMA 5.1.6. [**Brunotte et al., 2011**] *Let* $d \in \mathbb{N}$. *Then*

$$\mathcal{G}_d^{(0)} = \mathcal{G}_d \setminus \bigcup_{\pi \in \mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])} \setminus \{(\mathbf{0})\}} P_{\mathbb{C}}(\pi)$$

DEFINITION 5.1.7. *A set* $P \subseteq \mathbb{C}^d$ *(*$d \in \mathbb{N}$*) is called (complex, convex, d-dimensional) polyhedron iff it is the intersection of finitely many half-spaces of* $\mathbb{R}^{2d}$ *or* $\mathbb{R}^{2d}$ *itself. A polyhedron is considered nondegenerate iff it has positive and finite Lebesgue measure and degenerate otherwise. The set of all complex d-dimensional polyhedra shall be denoted by* $\mathcal{P}_d^{(\mathbb{C})}$.

LEMMA 5.1.8. [**Brunotte et al., 2011**] *Let* $d \in \mathbb{N}$ *and* $\mathbf{a} = (a_1 + \mathrm{i}b_1, \ldots, a_d + \mathrm{i}b_d), \mathbf{b} = (c_1 + \mathrm{i}d_1, \ldots, c_d + \mathrm{i}d_d) \in \mathbb{Z}[\mathrm{i}]^d$. *Then*

$$\begin{aligned}\left\{\mathbf{r} \in \mathbb{C}^d \mid \gamma_{\mathbf{r}}(\mathbf{a}) = \mathbf{b}\right\} = \{\mathbf{r} = (x_1 + \mathrm{i}y_1, \ldots, x_d + \mathrm{i}y_d) \in \mathbb{C}^d \mid{}& \forall\, i \in [\![1, d-1]\!] : b_i = a_{i+1} \\ & 0 \leq x_1 a_1 - y_1 b_1 + \ldots + x_d a_d - y_d b_d + c_d < 1 \\ & 0 \leq x_1 b_1 + y_1 a_1 + \ldots + x_d b_d + y_d a_d + d_d < 1\}.\end{aligned}$$

In particular: If $\pi \in \mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])}$ *then* $P_{\mathbb{C}}(\pi)$ *is a (possibly degenerate) convex polyhedron.*

LEMMA 5.1.9. [**Brunotte et al., 2011**] *Let* $d \in \mathbb{N}$, $\mathbf{r} \in \operatorname{int}(\mathcal{G}_d)$, $\rho \in (\rho(R(\mathbf{r})), 1)$, $\|\cdot\|_\rho$ *norm on* $\mathbb{C}^d$ *with* $\|R(\mathbf{r})\mathbf{a}\|_\rho \leq \rho \|\mathbf{a}\|_\rho$ *for all* $\mathbf{a} \in \mathbb{C}^d$ *(cf. proof of Theorem 3.4.2), and* $\mathbf{a} \in \mathbb{Z}[\mathrm{i}]^d$ *such that* $\gamma_{\mathbf{r}}^k(\mathbf{a}) = \mathbf{a}$ *for some* $k \in \mathbb{N}$. *Then*

$$\|\mathbf{a}\|_\rho \leq \frac{\sqrt{2}\,\|(0,\ldots,0,1)\|_\rho}{1-\rho}.$$

In particular: $\left\{\pi \in \mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])} \mid \mathbf{r} \in P_{\mathbb{C}}(\pi)\right\}$ *is a finite set.*

THEOREM 5.1.10. (**Weitzer**) *Let* $d \in \mathbb{N}$ *and* $\mathbf{r} \in \operatorname{int}(\mathcal{G}_d)$ *Then there is an open neighborhood* B *of* $\mathbf{r}$ *for which*

$$\left\{\pi \in \mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])} \mid B \cap P_{\mathbb{C}}(\pi) \neq \emptyset\right\}$$

is a finite set.

In particular: If $M \subseteq \operatorname{int}(\mathcal{G}_d)$ *with* $\operatorname{dist}(M, \partial\mathcal{G}_d) > 0$ *then* $\left\{\pi \in \mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])} \mid M \cap P_{\mathbb{C}}(\pi) \neq \emptyset\right\}$ *is a finite set.*

PROOF. Cf. proof of Theorem 3.5.6. □

Sets of witnesses can also be defined after slight adaptation:

DEFINITION 5.1.11. [**Brunotte et al., 2011**] *A set* $V \subseteq \mathbb{Z}[\mathrm{i}]^d$ *is called a set of witnesses for* $\mathbf{r} \in \mathbb{C}^d$ *(*$d \in \mathbb{N}$*) iff it is stable under* $\gamma_{\mathbf{r}}^{(1)} := \gamma_{\mathbf{r}}$, $\gamma_{\mathbf{r}}^{(2)} := -\gamma_{\mathbf{r}} \circ (-\mathrm{id}_{Z\mathrm{i}^d})$, $\gamma_{\mathbf{r}}^{(3)} := \operatorname{conj}_d \circ \gamma_{\mathbf{r}} \circ \operatorname{conj}_d$, *and* $\gamma_{\mathbf{r}}^{(4)} := -\operatorname{conj}_d \circ \gamma_{\mathbf{r}} \circ (-\operatorname{conj}_d)$ *(where* conj_d *is the function on* $\mathbb{C}^d$ *which replaces every entry of the input vector by its complex conjugate) and contains a generating set of the group* $(\mathbb{Z}[\mathrm{i}]^d, +)$ *which is closed under taking inverses.*

Lemma 5.1.12. [**Brunotte et al., 2011**] *Let* $d \in \mathbb{N}$, $\mathbf{r} \in \mathbb{C}^d$, *and* $V \subseteq \mathbb{Z}[\mathrm{i}]^d$ *a set of witnesses for* $\mathbf{r}$. *Then*

$$\mathbf{r} \in \mathcal{G}_d^{(0)} \Leftrightarrow \forall\, \mathbf{a} \in V : \exists\, n \in \mathbb{N} : \gamma_{\mathbf{r}}^n(\mathbf{a}) = \mathbf{0}.$$

Definition 5.1.13. [**Brunotte et al., 2011**] *For* $d \in \mathbb{N}$ *and* $\mathbf{r} \in \mathbb{C}^d$ *let*

$$\begin{aligned} V_{\mathbf{r},0} &:= \{(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1), (\pm \mathrm{i}, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm \mathrm{i})\} \\ \forall\, n \in \mathbb{N} : V_{\mathbf{r},n} &:= V_{\mathbf{r},n-1} \cup \gamma_{\mathbf{r}}^{(1)}(V_{\mathbf{r},n-1}) \cup \ldots \cup \gamma_{\mathbf{r}}^{(4)}(V_{\mathbf{r},n-1}) \\ V_{\mathbf{r}} &:= \bigcup_{n \in \mathbb{N}_0} V_{\mathbf{r},n} \end{aligned}$$

$V_{\mathbf{r}}$ *shall be referred to as the set of witnesses associated with* $\mathbf{r}$.

Theorem 5.1.14. [**Brunotte et al., 2011**] *Let* $d \in \mathbb{N}$ *and* $\mathbf{r} \in \mathrm{int}\,(\mathcal{G}_d)$. *Then* $V_{\mathbf{r}}$ *is a finite set of witnesses for* $\mathbf{r}$.

Definition 5.1.15. *For* $d \in \mathbb{N}$ *and* $M \subseteq \mathbb{C}^d$ *let*

$$\begin{aligned} \overline{\gamma}_M : \mathcal{P}(\mathbb{Z}[\mathrm{i}]^d) &\to \mathcal{P}(\mathbb{Z}[\mathrm{i}]^d). \\ V &\mapsto \{\gamma_{\mathbf{r}}(\mathbf{a}) \mid \mathbf{r} \in M \wedge \mathbf{a} \in V\} \end{aligned}$$

Definition 5.1.16. [**Brunotte et al., 2011**] *A set* $V \subseteq \mathbb{Z}[\mathrm{i}]^d$ *is called a set of witnesses for* $M \subseteq \mathbb{C}^d$ *(*$d \in \mathbb{N}$*) iff it is stable under* $\overline{\gamma}_M^{(1)} := \overline{\gamma}_M$, $\overline{\gamma}_M^{(2)} := -\overline{\gamma}_M \circ (-\mathrm{id}_{Z i^d})$, $\overline{\gamma}_M^{(3)} := \mathrm{conj}_d \circ \overline{\gamma}_M \circ \mathrm{conj}_d$, *and* $\overline{\gamma}_M^{(4)} := -\mathrm{conj}_d \circ \overline{\gamma}_M \circ (-\mathrm{conj}_d)$ *and contains a generating set of the group* $(\mathbb{Z}[\mathrm{i}]^d, +)$ *which is closed under taking inverses.*

Lemma 5.1.17. [**Brunotte et al., 2011**] *Let* $d \in \mathbb{N}$, $M \subseteq \mathbb{C}^d$, *and* $V \subseteq \mathbb{Z}[\mathrm{i}]^d$ *a set of witnesses for* M. *Then*

$$M \cap \mathcal{G}_d^{(0)} = M \setminus \bigcup_{\substack{\pi = (\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])} \\ \{\mathbf{a}_1, \ldots, \mathbf{a}_k\} \subseteq V}} P_{\mathbb{C}}(\pi)$$

Definition 5.1.18. [**Brunotte et al., 2011**] *For* $d \in \mathbb{N}$ *and* $M \subseteq \mathbb{C}^d$ *let*

$$\begin{aligned} V_{M,0} &:= \{(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1), (\pm \mathrm{i}, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm \mathrm{i})\} \\ \forall\, n \in \mathbb{N} : V_{M,n} &:= V_{M,n-1} \cup \overline{\gamma}_M^{(1)}(V_{M,n-1}) \cup \ldots \cup \overline{\gamma}_M^{(4)}(V_{M,n-1}) \\ V_M &:= \bigcup_{n \in \mathbb{N}_0} V_{M,n} \end{aligned}$$

V_M *shall be referred to as the set of witnesses associated with* M.

Theorem 5.1.19. (**Weitzer**) *Let* $d \in \mathbb{N}$ *and* $M \subseteq \mathrm{int}\,(\mathcal{G}_d)$ *with* $\mathrm{dist}\,(M, \partial\mathcal{G}_d) > 0$. *Then there is a* $k \in \mathbb{N}$ *and there are* $B_1, \ldots, B_k \subseteq \mathbb{C}^d$ *such that* $M = \bigcup_{i=1}^k B_i$ *and* V_{B_i} *is a finite set of witnesses for* B_i *for all* $i \in [\![1, k]\!]$.

Proof. Cf. proof of Theorem 3.6.9. □

The definition of critical points is completely analogous to the real case. Note that, again, only the first three types of points are defined in the cited source.

Definition 5.1.20. Cf. [**Brunotte et al., 2011**] *Let* $d \in \mathbb{N}$ *and* $\mathbf{r} \in \mathbb{C}^d$.

- $\mathbf{r}$ *is called a regular point (for* $\mathcal{G}_d^{(0)}$*) iff there exists an open neighborhood of* $\mathbf{r}$ *which intersects with only finitely many cutout polyhedra.*
- $\mathbf{r}$ *is called a weakly critical point (for* $\mathcal{G}_d^{(0)}$*) iff any open neighborhood of* $\mathbf{r}$ *intersects with infinitely many cutout polyhedra.*
- $\mathbf{r}$ *is called a critical point (for* $\mathcal{G}_d^{(0)}$*) iff for every open neighborhood* B *of* $\mathbf{r}$ *the set* $B \setminus \mathcal{G}_d^{(0)}$ *cannot be covered by finitely many cutout polyhedra.*
- $\mathbf{r}$ *is called a strongly critical point (for* $\mathcal{G}_d^{(0)}$*) iff for every open neighborhood* B *of* $\mathbf{r}$ *the set* $B \setminus \mathcal{G}_d^{(0)}$ *cannot be covered by finitely many polyhedra.*

Using the following results (cf. proofs of Lemma 4.1.3, Lemma 4.1.4, and Lemma 4.1.5) Algorithm 1 and Algorithm 2 from Chapter 4 can easily be adapted for Gaussian Shift Radix Systems. The analogue of Algorithm 1 will still terminate for all inputs and the analogue of Algorithm 2 is based on the slightly modified equivalence relation given in Definition 5.1.25.

Definition 5.1.21. [**Weitzer, 2015b**] *For $d \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{C}^d$, $\Pi_{\mathbf{r}}$ - the graph of witnesses associated with $\mathbf{r}$ - denotes the edge-colored multidigraph with vertex set $V_{\mathbf{r}}$ and an edge of color i from a vertex $\mathbf{a}$ to a vertex $\mathbf{b}$ iff $\gamma_{\mathbf{r}}^{(i)}(\mathbf{a}) = \mathbf{b}$ for $i \in [\![1,4]\!]$.*

If E_i is the set of all edges (ordered pairs) of color i for $i \in [\![1,4]\!]$ then the graph $\Pi_{\mathbf{r}}$ is completely characterized by the 4-tuple $(E_1, E_2, E_3, E_4) \in \mathcal{P}((\mathbb{Z}[\mathrm{i}]^d)^2)^4$ (as there are no isolated vertices) and thus the graph and the pair can be identified. For any such graph $\Pi = (E_1, E_2, E_3, E_4) \in \mathcal{P}((\mathbb{Z}[\mathrm{i}]^d)^2)^4$ let $P_{\mathbb{C}}(\Pi) := \left\{\mathbf{r} \in \mathbb{C}^d \mid \forall\, i \in [\![1,4]\!] : \forall\,(\mathbf{a},\mathbf{b}) \in E_i : \gamma_{\mathbf{r}}^{(i)}(\mathbf{a}) = \mathbf{b}\right\}$.

Lemma 5.1.22. [**Weitzer, 2015b**] *Let $d \in \mathbb{N}$, $\mathbf{a} = (a_1, \ldots, a_d), \mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{Z}[\mathrm{i}]^d$, and $\mathbf{r} \in \mathbb{C}^d$. Also let $\mathbf{z}_x := (\Re(z_1), \ldots, \Re(z_d))$ and $\mathbf{z}_y := (\Im(z_1), \ldots, \Im(z_d))$ for all $\mathbf{z} \in \mathbb{C}^d$. Then*

$$\text{(i)}\quad \left\{\mathbf{r} \in \mathbb{C}^d \mid \gamma_{\mathbf{r}}^{(1)}(\mathbf{a}) = \mathbf{b}\right\} = \{\mathbf{r} \in \mathbb{C}^d \mid \forall\, i \in [\![1,d-1]\!] : b_i = a_{i+1} \\ 0 \le \mathbf{r}_x\mathbf{a}_x - \mathbf{r}_y\mathbf{a}_y + \Re(b_d) < 1 \\ 0 \le \mathbf{r}_x\mathbf{a}_y + \mathbf{r}_y\mathbf{a}_x + \Im(b_d) < 1$$

$$\text{(ii)}\quad \left\{\mathbf{r} \in \mathbb{C}^d \mid \gamma_{\mathbf{r}}^{(2)}(\mathbf{a}) = \mathbf{b}\right\} = \{\mathbf{r} \in \mathbb{C}^d \mid \forall\, i \in [\![1,d-1]\!] : b_i = a_{i+1} \\ 0 \le -\mathbf{r}_x\mathbf{a}_x + \mathbf{r}_y\mathbf{a}_y - \Re(b_d) < 1 \\ 0 \le -\mathbf{r}_x\mathbf{a}_y - \mathbf{r}_y\mathbf{a}_x - \Im(b_d) < 1$$

$$\text{(iii)}\quad \left\{\mathbf{r} \in \mathbb{C}^d \mid \gamma_{\mathbf{r}}^{(3)}(\mathbf{a}) = \mathbf{b}\right\} = \{\mathbf{r} \in \mathbb{C}^d \mid \forall\, i \in [\![1,d-1]\!] : b_i = a_{i+1} \\ 0 \le \mathbf{r}_x\mathbf{a}_x + \mathbf{r}_y\mathbf{a}_y + \Re(b_d) < 1 \\ 0 \le -\mathbf{r}_x\mathbf{a}_y + \mathbf{r}_y\mathbf{a}_x - \Im(b_d) < 1$$

$$\text{(iv)}\quad \left\{\mathbf{r} \in \mathbb{C}^d \mid \gamma_{\mathbf{r}}^{(4)}(\mathbf{a}) = \mathbf{b}\right\} = \{\mathbf{r} \in \mathbb{C}^d \mid \forall\, i \in [\![1,d-1]\!] : b_i = a_{i+1} \\ 0 \le -\mathbf{r}_x\mathbf{a}_x - \mathbf{r}_y\mathbf{a}_y - \Re(b_d) < 1 \\ 0 \le \mathbf{r}_x\mathbf{a}_y - \mathbf{r}_y\mathbf{a}_x + \Im(b_d) < 1$$

$$\text{(v)}\quad \left\{\mathbf{s} \in \mathbb{C}^d \mid \forall\, i \in [\![1,4]\!] \gamma_{\mathbf{s}}^{(i)}(\mathbf{a}) = \gamma_{\mathbf{r}}^{(i)}(\mathbf{a})\right\} = \{\mathbf{s} \in \mathbb{C}^d \mid \\
\mathbf{s}_x\mathbf{a}_x + \mathbf{s}_y\mathbf{a}_y - \lfloor \mathbf{r}_x\mathbf{a}_x + \mathbf{r}_y\mathbf{a}_y \rfloor \ge 0 \wedge \mathbf{s}_x\mathbf{a}_x + \mathbf{s}_y\mathbf{a}_y + \lfloor -\mathbf{r}_x\mathbf{a}_x - \mathbf{r}_y\mathbf{a}_y \rfloor + 1 > 0 \\
\mathbf{s}_x\mathbf{a}_x - \mathbf{s}_y\mathbf{a}_y - \lfloor \mathbf{r}_x\mathbf{a}_x - \mathbf{r}_y\mathbf{a}_y \rfloor \ge 0 \wedge \mathbf{s}_x\mathbf{a}_x - \mathbf{s}_y\mathbf{a}_y + \lfloor -\mathbf{r}_x\mathbf{a}_x + \mathbf{r}_y\mathbf{a}_y \rfloor + 1 > 0 \\
-\mathbf{s}_x\mathbf{a}_x + \mathbf{s}_y\mathbf{a}_y - \lfloor -\mathbf{r}_x\mathbf{a}_x + \mathbf{r}_y\mathbf{a}_y \rfloor \ge 0 \wedge -\mathbf{s}_x\mathbf{a}_x + \mathbf{s}_y\mathbf{a}_y + \lfloor \mathbf{r}_x\mathbf{a}_x - \mathbf{r}_y\mathbf{a}_y \rfloor + 1 > 0 \\
-\mathbf{s}_x\mathbf{a}_x - \mathbf{s}_y\mathbf{a}_y - \lfloor -\mathbf{r}_x\mathbf{a}_x - \mathbf{r}_y\mathbf{a}_y \rfloor \ge 0 \wedge -\mathbf{s}_x\mathbf{a}_x - \mathbf{s}_y\mathbf{a}_y + \lfloor \mathbf{r}_x\mathbf{a}_x + \mathbf{r}_y\mathbf{a}_y \rfloor + 1 > 0 \\
\mathbf{s}_x\mathbf{a}_y + \mathbf{s}_y\mathbf{a}_x - \lfloor \mathbf{r}_x\mathbf{a}_y + \mathbf{r}_y\mathbf{a}_x \rfloor \ge 0 \wedge \mathbf{s}_x\mathbf{a}_y + \mathbf{s}_y\mathbf{a}_x + \lfloor -\mathbf{r}_x\mathbf{a}_y - \mathbf{r}_y\mathbf{a}_x \rfloor + 1 > 0 \\
\mathbf{s}_x\mathbf{a}_y - \mathbf{s}_y\mathbf{a}_x - \lfloor \mathbf{r}_x\mathbf{a}_y - \mathbf{r}_y\mathbf{a}_x \rfloor \ge 0 \wedge \mathbf{s}_x\mathbf{a}_y - \mathbf{s}_y\mathbf{a}_x + \lfloor -\mathbf{r}_x\mathbf{a}_y + \mathbf{r}_y\mathbf{a}_x \rfloor + 1 > 0 \\
-\mathbf{s}_x\mathbf{a}_y + \mathbf{s}_y\mathbf{a}_x - \lfloor -\mathbf{r}_x\mathbf{a}_y + \mathbf{r}_y\mathbf{a}_x \rfloor \ge 0 \wedge -\mathbf{s}_x\mathbf{a}_y + \mathbf{s}_y\mathbf{a}_x + \lfloor \mathbf{r}_x\mathbf{a}_y - \mathbf{r}_y\mathbf{a}_x \rfloor + 1 > 0 \\
-\mathbf{s}_x\mathbf{a}_y - \mathbf{s}_y\mathbf{a}_x - \lfloor -\mathbf{r}_x\mathbf{a}_y - \mathbf{r}_y\mathbf{a}_x \rfloor \ge 0 \wedge -\mathbf{s}_x\mathbf{a}_y - \mathbf{s}_y\mathbf{a}_x + \lfloor \mathbf{r}_x\mathbf{a}_y + \mathbf{r}_y\mathbf{a}_x \rfloor + 1 > 0\}.$$

Lemma 5.1.23. [**Weitzer, 2015b**] *Let $\mathbf{r} \in \operatorname{int}(\mathcal{G}_d)$. Then $P_{\mathbb{C}}(\Pi_{\mathbf{r}})$ is the intersection of a nondegenerate, open polyhedron and an affine subspace of $\mathbb{R}^{2d}$.*

Lemma 5.1.24. [**Weitzer, 2015b**] *Let $d \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{C}^d$. Then*

(i) $\mathbf{r} \in P_{\mathbb{C}}(\Pi_{\mathbf{r}})$, *and* $\mathbf{r} \in \mathcal{G}_d^{(0)} \Leftrightarrow P_{\mathbb{C}}(\Pi_{\mathbf{r}}) \subseteq \mathcal{G}_d^{(0)}$

(ii) $\mathcal{G}_d^{(0)} = \bigcup \left\{P_{\mathbb{C}}(\Pi_{\mathbf{r}}) \mid \mathbf{r} \in \mathcal{G}_d^{(0)}\right\}$ *and this union is disjoint.*

Definition 5.1.25. [**Weitzer, 2015b**]

$$\sim := \left\{(\mathbf{r}_1, \mathbf{r}_2) \in (\mathbb{C}^d)^2 \mid \forall\, i \in [\![1,4]\!] : \forall\, \mathbf{a} \in V : \gamma_{\mathbf{r}_1}^{(i)}(\mathbf{a}) = \gamma_{\mathbf{r}_2}^{(i)}(\mathbf{a})\right\}.$$

5.2. Pethő's Loudspeaker

In this section we will formulate a conjecture on the characterization of $\mathcal{G}_1^{(0)}$. In honor of Attila Pethő and because of its shape, $\mathcal{G}_1^{(0)}$ is known as Pethő's Loudspeaker (cf. Figure 1). Throughout the section we will identify $\mathbb{C}^1 \simeq \mathbb{R}^2$ and $\mathbb{Z}[\mathrm{i}]^1 \simeq \mathbb{Z}^2$.

Definition 5.2.1. [**Weitzer, 2015b**]

$$
\begin{array}{llll}
\textit{Let } P_0(1) & := (1,0), \quad P_0(2) := \left(\frac{22}{23}, \frac{4}{23}\right), \quad P_0(3) := \left(\frac{26}{27}, \frac{4}{27}\right) & & \\
P_1(n) & := \left(1 - \frac{2}{n^2-2}, \frac{n}{n^2-2}\right), & & n \in \mathbb{Z} \\
P_2(n) & := \left(1 - \frac{1}{n^2-n-1}, \frac{n-1}{n^2-n-1}\right), & & n \in \mathbb{Z} \\
P_3(n) & := \left(1 - \frac{1}{n^2-n}, \frac{n-1}{n^2-n}\right), & & n \in \mathbb{Z} \setminus \{0,1\} \\
P_4(n) & := \left(1 - \frac{1}{n^2}, \frac{n}{n^2}\right), & & n \in \mathbb{Z} \setminus \{0\} \\
P_5(n) & := \left(1 - \frac{1}{n^2+1}, \frac{n}{n^2+1}\right), & & n \in \mathbb{Z} \\
P_6(n) & := \left(1 - \frac{1}{n^2+n+1}, \frac{n+1}{n^2+n+1}\right), & & n \in \mathbb{Z} \\
P_7(n) & := \left(1 - \frac{1}{n^2+n+2}, \frac{n+1}{n^2+n+2}\right), & & n \in \mathbb{Z} \\
P_8(n) & := \left(1 - \frac{1}{n^2+2}, \frac{n}{n^2+2}\right), & & n \in \mathbb{Z} \\
P_9(n) & := \left(1 - \frac{1}{n^2+3}, \frac{n}{n^2+3}\right), & & n \in \mathbb{Z} \\
P_{10}(n) & := \left(1 - \frac{2}{n^2+n+6}, \frac{n+1}{n^2+n+6}\right), & & n \in \mathbb{Z}
\end{array}
$$

and let $\mathcal{G}_C$ denote the union of the region bounded by the following infinite polygonal chain and the same region reflected at the real axis. The boundary of $\mathcal{G}_C$ shall also be as given below where a solid line between two points indicates belonging of the corresponding line segment and an overline over a point indicates belonging of the corresponding vertex to $\mathcal{G}_C$.

$$
\begin{array}{cccccccccccccccccccc}
 & & & & & & P_0(1) & \text{—} & \overline{P_5(0)} & \text{—} & P_6(0) & \cdots & & & & & & & & \\
 & & & & & & & & \overline{P_5(1)} & \text{—} & P_6(1) & \text{—} & \overline{P_7(0)} & \cdots & P_7(1) & \cdots & & & & \\
 & & & & & & & & \overline{P_5(2)} & \text{—} & \overline{P_6(2)} & \cdots & P_7(2) & \cdots & \overline{P_8(2)} & \cdots & & & & \\
 & & & & & & P_4(3) & \cdots & \overline{P_5(3)} & \text{—} & P_6(3) & \cdots & P_7(3) & \cdots & \overline{P_8(3)} & \text{—} & & & & \\
 & & & & \overline{P_3(4)} & \cdots & P_4(4) & \cdots & \overline{P_5(4)} & \text{—} & P_6(4) & \cdots & P_7(4) & \cdots & \overline{P_8(4)} & \text{—} & & & & \\
 & & & & \overline{P_3(5)} & \cdots & P_4(5) & \cdots & \overline{P_5(5)} & \text{—} & P_6(5) & \cdots & P_7(5) & \cdots & \overline{P_8(5)} & \text{—} & \overline{P_9(5)} & \text{—} & & \\
\overline{P_0(2)} & \text{—} & \overline{P_2(6)} & \text{—} & \overline{P_3(6)} & \cdots & P_4(6) & \cdots & \overline{P_5(6)} & \text{—} & P_6(6) & \cdots & P_7(6) & \cdots & \overline{P_8(6)} & \text{—} & \overline{P_9(6)} & \text{—} & & \\
\overline{P_0(3)} & \text{—} & \overline{P_2(7)} & \text{—} & \overline{P_3(7)} & \cdots & P_4(7) & \cdots & \overline{P_5(7)} & \text{—} & P_6(7) & \cdots & P_7(7) & \cdots & \overline{P_8(7)} & \text{—} & \overline{P_9(7)} & \text{—} & & \\
\overline{P_1(8)} & \text{—} & \overline{P_2(8)} & \text{—} & \overline{P_3(8)} & \cdots & P_4(8) & \cdots & \overline{P_5(8)} & \text{—} & P_6(8) & \cdots & P_7(8) & \cdots & \overline{P_8(8)} & \text{—} & \overline{P_9(8)} & \text{—} & \overline{P_{10}(8)} & \cdots \\
 & & & & & & & & & & \vdots & & & & & & & & & \\
\overline{P_1(n)} & \text{—} & \overline{P_2(n)} & \text{—} & \overline{P_3(n)} & \cdots & P_4(n) & \cdots & \overline{P_5(n)} & \text{—} & P_6(n) & \cdots & P_7(n) & \cdots & \overline{P_8(n)} & \text{—} & \overline{P_9(n)} & \text{—} & \overline{P_{10}(n)} & \cdots \\
 & & & & & & & & & & \vdots & & & & & & & & &
\end{array}
$$

Conjecture 5.2.2. [**Weitzer, 2015b**] *If $\mathcal{G}_C$ is as defined above then $\mathcal{G}_1^{(0)} = \mathcal{G}_C$.*

Note that for all $i \in [\![1, 10]\!] : \lim_{n\to\infty} P_i(n) = P_0(1)$. Figure 1 shows the part of $\mathcal{G}_C$ which lies in the first quadrant of the unit disk and a magnification of the part where it gets "regular". It can be seen that ultimately the boundary of $\mathcal{G}_C$ consists of a sequence of pikes which have ten vertices each. For $n \in \mathbb{N}$ pike n shall refer to the pike which contains the vertex $P_5(n)$. The four infinite families of cutout polygons (yellow, orange, red, and blue) and the six additional cutout polygons (purple) form a chain from I to 1 in the complex plane and provide another way to define $\mathcal{G}_C$. Indeed, $\mathcal{G}_C$ is exactly that part of the first quadrant of the unit disk which lies left of the chain. Furthermore none of the elements of the chain are redundant. The corresponding cycles are given in the following definition and, by Lemma 3.5.8, have shapes as given in the subsequent theorem.

Definition 5.2.3. [**Weitzer, 2015b**] *Let*

$$
\begin{array}{lll}
C_0(1) & := & ((-2,0),(2,2),(0,-2),(-1,2),(2,0),(-1,-1),(0,2),(2,-1)) \\
C_0(2) & := & ((-1,-1),(1,2),(1,-2),(-1,1),(2,0)) \\
C_0(3) & := & ((-3,0),(3,2),(-1,-2),(1,3),(1,-3),(-2,3),(3,-1)) \\
C_0(4) & := & ((-4,-1),(4,3),(-2,-4),(1,5),(1,-4),(-2,4),(4,-3),(-4,2),(5,0)) \\
C_0(5) & := & ((-4,-1),(4,3),(-2,-3),(1,4),(1,-4),(-2,4),(4,-3),(-4,2),(5,0)) \\
C_0(6) & := & ((-4,0),(4,2),(-3,-3),(2,4),(0,-4),(-1,4),(3,-3),(-3,2),(4,-1))
\end{array}
$$

$$
\begin{aligned}
C_1(n) := (\ & (-n+k,-k),(n-k,k+1) && 1 \le k \le n-1\\
& (k,-n+k),(-k,n-k) && 1 \le k \le n-1\\
& (n,0)\\
),\ & n \ge 1
\end{aligned}
$$

$$
\begin{aligned}
C_2(n) := (\ & (-n,1),(n,1)\\
& (-n+k,-k-1),(n-k,k+2) && 1 \le k \le n-2\\
& (0,-n),(-1,n)\\
& (k+2,-n+k),(-k-2,n-k) && 1 \le k \le n-3\\
& (n,-2)\\
),\ & n \ge 6
\end{aligned}
$$

$$
\begin{aligned}
C_3(n) := (\ & (-n+k-1,-k+1),(n-k+1,k) && 1 \le k \le n\\
& (k,-n+k-1),(-k,n-k+1) && 1 \le k \le n-1\\
& (n,-1)\\
),\ & n \ge 1
\end{aligned}
$$

$$
\begin{aligned}
C_4(n) := (\ & (-n,0),(n,2)\\
& (-n+k,-k-2),(n-k,k+3) && 1 \le k \le n-4\\
& (-2,-n+1),(1,n),(1,-n),(-2,n)\\
& (k+3,-n+k),(-k-3,n-k) && 1 \le k \le n-5\\
& (n-1,-4),(-n+1,3),(n,-1)\\
),\ & n \ge 5
\end{aligned}
$$

THEOREM 5.2.4. *The cutout polygons corresponding to the cycles of the previous definition have the following vertices and boundaries (cf. Theorem 3.5.10 for notation):*

$$
\begin{aligned}
C_0(n):\ n=1:&\quad \overline{(\tfrac{2}{3},\tfrac{2}{3})}\\
n=2:&\quad (\tfrac{4}{5},\tfrac{3}{5})\text{- -}(\tfrac{2}{3},\tfrac{2}{3})\text{—}\overline{(\tfrac{3}{4},\tfrac{1}{2})}\text{—}\\
n=3:&\quad (\tfrac{12}{13},\tfrac{5}{13})\text{- -}(\tfrac{6}{7},\tfrac{3}{7})\text{—}\overline{(\tfrac{7}{8},\tfrac{3}{8})}\text{—}(\tfrac{10}{11},\tfrac{4}{11})\text{- -}\\
n=4:&\quad (\tfrac{16}{17},\tfrac{5}{17})\text{- -}(\tfrac{13}{14},\tfrac{5}{14})\text{—}\overline{(\tfrac{10}{11},\tfrac{7}{22})}\text{—}\\
n=5:&\quad (\tfrac{16}{17},\tfrac{5}{17})\text{- -}(\tfrac{12}{13},\tfrac{4}{13})\text{—}(\tfrac{13}{14},\tfrac{2}{7})\text{—}\\
n=6:&\quad (1,\tfrac{1}{3})\text{—}\overline{(\tfrac{13}{14},\tfrac{2}{7})}\text{—}\overline{(\tfrac{17}{18},\tfrac{5}{18})}\text{—}
\end{aligned}
$$

$$
\begin{aligned}
C_1(n):\ n=1:&\quad (0,0)\text{- -}(0,1)\text{- -}(-1,1)\text{—}\overline{(-1,0)}\text{—}\\
n=2:&\quad (\tfrac{3}{4},\tfrac{1}{2})\text{- -}(\tfrac{2}{3},\tfrac{2}{3})\text{- -}(\tfrac{1}{2},\tfrac{1}{2})\text{—}\\
n=3:&\quad (\tfrac{8}{9},\tfrac{1}{3})\text{- -}(\tfrac{7}{8},\tfrac{3}{8})\text{—}(\tfrac{5}{6},\tfrac{1}{3})\text{—}\\
n\ge 4:&\quad (1-\tfrac{1}{n^2},\tfrac{n}{n^2})\text{- -}(1-\tfrac{1}{n^2-n+2},\tfrac{n}{n^2-n+2})\text{—}(1-\tfrac{1}{n^2-2n+3},\tfrac{n-1}{n^2-2n+3})\text{- -}\\
&\quad (1-\tfrac{1}{n^2-n},\tfrac{n-1}{n^2-n})\text{—}
\end{aligned}
$$

$$
\begin{aligned}
C_2(n):\ n=6:&\quad (\tfrac{34}{35},\tfrac{6}{35})\text{—}(\tfrac{32}{33},\tfrac{2}{11})\text{- -}(\tfrac{22}{23},\tfrac{4}{23})\text{- -}\\
n=7:&\quad (\tfrac{47}{48},\tfrac{7}{48})\text{—}(\tfrac{44}{45},\tfrac{7}{45})\text{- -}(\tfrac{26}{27},\tfrac{4}{27})\text{- -}\\
n=8:&\quad (\tfrac{62}{63},\tfrac{8}{63})\text{—}(\tfrac{58}{59},\tfrac{8}{59})\text{- -}(\tfrac{30}{31},\tfrac{4}{31})\text{- -}\\
n\ge 9:&\quad (1-\tfrac{1}{n^2-1},\tfrac{n}{n^2-1})\text{—}(1-\tfrac{1}{n^2-n+3},\tfrac{n}{n^2-n+3})\text{- -}(1-\tfrac{2}{n^2-n+6},\tfrac{n}{n^2-n+6})\text{—}\\
&\quad (1-\tfrac{2}{n^2-2},\tfrac{n}{n^2-2})\text{- -}
\end{aligned}
$$

$$
\begin{aligned}
C_3(n):\ n=1:&\quad (1,\tfrac{1}{n})\text{—}(1-\tfrac{1}{n^2},\tfrac{n}{n^2})\text{—}(1-\tfrac{1}{n^2+1},\tfrac{n}{n^2+1})\text{- -}\\
n=2:&\quad (1,\tfrac{1}{n})\text{—}(1-\tfrac{1}{n^2},\tfrac{n}{n^2})\text{—}(1-\tfrac{1}{n^2+1},\tfrac{n}{n^2+1})\text{- -}\\
n\ge 3:&\quad (1,\tfrac{1}{n})\text{—}(1-\tfrac{1}{n^2},\tfrac{n}{n^2})\text{—}\overline{(1-\tfrac{1}{n^2+1},\tfrac{n}{n^2+1})}\text{- -}
\end{aligned}
$$

$$
\begin{aligned}
C_4(n):\ n=5:&\quad (1,\tfrac{1}{4})\text{—}(\tfrac{24}{25},\tfrac{7}{25})\text{- -}(\tfrac{18}{19},\tfrac{5}{19})\text{—}\overline{(\tfrac{21}{22},\tfrac{5}{22})}\text{—}\\
n=6:&\quad (1,\tfrac{1}{5})\text{—}(\tfrac{36}{37},\tfrac{8}{37})\text{- -}(\tfrac{28}{29},\tfrac{6}{29})\text{—}\overline{(\tfrac{31}{32},\tfrac{3}{16})}\text{—}\\
n=7:&\quad (1,\tfrac{1}{6})\text{—}(\tfrac{44}{45},\tfrac{8}{45})\text{- -}(\tfrac{39}{40},\tfrac{7}{40})\text{—}\overline{(\tfrac{43}{44},\tfrac{7}{44})}\text{—}\\
n=8:&\quad (1,\tfrac{1}{n-1})\text{—}(1-\tfrac{1}{n^2-n-3},\tfrac{n}{n^2-n-3})\text{—}\overline{(1-\tfrac{1}{n^2-n+2},\tfrac{n}{n^2-n+2})}\text{—}\\
n\ge 9:&\quad (1,\tfrac{1}{n-1})\text{—}\overline{(1-\tfrac{1}{n^2-n-3},\tfrac{n}{n^2-n-3})}\text{—}\overline{(1-\tfrac{1}{n^2-n+2},\tfrac{n}{n^2-n+2})}\text{—}
\end{aligned}
$$

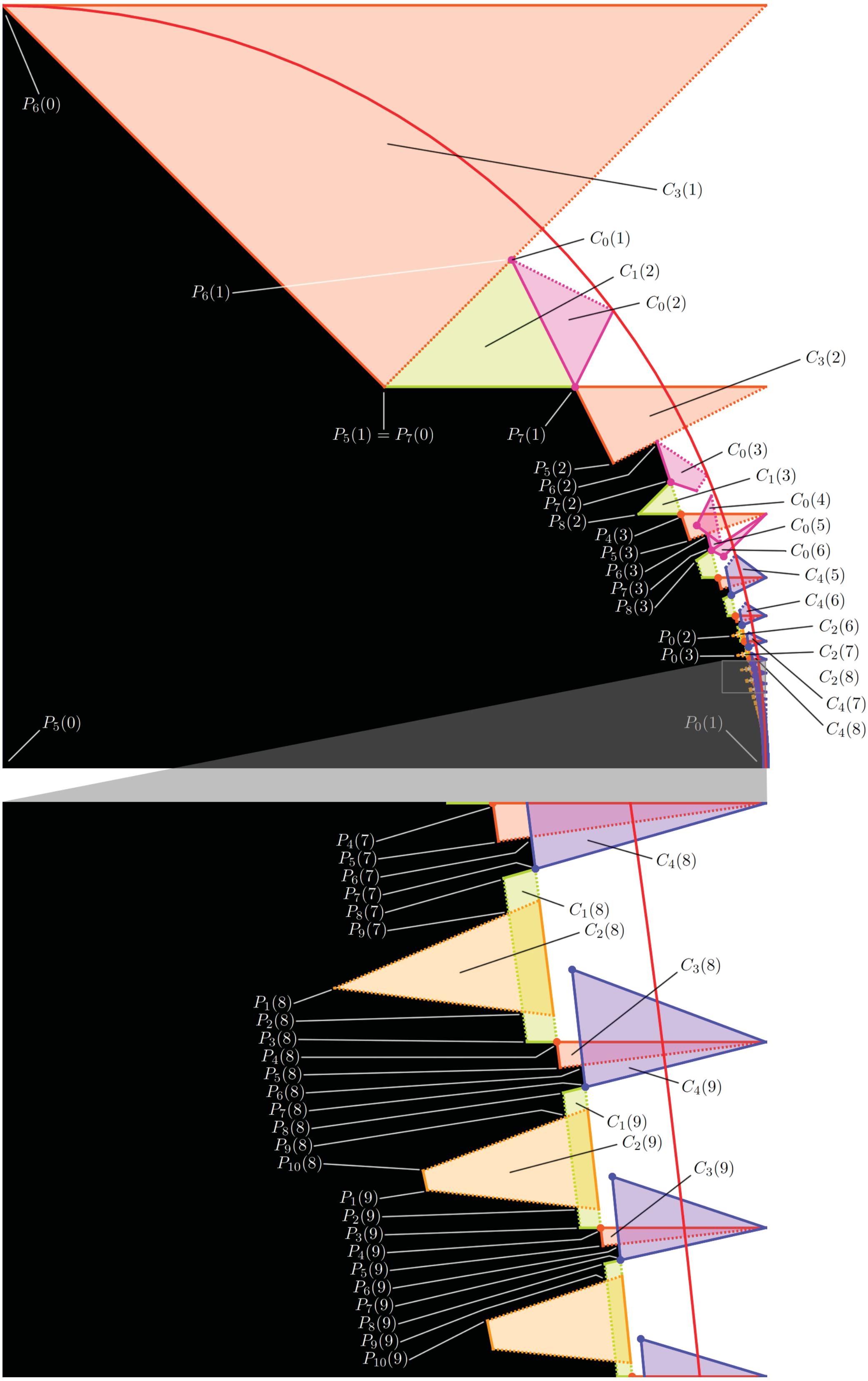


FIGURE 1. $\mathcal{G}_C$ in the first quadrant of the unit disk.

One might ask why we are only interested in the first quadrant of the unit disk. It follows from Theorem 5.1.3 that $\mathcal{G}_1^{(0)}$ is contained in the closed unit disk and from the cutout polygon of the cycle $C_1(1)$ it follows that it has empty intersection with the second quadrant. Furthermore it is symmetric with respect to the real axis according to the following lemma. Altogether it obviously suffices to consider only the first quadrant of the unit disk.

Lemma 5.2.5. [**Brunotte et al., 2011**] *Let* $d \in \mathbb{N}$, $\mathbf{r} \in \mathbb{C}^d$, $\mathbf{a}, \mathbf{b} \in \mathbb{Z}[\mathrm{i}]^d$, *and* $(\mathbf{a}_1, \dots, \mathbf{a}_k) \in \mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])}$. *Then*

$$\gamma_{\mathbf{r}}(\mathbf{a}) = \mathbf{b} \Leftrightarrow \gamma_{\overline{\mathbf{r}}}(\mathrm{i}\overline{\mathbf{a}}) = \mathrm{i}\overline{\mathbf{b}}.$$

In particular: $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ *is a cycle of* $\gamma_{\mathbf{r}}$ *iff* $(\mathrm{i}\overline{\mathbf{a}_1}, \dots, \mathrm{i}\overline{\mathbf{a}_k})$ *is a cycle of* $\gamma_{\overline{\mathbf{r}}}$.

5.3. Main result on Gaussian Shift Radix Systems: One inclusion

We shall now proof the following theorem which settles one inclusion of Conjecture 5.2.2.

Theorem 5.3.1. [**Weitzer, 2015b**] $\mathcal{G}_1^{(0)} \subseteq \mathcal{G}_C$.

The proof (see p. 74) will be done by identification of 20 infinite families of cutout polygons which cover everything outside $\mathcal{G}_C$ and inside the first quadrant of the unit disk.

Definition 5.3.2. [**Weitzer, 2015b**] *Let*

$$\begin{array}{rcl}
D_0(1) & := & ((-3,0),(3,3),(0,-4),(-2,3),(4,0),(-2,-2),(0,3),(3,-2)) \\
D_0(2) & := & ((-3,0),(3,3),(0,-4),(-2,3),(4,0),(-2,-2),(1,3),(2,-2),(-2,1),(3,1),(-1,-2),(0,3), \\
 & & (3,-2)) \\
D_0(3) & := & ((-3,-1),(3,3),(-1,-3),(0,4),(2,-3),(-3,2),(4,0)) \\
D_0(4) & := & ((-4,-2),(3,4),(0,-4),(-1,4),(3,-3),(-4,2),(5,0)) \\
D_0(5) & := & ((-5,-1),(5,3),(-3,-4),(2,5),(0,-5),(-1,5),(3,-4),(-4,3),(5,-1),(-5,0),(5,2), \\
 & & (-4,-3),(3,5),(-1,-5),(0,6),(2,-5),(-3,5),(5,-3),(-5,2),(6,0)) \\
D_0(6) & := & ((-5,0),(5,2),(-4,-3),(3,5),(-1,-5),(0,5),(2,-4),(-3,4),(5,-2),(-5,1),(5,1),(-4,-2), \\
 & & (4,4),(-2,-4),(1,5),(1,-5),(-2,5),(4,-4),(-4,3),(5,-1)) \\
D_0(7) & := & ((-15,-5),(13,10),(-9,-13),(5,15),(0,-15),(-4,15),(9,-12),(-12,9),(15,-4),(-15,0), \\
 & & (15,5),(-12,-9),(9,13),(-4,-15),(0,16),(5,-15),(-9,13),(13,-9),(-15,5),(16,0)) \\
D_0(8) & := & ((-4,0),(4,2),(-3,-2),(3,3),(-2,-3),(2,4),(0,-4),(-1,4),(3,-3),(-3,2),(4,-1)) \\
D_0(9) & := & ((-7,0),(7,2),(-6,-3),(6,5),(-4,-5),(3,6),(-1,-6),(0,7),(2,-6),(-3,6),(5,-5),(-5,4), \\
 & & (6,-2),(-6,1),(7,1),(-6,-2),(6,4),(-5,-5),(4,6),(-2,-6),(1,7),(1,-7),(-2,7),(4,-6), \\
 & & (-5,6),(6,-4),(-6,3),(7,-1)) \\
D_0(10) & := & ((-7,-1),(7,3),(-6,-4),(6,6),(-4,-6),(3,7),(-1,-7),(0,8),(2,-7),(-3,7),(5,-6), \\
 & & (-6,5),(7,-3),(-7,2),(8,0)) \\
D_0(11) & := & ((-10,0),(10,2),(-9,-3),(9,5),(-7,-6),(6,8),(-4,-8),(3,9),(-1,-9),(0,10),(2,-9), \\
 & & (-3,9),(5,-8),(-6,7),(8,-5),(-8,4),(9,-2),(-9,1),(10,1),(-9,-2),(9,4),(-8,-5), \\
 & & (7,7),(-5,-8),(4,9),(-2,-9),(1,10),(1,-10),(-2,10),(4,-9),(-5,9),(7,-7),(-8,6), \\
 & & (9,-4),(-9,3),(10,-1))
\end{array}$$

$$\begin{array}{rcll}
D_1(n,m) & := & (\ (-n-m,0) & \\
 & & (n+m-k+1,3k-1),(-n-m+k,-3k) & 1 \le k \le m+1 \\
 & & (n-k,3m+k+3),(-n+k+1,-3m-k-3) & 1 \le k \le n-3m-5 \\
 & & (3m-3k+7,n+k-2),(-3m+3k-5,-n-k+2) & 1 \le k \le m+2 \\
 & & (-3k+1,n+m-k+1),(3k+1,-n-m+k) & 1 \le k \le m+1 \\
 & & (-3m-k-3,n-k),(3m+k+4,-n+k+1) & 1 \le k \le n-3m-5 \\
 & & (-n-k+2,3m-3k+6),(n+k-1,-3m+3k-4) & 1 \le k \le m+1 \\
 & &),\ n \ge 2 \quad \wedge \quad -1 \le m \le (n-5)/3 &
\end{array}$$

$$\begin{array}{rcll}
D_2(n,m) & := & (\ (-n-m+k-1,-3k+1),(n+m-k+1,3k+1) & 1 \le k \le m+1 \\
 & & (-n+k,-3m-k-4),(n-k,3m+k+5) & 1 \le k \le n-3m-6 \\
 & & (-3m+3k-7,-n-k+2),(3m-3k+6,n+k-1) & 1 \le k \le m+2 \\
 & & (3k-1,-n-m+k-1),(-3k,n+m-k+1) & 1 \le k \le m+1 \\
 & & (3m+k+4,-n+k),(-3m-k-4,n-k) & 1 \le k \le n-3m-6 \\
 & & (n+k-2,-3m+3k-8),(-n-k+2,3m-3k+7) & 1 \le k \le m+2 \\
 & & (n+m+1,1) & \\
 & &),\ n \ge 3 \quad \wedge \quad -1 \le m \le (n-6)/3 &
\end{array}$$

$$\begin{array}{rcll}
D_3(n,m) & := & (\ (-n-m+k-1,-3k+3),(n+m-k+1,3k-1) & 1 \le k \le m+1
\end{array}$$

$$
\begin{array}{lll}
 & (-n+k,-3m-k-2),(n-k,3m+k+3) & 1 \le k \le n-3m-4 \\
 & (-3m-2,-n+1),(3m+2,n) & \textit{if } m=0 \\
 & (-3m-2,-n+1),(3m+2,n),(-3m,-n),(3m-1,n+1) & \textit{if } m \neq 0 \\
 & (-3m+3k,-n-k),(3m-3k-1,n+k+1) & 1 \le k \le m-1 \\
 & (3k-3,-n-m+k-1),(-3k+2,n+m-k+1) & 1 \le k \le m+1 \\
 & (3m+k+2,-n+k),(-3m-k-2,n-k) & 1 \le k \le n-3m-4 \\
 & (n-1,-3m-3),(-n+1,3m+2) & \textit{if } m=0 \\
 & (n-1,-3m-3),(-n+1,3m+2),(n,-3m-1),(-n,3m) & \textit{if } m \neq 0 \\
 & (n+k,-3m+3k-1),(-n-k,3m-3k) & 1 \le k \le m-1 \\
 & (n+m,-1) & \\
 &),\ n \ge 5 \quad \wedge \quad 0 \le m \le (n-5)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_4(n,m) := & (\ (-n-m+k-1,-3k+2),(n+m-k+1,3k) & 1 \le k \le m+1 \\
 & (-n+k,-3m-k-3),(n-k,3m+k+4) & 1 \le k \le n-3m-5 \\
 & (-3m-3,-n+1),(3m+3,n) & \\
 & (-3m+3k-4,-n-k+1),(3m-3k+3,n+k) & 1 \le k \le m+1 \\
 & (3k-1,-n-m+k-1),(-3k,n+m-k+1) & 1 \le k \le m+1 \\
 & (3m+k+4,-n+k),(-3m-k-4,n-k) & 1 \le k \le n-3m-6 \\
 & (n-1,-3m-5),(-n+1,3m+4) & \\
 & (n+k-1,-3m+3k-6),(-n-k+1,3m-3k+5) & 1 \le k \le m+1 \\
 & (n+m+1,0) & \\
 &),\ n \ge 6 \quad \wedge \quad 0 \le m \le (n-6)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_5(n,m) := & (\ (-n-m+k-1,-3k+3),(n+m-k+1,3k-1) & 1 \le k \le m+1 \\
 & (-n+k,-3m-k-1),(n-k,3m+k+2) & 1 \le k \le n-3m-2 \\
 & (-3m+3k-3,-n-k+1),(3m-3k+2,n+k) & 1 \le k \le m \\
 & (3k-3,-n-m+k-1),(-3k+2,n+m-k+1) & 1 \le k \le m+1 \\
 & (3m+k+1,-n+k),(-3m-k-1,n-k) & 1 \le k \le n-3m-2 \\
 & (n+k-1,-3m+3k-4),(-n-k+1,3m-3k+3) & 1 \le k \le m \\
 & (n+m,-1) & \\
 &),\ n \ge 2 \quad \wedge \quad 0 \le m \le (n-2)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_6(n,m) := & (\ (-n-m+k-1,-3k+2),(n+m-k+1,3k) & 1 \le k \le m+1 \\
 & (-n+k,-3m-k-2),(n-k,3m+k+3) & 1 \le k \le n-3m-3 \\
 & (-3m+3k-4,-n-k+1),(3m-3k+3,n+k) & 1 \le k \le m+1 \\
 & (3k-1,-n-m+k-1),(-3k,n+m-k+1) & 1 \le k \le m+1 \\
 & (3m+k+3,-n+k),(-3m-k-3,n-k) & 1 \le k \le n-3m-4 \\
 & (n+k-1,-3m+3k-6),(-n-k+1,3m-3k+5) & 1 \le k \le m+1 \\
 & (n+m+1,0) & \\
 &),\ n \ge 4 \quad \wedge \quad 0 \le m \le (n-4)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_7(n,m) := & (\ (-n-m,1) & \\
 & (n+m-k+1,3k-2),(-n-m+k,-3k+1) & 1 \le k \le m+1 \\
 & (n-k,3m+k+2),(-n+k+1,-3m-k-2) & 1 \le k \le n-3m-3 \\
 & (3m-3k+5,n+k-1),(-3m+3k-3,-n-k+1) & 1 \le k \le m+1 \\
 & (-3k+2,n+m-k+1),(3k,-n-m+k) & 1 \le k \le m+1 \\
 & (-3m-k-2,n-k),(3m+k+3,-n+k+1) & 1 \le k \le n-3m-3 \\
 & (-n-k+1,3m-3k+4),(n+k,-3m+3k-2) & 1 \le k \le m \\
 &),\ n \ge 5 \quad \wedge \quad 0 \le m \le (n-5)/5 &
\end{array}
$$

$$
\begin{array}{lll}
D_8(n,m) := & (\ (-n-m+k-1,-3k+2),(n+m-k+1,3k) & 1 \le k \le m+1 \\
 & (-n+k,-3m-k-3),(n-k,3m+k+4) & 1 \le k \le n-3m-4 \\
 & (-3m+3k-5,-n-k+1),(3m-3k+4,n+k) & 1 \le k \le m+1 \\
 & (3k-2,-n-m+k-1),(-3k+1,n+m-k+1) & 1 \le k \le m+1 \\
 & (3m+k+3,-n+k),(-3m-k-3,n-k) & 1 \le k \le n-3m-4 \\
 & (n+k-1,-3m+3k-6),(-n-k+1,3m-3k+5) & 1 \le k \le m+1 \\
 & (n+m+1,0) & \\
 &),\ n \ge 1 \quad \wedge \quad (m=-1 \quad \vee \quad 0 \le m \le (n-8)/5 \quad \vee \quad m=(n-4)/3) &
\end{array}
$$

$$
\begin{array}{lll}
D_9(n,m) := & (\ (-n-m,0) & \\
 & (n+m-k+1,3k-1),(-n-m+k,-3k) & 1 \le k \le m \\
 & (n-k+1,3m+k+1),(-n+k,-3m-k-1) & 1 \le k \le n-3m-3
\end{array}
$$

$$
\begin{array}{lll}
 & (3m+3,n-1),(-3m-1,-n+1) & \\
 & (3m-3k+4,n+k-1),(-3m+3k-2,-n-k+1) & 1\leq k\leq m+1\\
 & (-3k+1,n+m-k+1),(3k+1,-n-m+k) & 1\leq k\leq m\\
 & (-3m-k-1,n-k+1),(3m+k+2,-n+k) & 1\leq k\leq n-3m-3\\
 & (-n+1,3m+2),(n,-3m-1) & \\
 & (-n-k+1,3m-3k+3),(n+k,-3m+3k-1) & 1\leq k\leq m\\
 &),\ n\geq 4 \quad\wedge\quad (n-4)/5\leq m\leq (n-4)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_{10}(n,m) := & (\ (-n-m+k-1,-3k+1),(n+m-k+1,3k+1) & 1\leq k\leq m+1\\
 & (-n+k,-3m-k-3),(n-k,3m+k+4) & 1\leq k\leq n-3m-5\\
 & (-3m-3,-n+1),(3m+3,n) & \\
 & (-3m+3k-4,-n-k+1),(3m-3k+3,n+k) & 1\leq k\leq m+1\\
 & (3k-1,-n-m+k-1),(-3k,n+m-k+1) & 1\leq k\leq m+1\\
 & (3m+k+3,-n+k),(-3m-k-3,n-k) & 1\leq k\leq n-3m-5\\
 & (n-1,-3m-4),(-n+1,3m+3) & \\
 & (n+k-1,-3m+3k-5),(-n-k+1,3m-3k+4) & 1\leq k\leq m+1\\
 & (n+m+1,1) & \\
 &),\ n\geq 5 \quad\wedge\quad (n-6)/5\leq m\leq (n-5)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_{11}(n,m) := & (\ (-n-m,-2) & \\
 & (n+m-k+1,3k+1),(-n-m+k,-3k-2) & 1\leq k\leq m\\
 & (n-k+1,3m+k+2),(-n+k,-3m-k-2) & 1\leq k\leq n-3m-4\\
 & (3m+4,n-1),(-3m-2,-n+1) & \\
 & (3m-3k+5,n+k-1),(-3m+3k-3,-n-k+1) & 1\leq k\leq m+1\\
 & (-3k+2,n+m-k+1),(3k,-n-m+k) & 1\leq k\leq m+1\\
 & (-3m-k-2,n-k),(3m+k+3,-n+k+1) & 1\leq k\leq n-3m-3\\
 & (-n-k+1,3m-3k+4),(n+k,-3m+3k-2) & 1\leq k\leq m+1\\
 &),\ n\geq 4 \quad\wedge\quad (n-5)/5\leq m\leq (n-4)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_{12}(n,m) := & (\ (-n-m+k-1,-3k+1),(n+m-k+1,3k+1) & 1\leq k\leq m+1\\
 & (-n+k,-3m-k-3),(n-k,3m+k+4) & 1\leq k\leq n-3m-5\\
 & (-3m-3,-n+1),(3m+3,n) & \\
 & (-3m+3k-4,-n-k+1),(3m-3k+3,n+k) & 1\leq k\leq m+1\\
 & (3k-1,-n-m+k-1),(-3k,n+m-k+1) & 1\leq k\leq m+1\\
 & (3m+k+4,-n+k),(-3m-k-4,n-k) & 1\leq k\leq n-3m-6\\
 & (n+k-2,-3m+3k-8),(-n-k+2,3m-3k+7) & 1\leq k\leq m+2\\
 & (n+m+1,1) & \\
 &),\ n\geq 6 \quad\wedge\quad (n-7)/5\leq m\leq (n-6)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_{13}(n,m) := & (\ (-n-m+k-1,-3k+3),(n+m-k+1,3k-1) & 1\leq k\leq m+1\\
 & (-n+k,-3m-k-1),(n-k,3m+k+2) & 1\leq k\leq n-3m-3\\
 & (-3m-1,-n+1),(3m+1,n) & \\
 & (-3m+3k-2,-n-k+1),(3m-3k+1,n+k) & 1\leq k\leq m\\
 & (3k-2,-n-m+k-1),(-3k+1,n+m-k+1) & 1\leq k\leq m\\
 & (3m+k,-n+k-1),(-3m-k,n-k+1) & 1\leq k\leq n-3m-1\\
 & (n+k-1,-3m+3k-4),(-n-k+1,3m-3k+3) & 1\leq k\leq m\\
 & (n+m,-1) & \\
 &),\ n\geq 3 \quad\wedge\quad (n-3)/5\leq m\leq (n-3)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_{14}(n,m) := & (\ (-n-m+k-1,-3k+2),(n+m-k+1,3k) & 1\leq k\leq m+1\\
 & (-n+k,-3m-k-3),(n-k,3m+k+4) & 1\leq k\leq n-3m-4\\
 & (-3m+3k-5,-n-k+1),(3m-3k+4,n+k) & 1\leq k\leq m+1\\
 & (3k-2,-n-m+k-2),(-3k+1,n+m-k+2) & 1\leq k\leq m+1\\
 & (3m+k+3,-n+k-1),(-3m-k-3,n-k+1) & 1\leq k\leq n-3m-5\\
 & (n-1,-3m-5),(-n+1,3m+4) & \\
 & (n+k-1,-3m+3k-6),(-n-k+1,3m-3k+5) & 1\leq k\leq m+1\\
 & (n+m+1,0) & \\
 &),\ n\geq 2 \quad\wedge\quad (n-7)/5\leq m\leq (n-5)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_{15}(n,m) := & (\ (-n-m+k-1,-3k+2),(n+m-k+1,3k) & 1\leq k\leq m+1\\
 & (-n+k,-3m-k-3),(n-k,3m+k+4) & 1\leq k\leq n-3m-5\\
 & (-3m-3,-n+1),(3m+3,n) & \\
 & (-3m+3k-4,-n-k+1),(3m-3k+3,n+k) & 1\leq k\leq m+1\\
 & (3k-1,-n-m+k-1),(-3k,n+m-k+1) & 1\leq k\leq m+1
\end{array}
$$

$$
\begin{array}{lll}
 & (3m+k+3,-n+k),(-3m-k-3,n-k) & 1\le k\le n-3m-4\\
 & (n+k-1,-3m+3k-6),(-n-k+1,3m-3k+5) & 1\le k\le m+1\\
 & (n+m+1,0) & \\
 &),\ n\ge 4 \quad\wedge\quad ((n-7)/5\le m\le (n-6)/3 \quad\vee\quad m=(n-4)/3) &
\end{array}
$$

$$
\begin{array}{lll}
D_{16}(n,m) := & (\ (-n-m+k-1,-3k+3),(n+m-k+1,3k-1) & 1\le k\le m+1\\
 & (-n+k,-3m-k-1),(n-k,3m+k+2) & 1\le k\le n-3m-2\\
 & (-3m+3k-3,-n-k+1),(3m-3k+2,n+k) & 1\le k\le m\\
 & (3k-3,-n-m+k-1),(-3k+2,n+m-k+1) & 1\le k\le m+1\\
 & (3m+k+2,-n+k),(-3m-k-2,n-k) & 1\le k\le n-3m-4\\
 & (n-1,-3m-3),(-n+1,3m+2),(n,-3m-1),(-n,3m) & \\
 & (n+k,-3m+3k-1),(-n-k,3m-3k) & 1\le k\le m-1\\
 & (n+m,-1) & \\
 &),\ n\ge 7 \quad\wedge\quad (n-4)/5\le m\le (n-4)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_{17}(n,m) := & (\ (-n-m+k-1,-3k+1),(n+m-k+1,3k+1) & 1\le k\le m\\
 & (-n+k-1,-3m-k-1),(n-k+1,3m+k+2) & 1\le k\le n-3m-2\\
 & (-3m+3k-4,-n-k+1),(3m-3k+3,n+k) & 1\le k\le m+1\\
 & (3k-1,-n-m+k-1),(-3k,n+m-k+1) & 1\le k\le m+1\\
 & (3m+k+3,-n+k),(-3m-k-3,n-k) & 1\le k\le n-3m-5\\
 & (n-1,-3m-4),(-n+1,3m+3) & \\
 & (n+k-1,-3m+3k-5),(-n-k+1,3m-3k+4) & 1\le k\le m+1\\
 & (n+m+1,1) & \\
 &),\ n\ge 5 \quad\wedge\quad (n-5)/5\le m\le (n-5)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_{18}(n,m) := & (\ (-n-m+k-1,-3k+3),(n+m-k+1,3k-1) & 1\le k\le m+1\\
 & (-n+k,-3m-k-2),(n-k,3m+k+3) & 1\le k\le n-3m-4\\
 & (-3m+3k-5,-n-k+2),(3m-3k+4,n+k-1) & 1\le k\le m+1\\
 & (3k-2,-n-m+k-1),(-3k+1,n+m-k+1) & 1\le k\le m+1\\
 & (3m+k+2,-n+k),(-3m-k-2,n-k) & 1\le k\le n-3m-4\\
 & (n-1,-3m-3),(-n+1,3m+2) & \\
 & (n+k-1,-3m+3k-4),(-n-k+1,3m-3k+3) & 1\le k\le m\\
 & (n+m,-1) & \\
 &),\ n\ge 8 \quad\wedge\quad (n-4)/5\le m\le (n-5)/3 &
\end{array}
$$

$$
\begin{array}{lll}
D_{19}(n,m) := & (\ (-4n/3-3m+k,-3k+1),(4n/3+3m-k,3k+1) & 1\le k\le n/3-1\\
 & (-n-3m+3k-3,-n-3k+4),(n+3m-3k+2,n+3k-2) & 1\le k\le m\\
 & (-n+3k-3,-n-3m-k+2),(n-3k+2,n+3m+k-1) & 1\le k\le n/3\\
 & (3k-3,-4n/3-3m+k),(-3k+2,4n/3+3m-k) & 1\le k\le n/3\\
 & (n+3k-3,-n-3m+3k-2),(-n-3k+2,n+3m-3k+1) & 1\le k\le m\\
 & (n+3m+k-1,-n+3k-2),(-n-3m-k+1,n-3k+1) & 1\le k\le n/3\\
 & (4n/3+3m,1) & \\
 &),\ n\ge 3 \quad\wedge\quad n\equiv 0 \pmod 3 \quad\wedge\quad (m=0 \quad\vee\quad 1\le m\le (n-3/2)\cdot 2/9) & \\
 & (\ (-n-3m+k,-3k+1),(4n/3+3m-k+2/3,3k+1) & 1\le k\le (n-1)/3\\
 & (-n-3m+3k-2,-n-3k+2),(n+3m-3k+1,n+3k) & 1\le k\le m\\
 & (-n+3k-2,-n-3m-k),(n-3k+1,n+3m+k+1) & 1\le k\le (n-1)/3\\
 & (3k-3,-4n/3-3m+k-5/3),(-3k+2,4n/3+3m-k+5/3) & 1\le k\le (n-1)/3\\
 & (n+3k-4,-n-3m+3k-4),(-n-3k+3,n+3m-3k+3) & 1\le k\le m+1\\
 & (n+3m+k,-n+3k-1),(-n-3m-k,n-3k) & 1\le k\le (n-1)/3\\
 & (4n/3+3m+2/3,1) & \\
 &),\ n\ge 1 \quad\wedge\quad n\equiv 1 \pmod 3 \quad\wedge\quad (m=0 \quad\vee\quad 1\le m\le (n-5/2)\cdot 2/9) & \\
 & (\ (-4n/3-3m+k-7/3,-3k+3),(4n/3+3m-k+7/3,3k-1) & 1\le k\le (n+1)/3\\
 & (-n-3m+3k-3,-n-3k+2),(n+3m-3k+2,n+3k) & 1\le k\le m\\
 & (-n+3k-3,-n-3m-k),(n-3k+2,n+3m+k+1) & 1\le k\le (n+1)/3\\
 & (3k-2,-4n/3-3m+k-7/3),(-3k+1,4n/3+3m-k+7/3) & 1\le k\le (n-2)/3\\
 & (n+3k-4,-n-3m+3k-5),(-n-3k+3,n+3m-3k+4) & 1\le k\le m+1\\
 & (n+3m+k+1,-n+3k-2),(-n-3m-k-1,n-3k+1) & 1\le k\le (n-2)/3\\
 & (4n/3+3m+4/3,-1) & \\
 &),\ n\ge 5 \quad\wedge\quad n\equiv 2 \pmod 3 \quad\wedge\quad (m=0 \quad\vee\quad 1\le m\le (n-7/2)\cdot 2/9) &
\end{array}
$$

$$
\begin{array}{lll}
D_{20}(n,m) := & (\ (-4n/3-3m+k+2,-3k+2),(4n/3+3m-k-2,3k) & 1\le k\le n/3\\
 & (-n-3m+3k+1,-n-3k+2),(n+3m-3k-2,n+3k) & 1\le k\le m-1\\
 & (-n+3k-2,-n-3m-k+3),(n-3k+1,n+3m+k-2) & 1\le k\le n/3\\
 & (3k-2,-4n/3-3m+k+1),(-3k+1,4n/3+3m-k-1) & 1\le k\le n/3
\end{array}
$$

$(n+3k-2, -n-3m+3k), (-n-3k+1, n+3m-3k-1)$ $\quad 1 \le k \le m-1$
$(n+3m+k-3, -n+3k-3), (-n-3m-k+3, n-3k+2)$ $\quad 1 \le k \le n/3$
$(4n/3+3m-2, 0)$
$), n \ge 3 \quad \wedge \quad n \equiv 0 \pmod 3 \quad \wedge \quad (0 \le m \le n \cdot 2/9 \quad \vee \quad m = (n+3/2) \cdot 2/9)$
$(\; (-4n/3-3m+k+1/3, -3k+2), (4n/3+3m-k-1/3, 3k)$ $\quad 1 \le k \le (n-1)/3$
$(-n-3m+3k-1, -n-3k+3), (n+3m-3k, n+3k-1)$ $\quad 1 \le k \le m$
$(-n+3k-1, -n-3m-k+2), (n-3k, n+3m+k-1)$ $\quad 1 \le k \le (n-1)/3$
$(3k-2, -4n/3-3m+k+1/3), (-3k+1, 4n/3+3m-k-1/3)$ $\quad 1 \le k \le (n-1)/3$
$(n+3k-3, -n-3m+3k-2), (-n-3k+2, n+3m-3k+1)$ $\quad 1 \le k \le m$
$(n+3m+k-1, -n+3k-2), (-n-3m-k+1, n-3k+1)$ $\quad 1 \le k \le (n-1)/3$
$(4n/3+3m-1/3, 0)$
$), n \ge 4 \quad \wedge \quad n \equiv 1 \pmod 3 \quad \wedge \quad (0 \le m \le (n-1) \cdot 2/9 \quad \vee \quad m = (n+3/2) \cdot 2/9)$
$(\; (-4n/3-3m+k-1/3, -3k+1), (4n/3+3m-k+1/3, 3k+1)$ $\quad 1 \le k \le (n-2)/3$
$(-n-3m+3k-2, -n-3k+3), (n+3m-3k+1, n+3k-1)$ $\quad 1 \le k \le m$
$(-n+3k-2, -n-3m-k+1), (n-3k+1, n+3m+k)$ $\quad 1 \le k \le (n+1)/3$
$(3k-1, -4n/3-3m+k-1/3), (-3k, 4n/3+3m-k+1/3)$ $\quad 1 \le k \le (n+1)/3$
$(n+3k-3, -n-3m+3k-3), (-n-3k+2, n+3m-3k+2)$ $\quad 1 \le k \le m$
$(n+3m+k-1, -n+3k-3), (-n-3m-k+1, n-3k+2)$ $\quad 1 \le k \le (n+1)/3$
$(4n/3+3m+1/3, 1)$
$), n \ge 2 \quad \wedge \quad n \equiv 2 \pmod 3 \quad \wedge \quad (1 \le m \le (n-2) \cdot 2/9 \quad \vee \quad m = (n+3/2) \cdot 2/9)$

The following theorem then follows from Lemma 3.5.8.

THEOREM 5.3.3. [**Weitzer, 2015b**] *The cutout polygons corresponding to the cycles defined above have the following vertices and boundaries (cf. Theorem 3.5.10 for notation):*

$D_0(n)$:

$n=1$: $(\frac{3}{4},\frac{3}{4})$—$(\frac{2}{3},\frac{2}{3})$—
$n=2$: $(\frac{3}{4},\frac{2}{3})$- -$(\frac{3}{4},\frac{3}{4})$- -$(\frac{2}{3},\frac{2}{3})$- -
$n=3$: $(\frac{14}{15},\frac{2}{5})$- -$(\frac{5}{6},\frac{1}{2})$- -$(\frac{6}{7},\frac{3}{7})$—$(\frac{12}{13},\frac{5}{13})$—
$n=4$: $(\frac{23}{25},\frac{11}{25})$- -$(\frac{10}{11},\frac{5}{11})$—$(\frac{8}{9},\frac{4}{9})$—$(\frac{19}{21},\frac{3}{7})$- -
$n=5$: $(1,\frac{1}{3})$- -$(\frac{14}{15},\frac{1}{3})$- -$(\frac{16}{17},\frac{5}{17})$- -$(\frac{24}{25},\frac{7}{25})$—
$n=6$: $(\frac{16}{17},\frac{5}{17})$—$(\frac{15}{16},\frac{5}{16})$—
$n=7$: $(\frac{15}{16},\frac{5}{16})$
$n=8$: $(\frac{17}{18},\frac{5}{18})$—$(\frac{14}{15},\frac{4}{15})$—
$n=9$: $(\frac{48}{49},\frac{9}{49})$- -$(\frac{36}{37},\frac{7}{37})$—$(\frac{37}{38},\frac{7}{38})$—$(\frac{43}{44},\frac{2}{11})$- -
$n=10$: $(\frac{65}{66},\frac{2}{11})$- -$(\frac{35}{36},\frac{7}{36})$- -$(\frac{36}{37},\frac{7}{37})$—$(\frac{60}{61},\frac{11}{61})$—
$n=11$: $(\frac{87}{88},\frac{2}{11})$- -$(\frac{51}{52},\frac{5}{26})$—$(\frac{57}{58},\frac{5}{29})$—

$D_1(n,m)$:

$n=2 \wedge m=-1$: $(1,1)$—$(0,1)$—$(\frac{1}{2},\frac{1}{2})$- -
$n=3 \wedge m=-1$: $(1,\frac{1}{2})$—$(\frac{3}{4},\frac{1}{2})$—$(\frac{4}{5},\frac{2}{5})$- -
$n \ge 4 \wedge m=-1$: $(1,\frac{1}{n-1})$—$(1-\frac{1}{n^2-2n+1},\frac{n-1}{n^2-2n+1})$—$(1-\frac{1}{n^2-2n+2},\frac{n-1}{n^2-2n+2})$- -
$n=5 \wedge m=0$: $(1,\frac{1}{4})$—$(\frac{24}{25},\frac{7}{25})$- -$(\frac{18}{19},\frac{5}{19})$—$(\frac{21}{22},\frac{5}{22})$—
$n \ge 9 \wedge 0 \le m \le \frac{n-9}{5}$: $(1,\frac{1}{n-1})$—$(1-\frac{1}{n^2-n+nm-4m-3},\frac{n+m}{n^2-n+nm-4m-3})$—
$(1-\frac{1}{n^2-n+nm+2m+2},\frac{n+m}{n^2-n+nm+2m+2})$—
$n \ge 6 \wedge \frac{n-8}{5} \le m \le \frac{n-5}{5}$: $(1,\frac{1}{n-1})$—$(1-\frac{1}{8n+6nm-9m-11},\frac{6m+8}{8n+6nm-9m-11})$- -
$(1-\frac{1}{n^2-2n+nm+m+5},\frac{n+m}{n^2-2n+nm+m+5})$—$(1-\frac{1}{n^2-n+nm+2m+2},\frac{n+m}{n^2-n+nm+2m+2})$—
$n \ge 9 \wedge \frac{n-4}{5} \le m \le \frac{n-6}{3}$: $(1,\frac{1}{n-1})$—$(1-\frac{1}{8n+6nm-9m-11},\frac{6m+8}{8n+6nm-9m-11})$- -
$(1-\frac{1}{n^2-2n+nm+m+5},\frac{n+m}{n^2-2n+nm+m+5})$—$(1-\frac{1}{n^2+nm-3m-2},\frac{n+m}{n^2+nm-3m-2})$- -
$(1-\frac{1}{4n+6nm-3m-2},\frac{6m+4}{4n+6nm-3m-2})$—
$n \ge 8 \wedge m=\frac{n-5}{3}$: $(1,\frac{1}{n-1})$—$(1-\frac{1}{2n^2-6n+5},\frac{2n-3}{2n^2-6n+5})$- -$(1-\frac{3}{4n^2-10n+7},\frac{4n-5}{4n^2-10n+7})$—
$(1-\frac{3}{4n^2-8n+9},\frac{4n-5}{4n^2-8n+9})$- -$(1-\frac{1}{2n^2-7n+3},\frac{2n-6}{2n^2-7n+3})$—

$D_2(n,m):$

$n=3 \wedge m=-1: (\frac{7}{8},\frac{5}{8})\text{--}(\frac{5}{6},\frac{2}{3})\text{—}(\frac{2}{3},\frac{2}{3})\text{--}(\frac{4}{5},\frac{2}{5})\text{--}$

$n=4 \wedge m=-1: (1,\frac{1}{3})\text{--}(\frac{12}{13},\frac{5}{13})\text{--}(\frac{8}{9},\frac{1}{3})\text{--}(\frac{9}{10},\frac{3}{10})\text{--}$

$n\geq 7 \wedge -1\leq m\leq \frac{n-12}{5}: (1,\frac{1}{n-1})\text{--}(1-\frac{1}{n^2-n+nm-4m-5},\frac{n+m}{n^2-n+nm-4m-5})\text{--}$
$(1-\frac{1}{n^2-n+nm+2m+4},\frac{n+m}{n^2-n+nm+2m+4})\text{--}$

$n\geq 5 \wedge \frac{n-11}{5}\leq m\leq \frac{n-8}{5}: (1,\frac{1}{n-1})\text{--}(1-\frac{1}{12n+6nm-9m-17},\frac{6m+12}{12n+6nm-9m-17})\text{—}$
$(1-\frac{1}{n^2-2n+nm+m+7},\frac{n+m}{n^2-2n+nm+m+7})\text{--}(1-\frac{1}{n^2-n+nm+2m+4},\frac{n+m}{n^2-n+nm+2m+4})\text{--}$

$n\geq 11 \wedge \frac{n-7}{5}\leq m\leq \frac{n-8}{3}: (1,\frac{1}{n-1})\text{--}(1-\frac{1}{12n+6nm-9m-17},\frac{6m+12}{12n+6nm-9m-17})\text{—}$
$(1-\frac{1}{n^2-2n+nm+m+7},\frac{n+m}{n^2-2n+nm+m+7})\text{--}(1-\frac{1}{n^2+nm-3m-4},\frac{n+m}{n^2+nm-3m-4})\text{—}$
$(1-\frac{1}{8n+6nm-3m-4},\frac{6m+8}{8n+6nm-3m-4})\text{--}$

$n\geq 7 \wedge m=\frac{n-7}{3}: (1,\frac{1}{n-1})\text{--}(1-\frac{1}{2n^2-6n+5},\frac{2n-3}{2n^2-6n+5})\text{--}(1-\frac{3}{4n^2-12n+11},\frac{4n-7}{4n^2-12n+11})\text{--}$
$(1-\frac{3}{4n^2-10n+9},\frac{4n-7}{4n^2-10n+9})\text{—}(1-\frac{1}{2n^2-7n+3},\frac{2n-6}{2n^2-7n+3})\text{--}$

$n\geq 6 \wedge m=\frac{n-6}{3}: (1-\frac{1}{2n^2-4n+2},\frac{2n-1}{2n^2-4n+2})\text{--}(1-\frac{3}{4n^2-13n+9},\frac{4n-6}{4n^2-13n+9})\text{--}$
$(1-\frac{3}{4n^2-11n+12},\frac{4n-6}{4n^2-11n+12})\text{--}$

$D_3(n,m):$

$n\geq 5 \wedge 0\leq m\leq \frac{n-5}{5}: (1,\frac{1}{n-1})\text{--}(1-\frac{1}{n^2-n+nm+2m+2},\frac{n+m}{n^2-n+nm+2m+2})\text{—}$
$(1-\frac{1}{n^2-n+nm+2m+3},\frac{n+m}{n^2-n+nm+2m+3})\text{—}$

$n\geq 8 \wedge \frac{n-4}{5}\leq m\leq \frac{n-5}{3}: (1,\frac{1}{n-1})\text{--}(1-\frac{1}{4n+6nm-3m-2},\frac{6m+4}{4n+6nm-3m-2})\text{—}$
$(1-\frac{1}{5n+6nm-3m-2},\frac{6m+5}{5n+6nm-3m-2})\text{—}$

$D_4(n,m):$

$n\geq 8 \wedge 0\leq m\leq \frac{n-8}{5}: (1,\frac{1}{n-1})\text{—}(1-\frac{1}{n^2-n+nm+2m+4},\frac{n+m}{n^2-n+nm+2m+4})\text{--}$
$(1-\frac{1}{n^2-n+nm+2m+5},\frac{n+m}{n^2-n+nm+2m+5})\text{--}$

$n\geq 6 \wedge \frac{n-7}{5}\leq m\leq \frac{n-6}{3}: (1,\frac{1}{n-1})\text{—}(1-\frac{1}{7n+6nm-3m-3},\frac{6m+7}{7n+6nm-3m-3})\text{--}$
$(1-\frac{1}{8n+6nm-3m-3},\frac{6m+8}{8n+6nm-3m-3})\text{--}$

$D_5(n,m):$

$2\leq n\leq 3 \wedge m=0: (1,\frac{1}{n})\text{—}(1-\frac{1}{n^2+n-1},\frac{n+1}{n^2+n-1})\text{—}(1-\frac{1}{n^2},\frac{n}{n^2})\text{--}$

$n\geq 4 \wedge m=0: (1,\frac{1}{n})\text{—}(1-\frac{1}{n^2-1},\frac{n}{n^2-1})\text{—}(1-\frac{1}{n^2},\frac{n}{n^2})\text{--}$

$n\geq 9 \wedge 1\leq m\leq \frac{n-4}{5}: (1,\frac{1}{n})\text{—}(1-\frac{1}{n^2+nm-3m-1},\frac{n+m}{n^2+nm-3m-1})\text{—}$
$(1-\frac{1}{n^2+nm-3m},\frac{n+m}{n^2+nm-3m})\text{--}$

$n\geq 6 \wedge \frac{n-3}{5}\leq m\leq \frac{n-3}{3}: (1,\frac{1}{n})\text{—}(1-\frac{1}{4n+6nm-3m-1},\frac{6m+4}{4n+6nm-3m-1})\text{—}$
$(1-\frac{1}{3n+6nm-3m},\frac{6m+3}{3n+6nm-3m})\text{--}$

$n\geq 5 \wedge m=\frac{n-2}{3}: (1,\frac{1}{n})\text{—}(1-\frac{1}{2n^2-2n+1},\frac{2n-1}{2n^2-2n+1})\text{—}(1-\frac{1}{2n^2-3n+2},\frac{2n-2}{2n^2-3n+2})\text{--}$

$D_6(n,m):$

$n\geq 7 \wedge 0\leq m\leq \frac{n-7}{5}: (1,\frac{1}{n})\text{--}(1-\frac{1}{n^2+nm-3m-3},\frac{n+m}{n^2+nm-3m-3})\text{--}$
$(1-\frac{1}{n^2+nm-3m-2},\frac{n+m}{n^2+nm-3m-2})\text{—}$

$n\geq 5 \wedge \frac{n-6}{5}\leq m\leq \frac{n-5}{3}: (1,\frac{1}{n})\text{--}(1-\frac{1}{7n+6nm-3m-3},\frac{6m+7}{7n+6nm-3m-3})\text{--}$
$(1-\frac{1}{6n+6nm-3m-2},\frac{6m+6}{6n+6nm-3m-2})\text{—}$

$n\geq 4 \wedge m=\frac{n-4}{3}: (1,\frac{1}{n})\text{--}(1-\frac{1}{2n^2-2n+1},\frac{2n-1}{2n^2-2n+1})\text{—}(1-\frac{1}{2n^2-3n+2},\frac{2n-2}{2n^2-3n+2})\text{—}$

$D_7(n,m):$

$n\geq 9 \wedge 0\leq m\leq \frac{n-9}{11}: (1-\frac{1}{n^2+nm-3m-1},\frac{n+m}{n^2+nm-3m-1})\text{—}$

$(1-\frac{1}{n^2-n+nm+2m+3},\frac{n+m}{n^2-n+nm+2m+3})$- -$(1-\frac{2}{n^2-n+nm+5m+6},\frac{n+m}{n^2-n+nm+5m+6})$—
$(1-\frac{2}{n^2+nm-6m-2},\frac{n+m}{n^2+nm-6m-2})$- -
$n\geq 5\wedge\frac{n-8}{11}\leq m\leq\frac{n-5}{5}:(1-\frac{1}{n^2+nm-3m-1},\frac{n+m}{n^2+nm-3m-1})$—
$(1-\frac{1}{n^2-n+nm+2m+3},\frac{n+m}{n^2-n+nm+2m+3})$- -$(1-\frac{1}{4n+6nm-3m-1},\frac{6m+4}{4n+6nm-3m-1})$- -

$D_8(n,m)$:
$n=1\wedge m=-1:(0,0)$- -$(0,1)$- -$(-1,1)$—$\overline{(-1,0)}$—
$n=2\wedge m=-1:(\frac{3}{4},\frac{1}{2})$- -$(\frac{2}{3},\frac{2}{3})$- -$(\frac{1}{2},\frac{1}{2})$—
$n=3\wedge m=-1:(\frac{8}{9},\frac{1}{3})$- -$(\frac{7}{8},\frac{3}{8})$—$(\frac{5}{6},\frac{1}{3})$—
$n\geq 4\wedge m=-1:(1-\frac{1}{n^2},\frac{1}{n})$- -$(1-\frac{1}{n^2-n+2},\frac{n}{n^2-n+2})$—$(1-\frac{1}{n^2-2n+3},\frac{n-1}{n^2-2n+3})$- -
$(1-\frac{1}{n^2-n},\frac{n-1}{n^2-n})$—
$n\geq 16\wedge 0\leq m\leq\frac{n-16}{11}:(1-\frac{1}{n^2+n+nm-3m-3},\frac{n+m+1}{n^2+n+nm-3m-3})$- -
$(1-\frac{1}{n^2+nm+2m+4},\frac{n+m+1}{n^2+nm+2m+4})$—$(1-\frac{2}{n^2-n+nm+5m+10},\frac{n+m}{n^2-n+nm+5m+10})$- -
$(1-\frac{2}{n^2+nm-6m-6},\frac{n+m}{n^2+nm-6m-6})$—
$n\geq 8\wedge\frac{n-15}{11}\leq m\leq\frac{n-8}{5}:(1-\frac{1}{n^2+n+nm-3m-3},\frac{n+m+1}{n^2+n+nm-3m-3})$- -
$(1-\frac{1}{n^2+nm+2m+4},\frac{n+m+1}{n^2+nm+2m+4})$—$\overline{(1-\frac{1}{8n+6nm-3m-3},\frac{6m+8}{8n+6nm-3m-3})}$—
$n\geq 4\wedge m=\frac{n-4}{3}:(1-\frac{3}{4n^2-4n+3},\frac{4n-1}{4n^2-4n+3})$- -$(1-\frac{3}{4n^2-6n+2},\frac{4n-1}{4n^2-6n+2})$—
$\overline{(1-\frac{2}{2n^2-3n+2},\frac{2n-1}{2n^2-3n+2})}$—

$D_9(n,m)$:
$n\geq 4\wedge m=\frac{n-4}{5}:(1-\frac{5}{6n^2-7n+7},\frac{6n-4}{6n^2-7n+7})$- -$(1-\frac{5}{6n^2-2n+2},\frac{6n+1}{6n^2-2n+2})$- -
$(1-\frac{5}{6n^2-7n+2},\frac{6n-4}{6n^2-7n+2})$—
$n\geq 7\wedge\frac{n-3}{5}\leq m\leq\frac{n-4}{3}:(1-\frac{1}{4n+6nm-3m-1},\frac{6m+4}{4n+6nm-3m-1})$- -
$(1-\frac{1}{5n+6nm-3m-2},\frac{6m+5}{5n+6nm-3m-2})$- -$(1-\frac{1}{4n+6nm-3m-2},\frac{6m+4}{4n+6nm-3m-2})$—
$\overline{(1-\frac{1}{3n+6nm-3m-1},\frac{6m+3}{3n+6nm-3m-1})}$—

$D_{10}(n,m)$:
$n\geq 6\wedge m=\frac{n-6}{5}:(1-\frac{5}{6n^2-9n+8},\frac{6n-6}{6n^2-9n+8})$—$\overline{(1-\frac{5}{6n^2-4n+3},\frac{6n-1}{6n^2-4n+3})}$—
$(1-\frac{5}{6n^2-9n+3},\frac{6n-6}{6n^2-9n+3})$- -
$n\geq 5\wedge\frac{n-5}{5}\leq m\leq\frac{n-5}{3}:(1-\frac{1}{6n+6nm-3m-2},\frac{6m+6}{6n+6nm-3m-2})$—
$\overline{(1-\frac{1}{7n+6nm-3m-3},\frac{6m+7}{7n+6nm-3m-3})}$—$(1-\frac{1}{6n+6nm-3m-3},\frac{6m+6}{6n+6nm-3m-3})$- -
$(1-\frac{1}{5n+6nm-3m-2},\frac{6m+5}{5n+6nm-3m-2})$- -

$D_{11}(n,m)$:
$n\geq 5\wedge m=\frac{n-5}{5}:(1-\frac{5}{6n^2-8n+10},\frac{6n-5}{6n^2-8n+10})$- -$(1-\frac{5}{6n^2-3n+5},\frac{6n}{6n^2-3n+5})$—
$(1-\frac{5}{6n^2-8n+5},\frac{6n-5}{6n^2-8n+5})$- -
$n\geq 4\wedge\frac{n-4}{5}\leq m\leq\frac{n-4}{3}:(1-\frac{1}{5n+6nm-3m-1},\frac{6m+5}{5n+6nm-3m-1})$- -
$(1-\frac{1}{6n+6nm-3m-2},\frac{6m+6}{6n+6nm-3m-2})$—$(1-\frac{1}{5n+6nm-3m-2},\frac{6m+5}{5n+6nm-3m-2})$- -
$(1-\frac{1}{4n+6nm-3m-1},\frac{6m+4}{4n+6nm-3m-1})$- -

$D_{12}(n,m)$:
$n\geq 7\wedge m=\frac{n-7}{5}:(1-\frac{5}{6n^2-10n+6},\frac{6n-7}{6n^2-10n+6})$- -$(1-\frac{5}{6n^2-5n+1},\frac{6n-2}{6n^2-5n+1})$- -
$(1-\frac{5}{6n^2-10n+1},\frac{6n-7}{6n^2-10n+1})$- -
$n\geq 6\wedge\frac{n-6}{5}\leq m\leq\frac{n-6}{3}:(1-\frac{1}{7n+6nm-3m-3},\frac{6m+7}{7n+6nm-3m-3})$- -
$(1-\frac{1}{8n+6nm-3m-4},\frac{6m+8}{8n+6nm-3m-4})$- -$(1-\frac{1}{7n+6nm-3m-4},\frac{6m+7}{7n+6nm-3m-4})$- -
$(1-\frac{1}{6n+6nm-3m-3},\frac{6m+6}{6n+6nm-3m-3})$- -

$D_{13}(n,m):$

$n \geq 3 \wedge m = \frac{n-3}{5} : (1 - \frac{5}{6n^2-6n+9}, \frac{6n-3}{6n^2-6n+9})\text{-\,-}(1 - \frac{5}{6n^2-n+4}, \frac{6n+2}{6n^2-n+4})\text{-\,-}$
$(1 - \frac{5}{6n^2-6n+4}, \frac{6n-3}{6n^2-6n+4})\text{—}$

$n \geq 6 \wedge \frac{n-2}{5} \leq m \leq \frac{n-3}{3} : (1 - \frac{1}{3n+6nm-3m}, \frac{6m+3}{3n+6nm-3m})\text{-\,-}$
$(1 - \frac{1}{4n+6nm-3m-1}, \frac{6m+4}{4n+6nm-3m-1})\text{-\,-}(1 - \frac{1}{3n+6nm-3m-1}, \frac{6m+3}{3n+6nm-3m-1})\text{-\,-}$
$(1 - \frac{1}{2n+6nm-3m}, \frac{6m+2}{2n+6nm-3m})\text{-\,-}$

$D_{14}(n,m):$

$n = 2 \wedge m = -1 : \overline{(\frac{3}{4}, \frac{1}{2})}\text{—}(\frac{4}{5}, \frac{3}{5})\text{-\,-}(\frac{2}{3}, \frac{2}{3})\text{—}$

$n \geq 6 \wedge \frac{n-7}{5} \leq m \leq \frac{n-6}{3} : \overline{(1 - \frac{1}{8n+6nm-3m-3}, \frac{6m+8}{8n+6nm-3m-3})}\text{—}$
$(1 - \frac{1}{9n+6nm-3m-4}, \frac{6m+9}{9n+6nm-3m-4})\text{—}$

$n \geq 5 \wedge m = \frac{n-5}{3} : \overline{(1 - \frac{1}{2n^2-3n+2}, \frac{2n-2}{2n^2-3n+2})}\text{—}(\frac{2n^2-2n}{2n^2-2n+1}, \frac{2n-1}{2n^2-2n+1})\text{-\,-}$
$(1 - \frac{1}{2n^2-3n+1}, \frac{2n-2}{2n^2-3n+1})\text{-\,-}(1 - \frac{1}{2n^2-4n+2}, \frac{2n-3}{2n^2-4n+2})\text{—}$

$D_{15}(n,m):$

$n \geq 6 \wedge \frac{n-7}{5} \leq m \leq \frac{n-6}{3} : (1 - \frac{1}{8n+6nm-3m-3}, \frac{6m+8}{8n+6nm-3m-3})\text{—}$
$(1 - \frac{1}{7n+6nm-3m-3}, \frac{6m+7}{7n+6nm-3m-3})\text{—}$

$n \geq 4 \wedge m = \frac{n-4}{3} : (1, \frac{1}{n})\text{-\,-}(1 - \frac{1}{2n^2-2n+1}, \frac{2n-1}{2n^2-2n+1})\text{—}\overline{(1 - \frac{1}{2n^2-3n+2}, \frac{2n-2}{2n^2-3n+2})}\text{—}$

$D_{16}(n,m):$

$n \geq 7 \wedge \frac{n-4}{5} \leq m \leq \frac{n-4}{3} : (1 - \frac{1}{5n+6nm-3m-2}, \frac{6m+5}{5n+6nm-3m-2})\text{—}$
$(1 - \frac{1}{4n+6nm-3m-1}, \frac{6m+4}{4n+6nm-3m-1})\text{—}$

$D_{17}(n,m):$

$n \geq 5 \wedge \frac{n-5}{5} \leq m \leq \frac{n-5}{3} : \overline{(1 - \frac{1}{6n+6nm-3m-2}, \frac{6m+6}{6n+6nm-3m-2})}$

$D_{18}(n,m):$

$n \geq 8 \wedge \frac{n-4}{5} \leq m \leq \frac{n-5}{3} : \overline{(1 - \frac{1}{4n+6nm-3m-2}, \frac{6m+4}{4n+6nm-3m-2})}$

$D_{19}(n,m):$

$n = 1 \wedge m = 0 : \overline{(\frac{1}{2}, \frac{3}{4})}\text{—}(\frac{1}{2}, 1)\text{—}\overline{(\frac{2}{5}, \frac{4}{5})}\text{—}$

$n \geq 3 \wedge n \equiv 0 \bmod 3 \wedge m = 0 : (1 - \frac{1}{2n^2-3n+2}, \frac{2n-2}{2n^2-3n+2})\text{—}\overline{(1 - \frac{1}{2n^2-2n+1}, \frac{2n-1}{2n^2-2n+1})}\text{—}$
$(1 - \frac{3}{4n^2-6n+3}, \frac{4n-3}{4n^2-6n+3})\text{-\,-}(1 - \frac{3}{4n^2-6n+6}, \frac{4n-3}{4n^2-6n+6})\text{-\,-}$

$n \geq 9 \wedge n \equiv 0 \bmod 3 \wedge 1 \leq m \leq \frac{2n-6}{9} : (1 - \frac{3}{4n^2-4n+9nm+3}, \frac{4n+9m-3}{4n^2-4n+9nm+3})\text{—}$
$\overline{(1 - \frac{1}{2n^2-2n+6nm-3m+1}, \frac{2n+6m-1}{2n^2-2n+6nm-3m+1})}\text{—}$
$(1 - \frac{3}{4n^2-6n+9nm-9m+3}, \frac{4n+9m-3}{4n^2-6n+9nm-9m+3})\text{-\,-}$

$n \geq 4 \wedge n \equiv 1 \bmod 3 \wedge 0 \leq m \leq \frac{2n-8}{9} : (1 - \frac{3}{4n^2-2n+9nm+4}, \frac{4n+9m-1}{4n^2-2n+9nm+4})\text{—}$
$\overline{(1 - \frac{1}{2n^2+6nm-3m}, \frac{2n+6m+1}{2n^2+6nm-3m})}\text{—}(1 - \frac{3}{4n^2-4n+9nm-9m}, \frac{4n+9m-1}{4n^2-4n+9nm-9m})\text{-\,-}$

$n \geq 5 \wedge n \equiv 2 \bmod 3 \wedge 0 \leq m \leq \frac{2n-10}{9} : \overline{(1 - \frac{3}{4n^2+3n+9nm+2}, \frac{4n+9m+4}{4n^2+3n+9nm+2})}\text{—}$
$\overline{(1 - \frac{1}{2n^2+2n+6nm-3m-1}, \frac{2n+6m+3}{2n^2+2n+6nm-3m-1})}\text{—}$
$\overline{(1 - \frac{3}{4n^2+n+9nm-9m-3}, \frac{4n+9m+4}{4n^2+n+9nm-9m-3})}\text{—}$

$n \geq 6 \wedge m = \frac{2n-3}{9} : (1 - \frac{3}{6n^2-7n+3}, \frac{6n-6}{6n^2-7n+3})\text{—}\overline{(1 - \frac{3}{10n^2-14n+6}, \frac{10n-9}{10n^2-14n+6})}\text{—}$
$(1 - \frac{3}{10n^2-20n+6}, \frac{10n-15}{10n^2-20n+6})\text{-\,-}(1 - \frac{2}{4n^2-6n+1}, \frac{4n-4}{4n^2-6n+1})\text{-\,-}$

$n \geq 7 \wedge m = \frac{2n-5}{9} : (1 - \frac{3}{6n^2-7n+4}, \frac{6n-6}{6n^2-7n+4})\text{—}\overline{(1 - \frac{3}{10n^2-15n+8}, \frac{10n-10}{10n^2-15n+8})}\text{—}$
$(1 - \frac{1}{2n^2-3n+2}, \frac{2n-2}{2n^2-3n+2})\text{-\,-}$

$n \geq 8 \wedge m = \frac{2n-7}{9} : \overline{(1 - \frac{3}{6n^2-4n+2}, \frac{6n-3}{6n^2-4n+2})}\text{—}(1 - \frac{3}{10n^2-10n+4}, \frac{10n-5}{10n^2-10n+4})\text{-\,-}$

$(1-\frac{1}{2n^2-2n+1},\frac{2n-1}{2n^2-2n+1})$—

$D_{20}(n,m)$:

$n \geq 4 \wedge n \equiv 1 \bmod 3 \wedge m = 0 : (1-\frac{1}{2n^2-5n+2},\frac{2n-4}{2n^2-5n+2})$—$(1-\frac{1}{2n^2-4n+1},\frac{2n-3}{2n^2-4n+1})$--
$(1-\frac{3}{4n^2-7n+3},\frac{4n-4}{4n^2-7n+3})$—$(1-\frac{3}{4n^2-7n+6},\frac{4n-4}{4n^2-7n+6})$--

$n \geq 6 \wedge n \equiv 0 \bmod 3 \wedge 1 \leq m \leq \frac{2n}{9} : (1-\frac{3}{4n^2-7n+9nm+3},\frac{4n+9m-6}{4n^2-7n+9nm+3})$--
$(1-\frac{1}{2n^2-4n+6nm-3m+2},\frac{2n+6m-3}{2n^2-4n+6nm-3m+2})$--
$(1-\frac{3}{4n^2-9n+9nm-9m+6},\frac{4n+9m-6}{4n^2-9n+9nm-9m+6})$—

$n \geq 7 \wedge n \equiv 1 \bmod 3 \wedge 1 \leq m \leq \frac{2n-5}{9} : (1-\frac{3}{4n^2-5n+9nm+4},\frac{4n+9m-4}{4n^2-5n+9nm+4})$--
$(1-\frac{1}{2n^2-2n+6nm-3m+1},\frac{2n+6m-1}{2n^2-2n+6nm-3m+1})$--
$(1-\frac{3}{4n^2-7n+9nm-9m+3},\frac{4n+9m-4}{4n^2-7n+9nm-9m+3})$—

$n \geq 5 \wedge n \equiv 2 \bmod 3 \wedge 0 \leq m \leq \frac{2n-7}{9} : (1-\frac{3}{4n^2-3n+9nm+5},\frac{4n+9m-2}{4n^2-3n+9nm+5})$--
$(1-\frac{1}{2n^2+6nm-3m},\frac{2n+6m+1}{2n^2+6nm-3m})$--$(1-\frac{3}{4n^2-5n+9nm-9m},\frac{4n+9m-2}{4n^2-5n+9nm-9m})$--

$n \geq 3 \wedge m = \frac{2n+3}{9} : (1-\frac{3}{6n^2-4n+3},\frac{6n-3}{6n^2-4n+3})$--$(1-\frac{3}{10n^2-8n+3},\frac{10n-3}{10n^2-8n+3})$—
$(1-\frac{1}{2n^2-2n+1},\frac{2n-1}{2n^2-2n+1})$—

$n \geq 10 \wedge m = \frac{2n-2}{9} : (1-\frac{3}{6n^2-7n+4},\frac{6n-6}{6n^2-7n+4})$--$(1-\frac{3}{10n^2-12n+5},\frac{10n-7}{10n^2-12n+5})$--
$(1-\frac{3}{10n^2-15n+5},\frac{10n-10}{10n^2-15n+5})$--$(1-\frac{1}{2n^2-3n+1},\frac{2n-2}{2n^2-3n+1})$—

$n \geq 2 \wedge m = \frac{2n-4}{9} : (1-\frac{3}{6n^2-7n+5},\frac{6n-6}{6n^2-7n+5})$--$(1-\frac{3}{10n^2-10n+4},\frac{10n-5}{10n^2-10n+4})$--
$(1-\frac{3}{10n^2-13n+4},\frac{10n-8}{10n^2-13n+4})$—$(1-\frac{1}{2n^2-3n+1},\frac{2n-2}{2n^2-3n+1})$--

Note that in the following cases the given points coincide:

$D_1(n,m)$, $n \geq 8 \wedge m = \frac{n-8}{5}$:	Points 2 and 3.
$D_1(n,m)$, $n \geq 9 \wedge m = \frac{n-4}{5}$:	Points 4 and 5.
$D_8(n,m)$, $n \geq 16 \wedge m = \frac{n-16}{11}$:	Points 3 and 4.
$D_{20}(n,m)$, $n = 4 \wedge m = 0$:	Points 1 and 4.

Also note that:

$C_0(2) = D_{14}(2,-1)$	
$C_0(4) = D_8(4,0)$	
$C_0(5) = D_{20}(4,0)$	
$C_1(n) = D_8(n,-1)$	$(n \geq 1)$
$C_2(n) = D_7(n,0)$	$(n \geq 6)$
$C_3(n) = D_1(n+1,-1)$	$(n \geq 1)$
$C_4(n) = D_1(n,0)$	$(n \geq 5)$

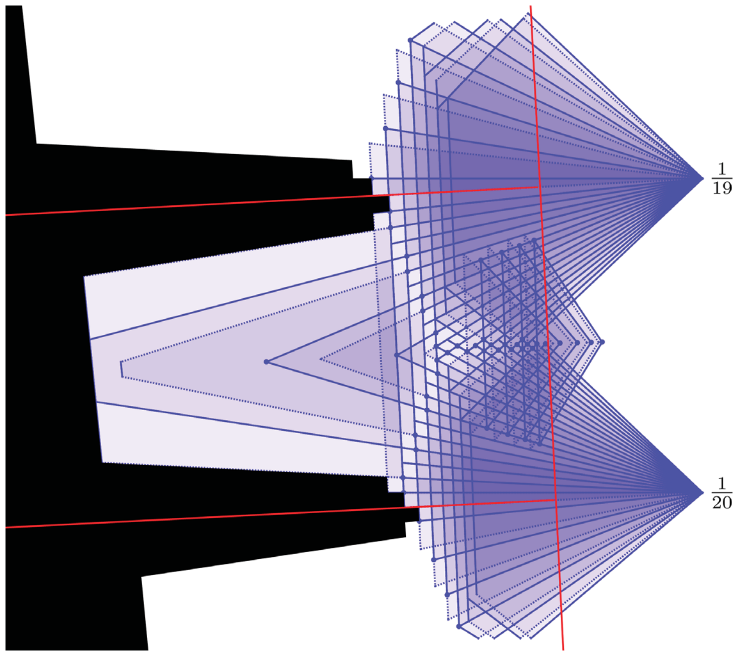
PROOF OF THEOREM 5.3.1. The following selection of cycles cuts out the whole region right of the chain, which proves that the Loudspeaker is actually contained in $\mathcal{G}_C$:

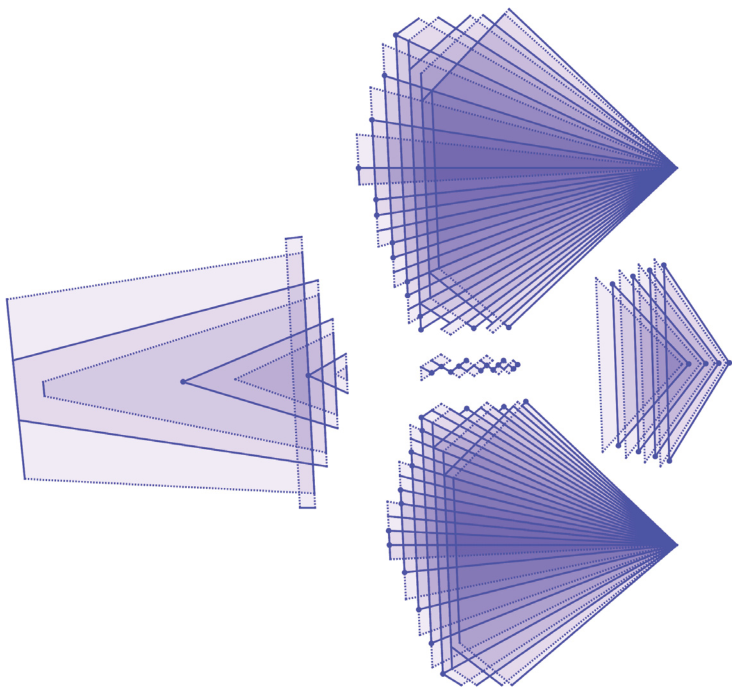
$C_0(1), C_0(3), C_0(6)$, $D_0(1), \ldots, D_0(11)$, $D_8(2,-1)$, $D_2(3,-1)$, $D_{19}(3,0)$, $D_4(6,0)$

$D_1(n,m)$:	$n \geq 2 \wedge -1 \leq m \leq \frac{n-5}{3}$	$D_{11}(n,m)$:	$n \geq 4 \wedge \frac{n-5}{5} \leq m \leq \frac{n-4}{3}$
$D_2(n,m)$:	$n \geq 4 \wedge -1 \leq m \leq \frac{n-7}{3}$	$D_{12}(n,m)$:	$n \geq 6 \wedge \frac{n-7}{5} \leq m \leq \frac{n-6}{3}$
$D_3(n,m)$:	$n \geq 5 \wedge 0 \leq m \leq \frac{n-5}{3}$	$D_{13}(n,m)$:	$n \geq 3 \wedge \frac{n-3}{5} \leq m \leq \frac{n-3}{3}$
$D_4(n,m)$:	$n \geq 7 \wedge 0 \leq m \leq \frac{n-7}{3}$	$D_{14}(n,m)$:	$n \geq 6 \wedge \frac{n-7}{5} \leq m \leq \frac{n-6}{3}$
$D_5(n,m)$:	$n \geq 2 \wedge 0 \leq m \leq \frac{n-2}{3}$	$D_{15}(n,m)$:	$n \geq 6 \wedge \frac{n-7}{5} \leq m \leq \frac{n-6}{3}$
$D_6(n,m)$:	$n \geq 4 \wedge 0 \leq m \leq \frac{n-4}{3}$	$D_{16}(n,m)$:	$n \geq 7 \wedge \frac{n-4}{5} \leq m \leq \frac{n-4}{3}$
$D_7(n,m)$:	$n \geq 5 \wedge 0 \leq m \leq \frac{n-5}{5}$	$D_{17}(n,m)$:	$n \geq 5 \wedge \frac{n-5}{5} \leq m \leq \frac{n-5}{3}$
$D_8(n,m)$:	$n \geq 3 \wedge -1 \leq m \leq \frac{n-8}{5}$	$D_{18}(n,m)$:	$n \geq 8 \wedge \frac{n-4}{5} \leq m \leq \frac{n-5}{3}$
$D_9(n,m)$:	$n \geq 4 \wedge \frac{n-4}{5} \leq m \leq \frac{n-4}{3}$	$D_{19}(n,m)$:	$n \geq 4 \wedge \frac{1-n \bmod 3}{2} \leq m \leq \frac{2n-2(n \bmod 3)-5}{9}$
$D_{10}(n,m)$:	$n \geq 5 \wedge \frac{n-6}{5} \leq m \leq \frac{n-5}{3}$	$D_{20}(n,m)$:	$n \geq 5 \wedge \frac{2-n \bmod 3}{2} \leq m \leq \frac{2n-4(n \bmod 3)+1}{9}$

□

The figures below show a regular sector (where the polygons of the infinite families are sufficient to cut out the respective part). In the first figure it can be seen for $n = 19$ that the whole region outside $\mathcal{G}_C$ in the sector $\frac{1}{n+1} < \arctan\phi \leq \frac{1}{n}$ of the unit disk is being cut out. It can be shown by comparing the coordinates of the vertices of the polygons that this is the case for every $n \geq 6$. The subsequent figures show the polygons moved apart in groups to illustrate how the polygons fit together. It can be seen that the dotted lines of one group hit soild ones of the other group and vice versa, and that single missing points are also complemented. Note that the polygons from the families 19 and 20 are needed to cut out a small region remaining in the respective sector if only the families one to 18 are considered. In fact a single (but not arbitrary) polygon of these two families would be sufficient.





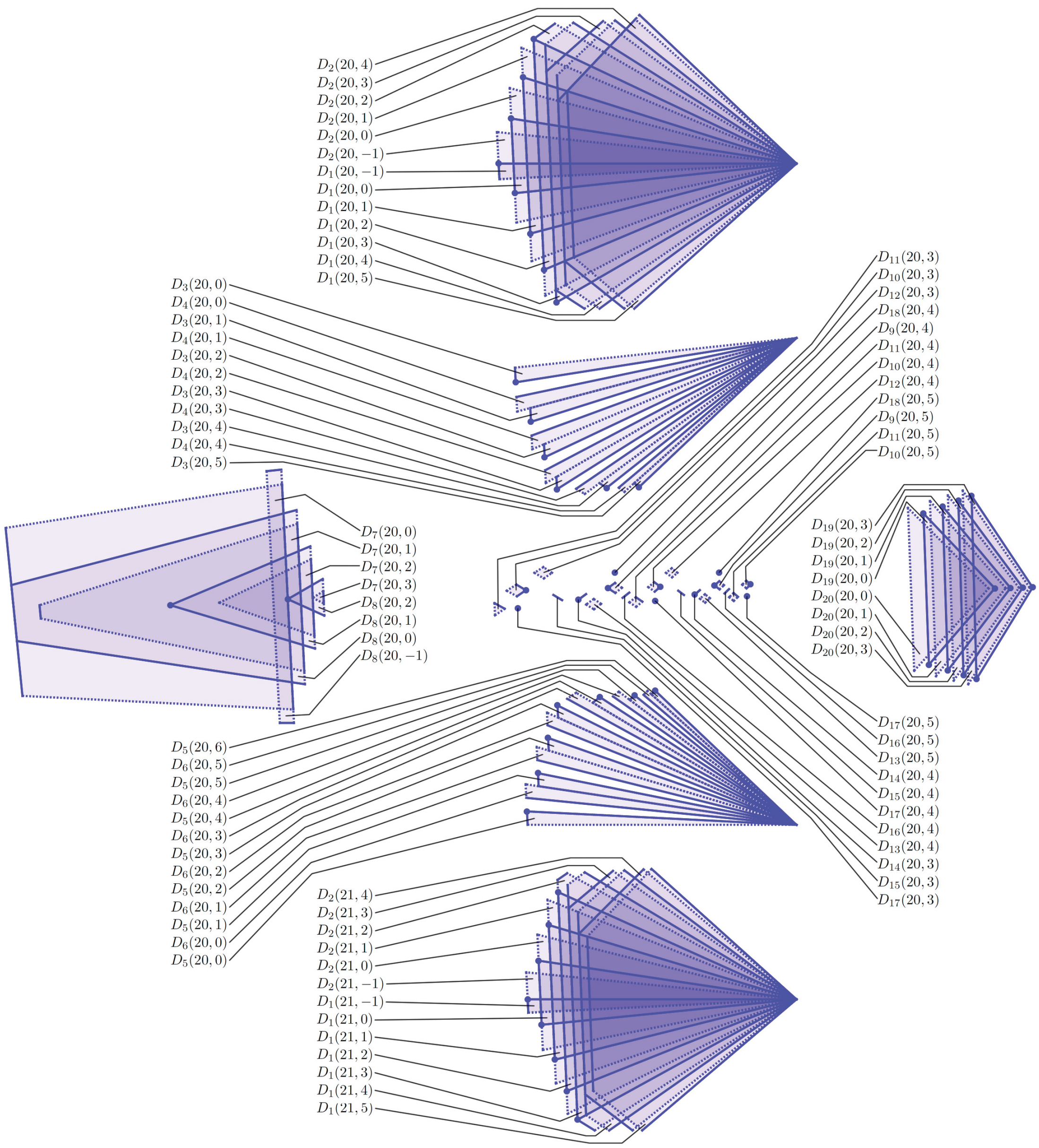


FIGURE 2. 20 families of cutout polygons.

5.4. Critical points

One consequence of Theorem 5.3.1 is the following corollary.

COROLLARY 5.4.1. [**Weitzer, 2015b**] *$r = 1$ is the only critical and $r = 1$, $r = \mathrm{i}$, and $r = -\mathrm{i}$ are the only weakly critical points of $\mathcal{G}_1^{(0)}$ satisfying $r \in \overline{\mathcal{G}_1^{(0)}}$.*

PROOF. For $n \in \mathbb{N}$ the line through $P_5(n)$ and $P_6(n)$ hits the origin and has a gradient of $\frac{1}{n}$. Let $r = (x, y) \in \mathbb{C}$ such that $|r| = 1$ and $0 < y(n-1) \leq x$, and $z = (a, b) \in \mathbb{Z}[\mathrm{i}]$ such that $|a| + |b| \leq n$ and $\max\{|a|, |b|\} < n$. Then r lies on the unit circle in the sector between the real axis and the line through $P_5(n-1)$ and $P_6(n-1)$ and one can deduce the following cases for the product $rz = (xa - yb, xb + ya)$:

$$\begin{array}{lllllll} a > 0 & \wedge & b \geq 0 & \Rightarrow & a-1 \leq xa - yb < a & \wedge & b < xb + ya < b+1 \\ a \leq 0 & \wedge & b > 0 & \Rightarrow & a-1 < xa - yb < a & \wedge & b-1 \leq xb + ya < b \\ a < 0 & \wedge & b \leq 0 & \Rightarrow & a < xa - yb \leq a+1 & \wedge & b-1 < xb + ya < b \\ a \geq 0 & \wedge & b < 0 & \Rightarrow & a < xa - yb < a+1 & \wedge & b < xb + ya \leq b+1 \end{array}$$

So the product, which is just z rotated by the argument of r, is contained in the unit square lying next to z in rotational direction. This implies a specific behavior of $\gamma_r^2(z) = \lceil r \lfloor rz \rfloor \rceil$ if $(a < 0 \vee b < 0) \Rightarrow |a| + |b| < n$:

$$\begin{array}{llllllllllll} a > 1 & \wedge & b \geq 0 & \Rightarrow & \gamma_r^2(z) = z + (-1, 1) & & a = 1 & \wedge & b \geq 0 & \Rightarrow & \gamma_r^2(z) = z + (-1, 0) \\ a \leq 0 & \wedge & b > 1 & \Rightarrow & \gamma_r^2(z) = z + (-1, -1) & & a \leq 0 & \wedge & b = 1 & \Rightarrow & \gamma_r^2(z) = z + (0, -1) \\ a < 0 & \wedge & b \leq 0 & \Rightarrow & \gamma_r^2(z) = z + (1, -1) \\ a \geq 0 & \wedge & b < 0 & \Rightarrow & \gamma_r^2(z) = z + (1, 1) \end{array}$$

Therefore the orbits of $(n-1, 1)$ and $(-n+1, 0)$ both end up in $(0,0)$ and cover the set

$$M_n := \{(a,b) \in \mathbb{Z}[\mathrm{i}] \mid |a| + |b| \leq n \wedge ((a \leq 0 \vee b \leq 0) \Rightarrow |a| + |b| < n)\}.$$

The figure below shows the orbits in black for $n = 10$.

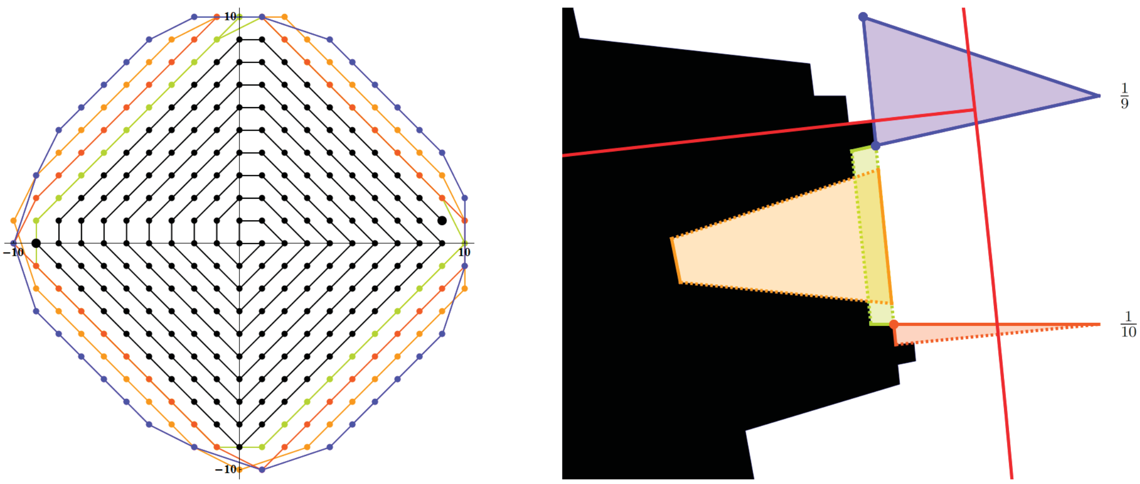


FIGURE 3. Orbits of $(9,1)$ and $(-9,0)$.

A case analysis shows that $\left\|\gamma_{\lambda r}^2(z)\right\|_1 \leq \left\|\gamma_r^2(z)\right\|_1$ for any $z \in M_n$ and $\lambda \in [0,1]$ which implies that the orbit of any element of M_n ends up in $(0,0)$ even if $|r| \leq 1$. In conclusion:

$$\forall\, n \in \mathbb{N} : \exists\, m \in \mathbb{N} : \forall\, r = (x,y) \in \mathbb{C} : (0 < y(n-1) \leq x \ \wedge\ |r| \leq 1 \Rightarrow \gamma_r^m(M_n) = \{(0,0)\})$$

$(m = 2n^2 - n - 1$ possible).

M_n is even maximal with respect to this property in the sense that there is no proper superset of M_n that is connected where two Gaussian integers are considered neighbors if their distance is 1. This can also be seen in the figure above which shows the cycles $C_1(n), \ldots C_4(n)$ (almost) encasing M_n. The missing point $(0,n)$ can be proven to be mapped to $(-1, n-1)$ under γ_r^2 for $r = \frac{1}{2}(P_3(n) + P_4(n))$ which belongs to the polygon corresponding to $C_1(n)$. Since $(-1, n-1)$ is contained in $C_1(n)$, the orbit of $(0,n)$ does not end up in $(0,0)$, which closes the gap.

Though noteworthy, the maximality of M_n is not needed to deduce that all cycles of any r having the properties above have empty intersection with M_n. They are forced to grow beyond all bounds as n increases and thus infinitely many cycles are needed to cut out, say, the set

$\overline{\{P_4(n) \mid n \geq 3\}}$ from the Loudspeaker which is actually being cut out entirely. It follows that $1 \in \overline{\mathcal{G}_1^{(0)}}$ is a critical point.

Since for all $(a, b) \in \mathbb{Z}[\mathrm{i}]$

$$((a, b), (b, -a), (-a, -b), (-b, a))$$

is a cycle of $(0, 1)$ and

$$((a, b), (-b, a), (-a, -b), (b, -a))$$

is a cycle of $(0, -1)$, it follows that $\mathrm{i} \in \overline{\mathcal{G}_1^{(0)}}$ and $-\mathrm{i} \in \overline{\mathcal{G}_1^{(0)}}$ are weakly critical points. It follows from Theorem 5.3.1 that the intersection of the topological closure of $\mathcal{G}_1^{(0)}$ and the boundary of $\mathcal{G}_1$ (unit circle) consists of these three points which implies that there are no other critical or weakly critical points in $\overline{\mathcal{G}_1^{(0)}}$ as all weakly critical points in $\overline{\mathcal{G}_1^{(0)}}$ lie on the boundary of $\mathcal{G}_1$ (Theorem 5.1.10). $\square$

5.5. The other inclusion and a more general conjecture

After proving one inclusion of Conjecture 5.2.2 in Section 5.3 we shall now try to get a grip on the other one.

Theorem 5.5.1. [**Weitzer, 2015b**] *Let* $D := \left\{z \in \mathbb{C} \mid |z| \leq \frac{2047}{2048}\right\}$. *Then*

$$\mathcal{G}_1^{(0)} \cap D = \mathcal{G}_C \cap D.$$

Proof. The result could be achieved by application of the GSRS analogues (cf. end of Section 5.1) of Algorithm 1 and Algorithm 2 of Chapter 4 (cf. proof of Theorem 4.5.1). $\square$

Corollary 5.5.2. (**Weitzer**) *Let* $D := \left\{z \in \mathbb{C} \mid \frac{2047}{2048} < |z| \leq 1\right\} \cap \left\{(x, y) \in \mathbb{C} \mid |y| \leq \frac{x}{31}\right\} \cap \mathcal{G}_C$. *Then*

$$\mathcal{G}_1^{(0)} \cap (\mathbb{C} \setminus D) = \mathcal{G}_C \cap (\mathbb{C} \setminus D).$$

Proof. Follows mostly from Theorem 5.3.1 and from Theorem 5.5.1. Missing parts in the neighborhood of $(0, 1)$ and $(0, -1)$ could again be settled by application of the GSRS analogues of Algorithm 1 and Algorithm 2 of Chapter 4. $\square$

The disk D contains all pikes up to and including the 30th. Since the 8th pike is already regular and the general regular structure of the Loudspeaker is therefore verified for quite many pikes, it appears reasonable to assume that Conjecture 5.2.2 is in fact true. Despite best efforts a general proof could not be given by now. Yet in addition to the computational evidence there are other observations supporting the believe in the truth of Conjecture 5.2.2. To understand them we recapitulate what it would mean if the conjecture were true. Since $\mathcal{G}_d^{(0)} = \mathcal{G}_d \setminus \bigcup_{\pi \in \mathcal{C}_d^{(\mathbb{Z}[\mathrm{i}])} \setminus \{(\mathbf{0})\}} P_{\mathbb{C}}(\pi)$ by Lemma 5.1.6 it means that the cutout polygon of every cycle has empty intersection with $\mathcal{G}_C$. Cutout polygons can be computed by Lemma 5.1.8. For $d = 1$ it states the following for all $(a, b), (A, B) \in \mathbb{Z}[\mathrm{i}]$:

$$\left\{(x, y) \in \mathbb{C} \mid \gamma_{(x,y)}((a, b)) = (A, B)\right\} = \{(x, y) \in \mathbb{C} \mid 0 \leq xa - yb + A < 1 \wedge 0 \leq xb + ya + B < 1\}.$$

and thus we define

Definition 5.5.3. *For* $z = (a, b), Z = (A, B) \in \mathbb{Z}[\mathrm{i}]$ *let*

$$S(z, Z) := \{(x, y) \in \mathbb{C} \mid 0 \leq xa - yb + A < 1 \wedge 0 \leq xb + ya + B < 1\}.$$

So $S(z, Z)$ consists of exactly those parameters the corresponding Gaussian Shift Radix Systems of which map z to Z. The reason why we use the letter S to denote this set is because it is in fact a half-open square in arbitrary position with side length $\frac{1}{|z|}$ (if $z \neq 0$, otherwise it is equal to $\mathbb{C}$ or $\emptyset$ if $Z = 0$ or $Z \neq 0$ respectively). If $\mathcal{G}_C \subseteq \mathcal{G}_1^{(0)}$ holds then for every cycle the intersection of all those squares (one for each step in the cycle) has to have empty intersection with $\mathcal{G}_C$. But observation supports that an even stronger property seems to hold at least for a certain subset of $\mathcal{G}_C$.

DEFINITION 5.5.4. *For $n \in \mathbb{N}_0$ let*

$$S_n := \{(x, y) \in \mathbb{C} \mid x - ny \geq 0 \wedge -x + (n+1)y > 0 \wedge (-n-1)x - 2y + n + 1 \geq 0\}.$$

S_n shall be referred to as the nth sector or sector n. Furthermore let

$$\mathcal{G}_C{}' := \bigcup_{n \in \mathbb{N}_0} S_n.$$

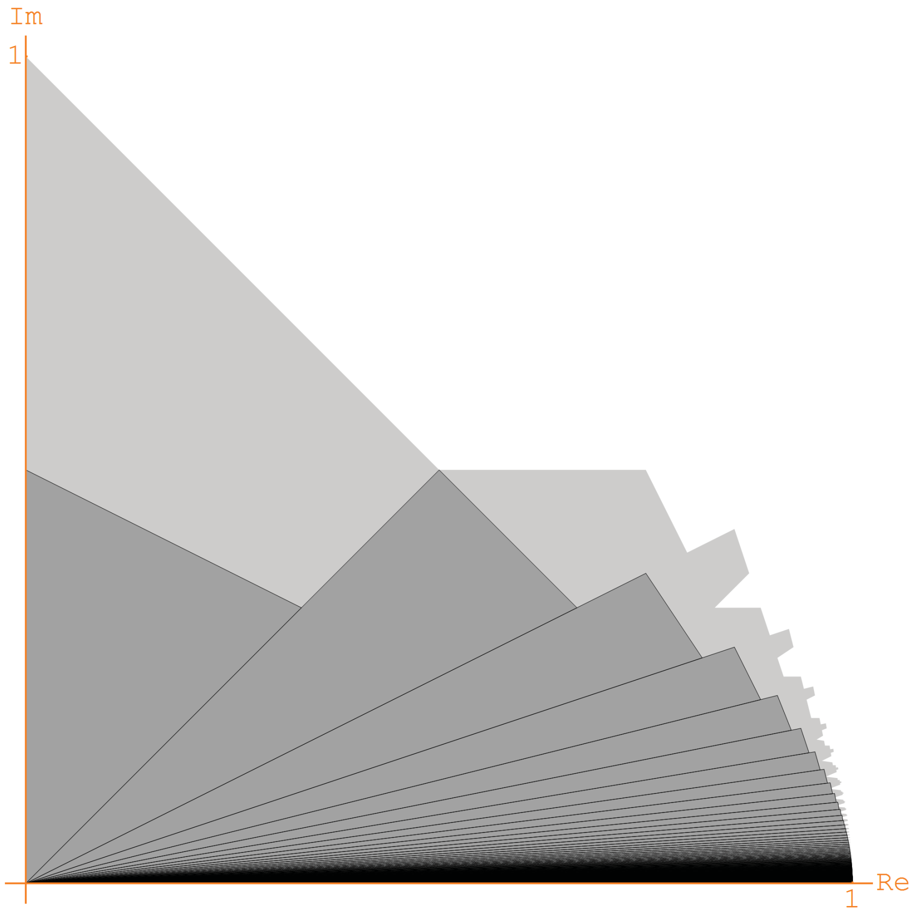


FIGURE 4. $\mathcal{G}_C{}'$ (dark gray) inside $\mathcal{G}_C$ (light gray) and the sectors S_n.

Note that sector n lies between pike n and pike $n+1$. $\mathcal{G}_C{}'$ is almost identical to $\mathcal{G}_C$ (at least in the regular sectors but the initial irregular ones are already settled anyway) but all the pikes are being cut off along the edge between $P_{10}(n)$ and $P_1(n+1)$. 5898030 randomly chosen, distinct cycles satisfied the following conjecture.

CONJECTURE 5.5.5. (**Weitzer**) *Let $\pi = (a_1, \ldots, a_k) \in \mathcal{C}_1^{(\mathbb{Z}[i])}$ non-trivial such that $P_{\mathbb{C}}(\pi) \neq \emptyset$. Then there exists an $i \in [\![1, k]\!]$ such that*

$$S(a_i, a_{i\%k+1}) \cap \mathcal{G}_C{}' = \emptyset.$$

So among all the squares which contribute to the cutout polygon of a cycle there seems to be always at least one which does not intersect with $\mathcal{G}_C{}'$. If the conjecture from above holds this would of course imply that $\mathcal{G}_C{}' \subseteq \mathcal{G}_1^{(0)}$. But analysis of the same 5898030 cycles suggests an even stronger conjecture which does explicitly state where in the cycle such a decisive step leading to an empty intersection with $\mathcal{G}_C{}'$ is to be found.

Conjecture 5.5.6. (**Weitzer**) *Let* $\pi = (a_1, \ldots, a_k) \in \mathcal{C}_1^{(\mathbb{Z}[\mathrm{i}])}$ *non-trivial and* $n \in \mathbb{N}_{\geq 4}$ *such that*

$$P_{\mathbb{C}}(\pi) \cap \{(x,y) \in \mathbb{C} \mid x - ny \geq 0 \wedge -x + (n+1)y > 0 \wedge |(x,y)| \leq 1\} \neq \emptyset.$$

Then there exists an $i \in [\![1,k]\!]$ *such that for* $a_i = (a,b)$ *and* $a_{i\%k+1} = (A,B)$ *one of the following statements holds:*

(i) $a > 0 \wedge b \geq 0 \wedge (n+1)|a| \leq 2|b| \wedge |b| \leq |B|$

(ii) $a \leq 0 \wedge b > 0 \wedge (n+1)|b| \leq 2|a| \wedge |a| < |A|$

(iii) $a < 0 \wedge b \leq 0 \wedge (n+1)|a| \leq 2|b| \wedge |b| < |B|$

(iv) $a \geq 0 \wedge b < 0 \wedge (n+1)|b| \leq 2|a| \wedge |a| \leq |A|$.

In all four cases we get $S(a_i, a_{i\%k+1}) \cap \mathcal{G}_C{}' = \emptyset$ and thus Conjecture 5.5.6 implies Conjecture 5.5.5. For $n \leq 3$ counterexamples can be found (which still respect Conjecture 5.5.5) but this region is already settled anyway. Also, for all $4 \leq n \leq 64$ there have been found cycles which contain only a single element for which the condition of Conjecture 5.5.5 is met and this single element also always met the condition of Conjecture 5.5.6. We close the section with the following theorem which provides first results on Conjecture 5.5.6.

Theorem 5.5.7. (**Weitzer**) *Let* $\pi = (a_1, \ldots, a_k) \in \mathcal{C}_1^{(\mathbb{Z}[\mathrm{i}])}$ *non-trivial and* $n \in \mathbb{N}_{\geq 6}$ *such that*

$$P_{\mathbb{C}}(\pi) \cap \{(x,y) \in \mathbb{C} \mid x - ny \geq 0 \wedge y > 0\} \neq \emptyset.$$

Then there exists an $i \in [\![1,k]\!]$ *such that for* $a_i = (a,b)$ *one of the following statements holds:*

(i) $a > 0 \wedge b \geq 0 \wedge (n+1)|a| \leq 2|b|$

(iii) $a < 0 \wedge b \leq 0 \wedge (n+1)|a| \leq 2|b|$

and there exists an $i \in [\![1,k]\!]$ *such that for* $a_i = (a,b)$ *one of the following statements holds:*

(ii) $a \leq 0 \wedge b > 0 \wedge (n+1)|b| \leq 2|a|$

(iv) $a \geq 0 \wedge b < 0 \wedge (n+1)|b| \leq 2|a|$.

Proof. We prove only the first part of the statement as the second one follows completely analogously. Assume that there is no $i \in [\![1,k]\!]$ such that (i) holds and let $(x,y) \in P_{\mathbb{C}}(\pi)$ with

$$0 < y \leq \frac{x}{n}.$$

At first we observe that for geometric reasons $k \geq 3$ ($\gamma^2_{(x,y)}$ turns its input roughly by $2\tan(1/n)$) and we can assume w.l.o.g. that a_1 is contained in the first and a_3 in the second quadrant. If $a_1 = (a,b)$ we get by our assumption that

$$b < \frac{n+1}{2}a.$$

Next we use Lemma 5.1.22 to compute that

$$P_{\mathbb{C}}\left(\Pi_{\left(\frac{1}{2},\frac{1}{4}\right)}\right) \cup P_{\mathbb{C}}\left(\Pi_{\left(\frac{1}{4},\frac{1}{2}\right)}\right) \cup P_{\mathbb{C}}\left(\Pi_{\left(\frac{1}{4},\frac{1}{4}\right)}\right) = \{(x,y) \in \mathbb{C} \mid x > 0 \wedge y > 0 \wedge y < -x + 1\}$$

and since $\left\{\left(\frac{1}{2},\frac{1}{4}\right), \left(\frac{1}{4},\frac{1}{2}\right), \left(\frac{1}{4},\frac{1}{4}\right)\right\} \subseteq \mathcal{G}_1^{(0)}$ we get

$$y \geq -x + 1.$$

Together with $y \leq \frac{x}{n}$ this gives

$$x \geq \frac{n}{n+1}.$$

Assume that a_2 lies in the fourth quadrant. Then

$$\Re(a_2) = \Re\left(\gamma_{(x,y)}(a,b)\right) = \Re\left(-\lfloor xa - yb \rfloor, -\lfloor xb + ya \rfloor\right) = -\lfloor xa - yb \rfloor \geq 0$$

and thus

$$xa - yb < 1 \Rightarrow xa - y\frac{n+1}{2}a < 1 \Rightarrow xa - \frac{x}{n}\frac{n+1}{2}a < 1 \Rightarrow ax\left(1 - \frac{n+1}{2n}\right) < 1$$
$$\Rightarrow a\frac{n}{n+1}\left(1 - \frac{n+1}{2n}\right) < 1 \Rightarrow a < 2\frac{n+1}{n-1} \Rightarrow a \leq 2 \wedge b \leq n.$$

If $a < 2$ or $b < n$ we get by the proof of Corollary 5.4.1 that the orbit of (a, b) under $\gamma_{(x,y)}$ ends up in 0 which is a contradiction. Therefore $a = 2$ and $b = n$. But then we get

$$2x - ny < 1 \Rightarrow y > \frac{2x-1}{n}$$

which implies together with $y \leq \frac{x}{n}$ that

$$x < 1 \wedge y < \frac{1}{n} < \frac{1}{2} \Rightarrow x < \frac{-2y+n+1}{n} \Rightarrow nx + 2y < n + 1$$
$$\Rightarrow \Im(a_2) = \Im\left(\gamma_{(x,y)}(2, n)\right) = -\lfloor nx + 2y \rfloor \geq -n.$$

Furthermore

$$y \leq \frac{x}{n} < \frac{2x}{n} \Rightarrow 0 < 2x - ny < 1 \Rightarrow \Re(a_2) = 0.$$

Thus the proof of Corollary 5.4.1 implies that the orbit of a_2 under $\gamma_{(x,y)}$ ends up in 0 which is also a contradiction. So a_2 does not lie in the fourth but in the third quadrant. Assume that $i = 2$ does not satisfy (iii). Then by the same argument as before we get that a_3 lies in the first quadrant which is a contradiction. Thus $i = 2$ satisfies (iii) which completes the proof. □

5.6. Properties of $\mathcal{G}_C$

The perimeter of $\mathcal{G}_C$ is two times the sum of all distances of successive vertices of the boundary of the intersection of $\mathcal{G}_C$ and the first quadrant.

THEOREM 5.6.1. [**Weitzer, 2015b**] *The perimeter of $\mathcal{G}_C$ is given by*

$$2\sum_{n=8}^{\infty}\left(\frac{(n-2)\sqrt{n^2+1}}{(n^2-n-1)(n^2-2)} + \frac{\sqrt{n^2+1}}{(n^2+1)(n^2+n+1)} + \frac{\sqrt{n^2+4}}{(n^2+2)(n^2+n+2)} + \frac{\sqrt{n^2-2n+2}}{n^4-2n^3+n} + \right.$$
$$\frac{\sqrt{n^2+1}}{n^4+5n^2+6} + \frac{\sqrt{n^6+n^4}}{n^6+n^4} + \frac{(n-1)\sqrt{n^2+9}}{(n^2+3)(n^2+n+6)} + \frac{\sqrt{n^2+2n+2}}{(n^2+n+1)(n^2+n+2)} +$$
$$\left.\frac{(n-7)\sqrt{n^2+2n+5}}{(n^2+n+6)(n^2+2n-1)}\right) - \frac{\pi^2}{3} + 6 + \frac{3845467959583\sqrt{2}}{2154669737220} + \frac{48281\sqrt{5}}{270270} + \frac{28279\sqrt{10}}{311220} +$$
$$\frac{\sqrt{13}}{77} + \frac{2789\sqrt{17}}{79560} + \frac{18018457\sqrt{26}}{1214863650} + \frac{\sqrt{29}}{432} + \frac{\sqrt{34}}{322} + \frac{3453570319189\sqrt{37}}{335814194609712} + \frac{\sqrt{53}}{1479} + \frac{3\sqrt{58}}{806} + \frac{\sqrt{65}}{1653}$$

which is approximately 7.03170158145510089909924300354696922102694215887253(7).

The area can easily be calculated using the fact that $\mathcal{G}_C$ is star-shaped with respect to the origin which can be seen in Figure 5. The total area is just two times the sum of the areas of all triangles where two vertices are successive vertices of the boundary of the intersection of $\mathcal{G}_C$ and the first quadrant and the third one is $(0, 0)$.

THEOREM 5.6.2. [**Weitzer, 2015b**] *If ψ denotes the digamma function then the area of $\mathcal{G}_C$ is given by*

$$\frac{1}{2}\Bigg(\psi(9-i)+\psi(9+i)-\frac{1}{3}\left(3-i\sqrt{3}\right)\psi\left(\frac{1}{2}\left(17-i\sqrt{3}\right)\right)-\frac{1}{3}\left(3+i\sqrt{3}\right)\psi\left(\frac{1}{2}\left(17+i\sqrt{3}\right)\right)-$$
$$\psi\left(9-\sqrt{2}\right)-\psi\left(9+\sqrt{2}\right)-\frac{1}{2}\psi\left(8-i\sqrt{2}\right)-\frac{1}{2}\psi\left(8+i\sqrt{2}\right)+\frac{1}{3}\psi\left(8-i\sqrt{3}\right)+$$
$$\frac{1}{3}\psi\left(8+i\sqrt{3}\right)+\frac{1}{5}\left(5-\sqrt{5}\right)\psi\left(\frac{1}{2}\left(17-\sqrt{5}\right)\right)+\frac{1}{5}\left(5+\sqrt{5}\right)\psi\left(\frac{1}{2}\left(17+\sqrt{5}\right)\right)+$$
$$\frac{1}{14}\left(7-i\sqrt{7}\right)\psi\left(\frac{1}{2}\left(17-i\sqrt{7}\right)\right)+\frac{1}{14}\left(7+i\sqrt{7}\right)\psi\left(\frac{1}{2}\left(17+i\sqrt{7}\right)\right)-$$
$$\frac{1}{69}\left(23-i\sqrt{23}\right)\psi\left(\frac{1}{2}\left(17-i\sqrt{23}\right)\right)-\frac{1}{69}\left(23+i\sqrt{23}\right)\psi\left(\frac{1}{2}\left(17+i\sqrt{23}\right)\right)-$$
$$2\psi'(1)+\psi''(1)+\frac{6459645509579599739}{831140131659037200}\Bigg)$$

which is approximately 1.16162449638415389252015605647076743460822751979981(8).

The following figure shows that the boundary of $\mathcal{G}_C$ can be completely described using segments of lines from six families.

DEFINITION 5.6.3.

$$\begin{aligned}
\mathcal{F}_\infty(n) &:= \{(0,1/n)+t(1,0)\mid t\in\mathbb{R}\}=\{(x,y)\in\mathbb{C}\mid ny=1\}, && n\in\mathbb{Z}\setminus\{0\}\\
\mathcal{F}_0(n) &:= \{(0,0)+t(n,1)\mid t\in\mathbb{R}\}=\{(x,y)\in\mathbb{C}\mid ny=x\}, && n\in\mathbb{Z}\\
\mathcal{F}_{\frac{1}{2}}(n) &:= \{1/2,0)+t(n,2)\mid t\in\mathbb{R}\}=\{(x,y)\in\mathbb{C}\mid ny=2x-1\}, && n\in\mathbb{Z}\\
\mathcal{F}_{\frac{2}{3}}(n) &:= \{2/3,0)+t(n,3)\mid t\in\mathbb{R}\}=\{(x,y)\in\mathbb{C}\mid ny=3x-2\}, && n\in\mathbb{Z}\\
\mathcal{F}_1(n) &:= \{(1,0)+t(-2,n)\mid t\in\mathbb{R}\}=\{(x,y)\in\mathbb{C}\mid 2y=-nx+n\}, && n\in\mathbb{Z}\\
\mathcal{F}_2(n) &:= \{(2,0)+t(n,-1)\mid t\in\mathbb{R}\}=\{(x,y)\in\mathbb{C}\mid ny=-x+2\}, && n\in\mathbb{Z}.
\end{aligned}$$

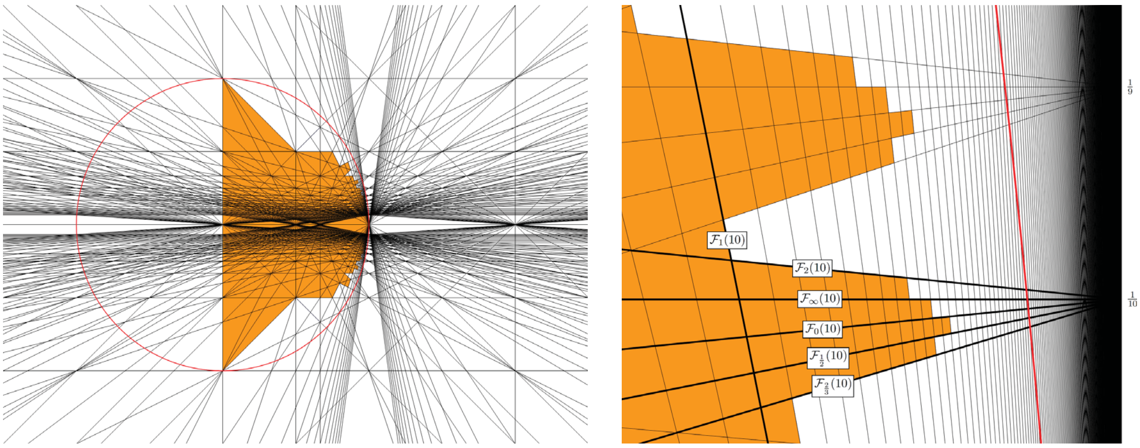


FIGURE 5. Six families of lines.

5.7. Hidden patterns - A kind of "self-similarity"

In the last section of this chapter we will explain a kind of "self-similarity" in patterns revealed by the GSRS analogue of Algorithm 1. By Lemma 5.1.24, which is what the algorithm essentially bases on, $\mathcal{G}_1^{(0)}$ is given by the disjoint union $\bigcup\left\{P_{\mathbb{C}}\left(\Pi_{\mathbf{r}}\right)\mid \mathbf{r}\in\mathcal{G}_1^{(0)}\right\}$. By Lemma 5.1.23 the sets

$P_{\mathbb{C}}(\Pi_{\mathbf{r}})$ are either singletons, open line segments or nondegenerate and open polygons. Figure 6 below shows a four coloring of those nondegenerate polygons covering a part of the Loudspeaker (the singletons and open line segments fill the space in between and are not shown). Figure 7 shows a magnification of the area around pike 7. It can be seen that there seems to be a repeating pattern which gets finer the closer one gets to the unit circle. That the polygons get smaller the larger the absolute value, say, of their barycenter gets, is not surprising as the corresponding sets of witnesses grow with the absolute value. The similarities between different parts on the other hand, are not so easy to comprehend. Not only does a recognizable pattern occur in different parts, but it also seems to encode the shape of the Loudspeaker itself. Figure 8 shows the tips (i.e. the quadrilateral $\operatorname{conv}(\{P_5(n), P_6(n), P_7(n), P_8(n)\})$) of the pikes 2 to 7 affinely transformed to the unit square and Figure 9 shows an overlapped image of the tip of pike 7 and the Loudspeaker. It can be seen that the boundary of the Loudspeaker follows line segments which are present in the transformed partition of the tip. The further down the tip is, the finer the partition gets and the more parts of the boundary are traced. It is this kind of “self-similarity” that is explained here.

FIGURE 6. The pattern of sets of witnesses revealed by the GSRS analogue of Algorithm 1 from Chapter 4.

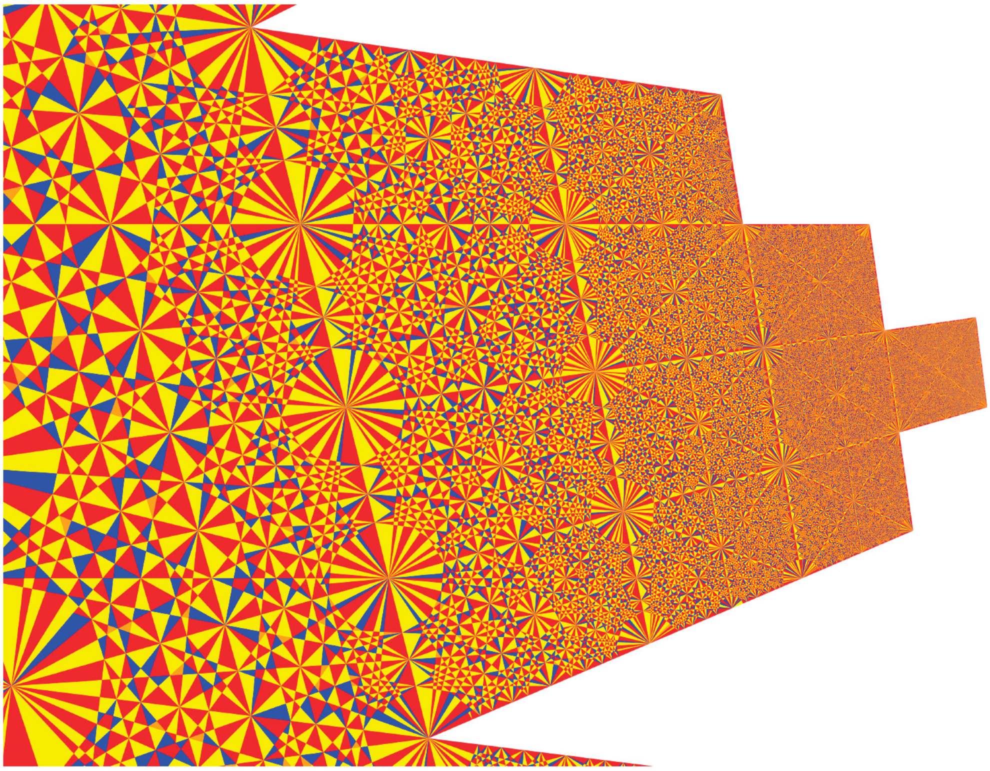


FIGURE 7. A magnification of pike 7.

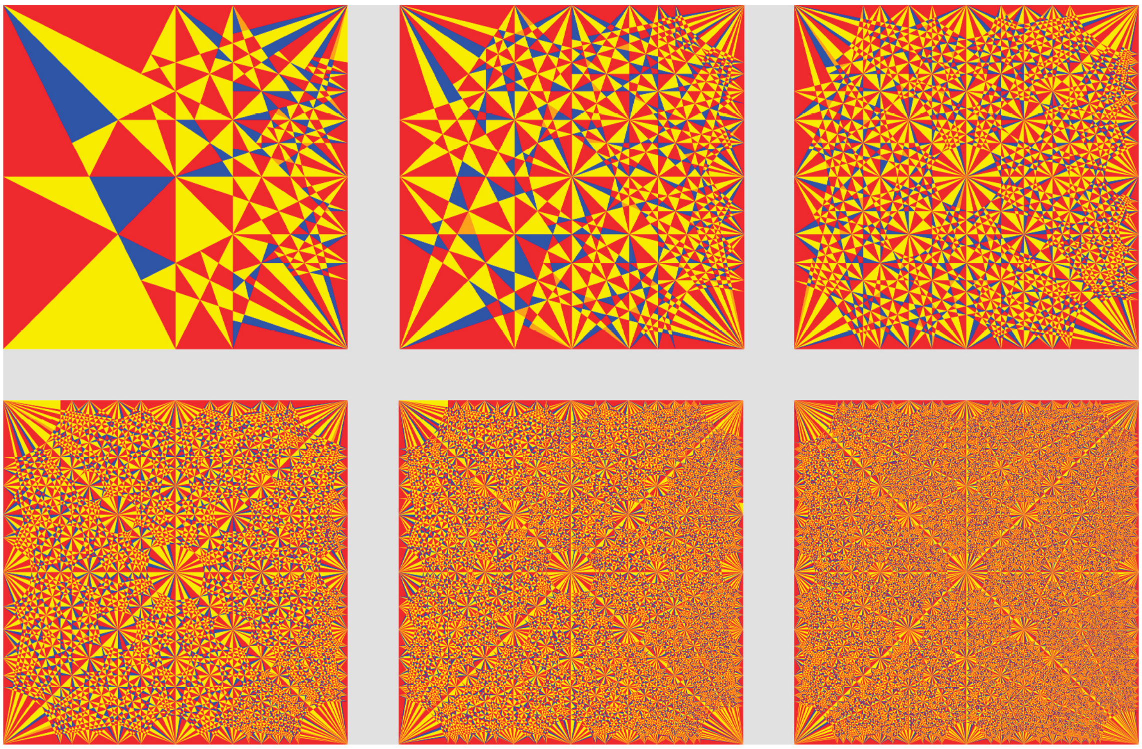


FIGURE 8. The transformed tips of pikes 2 to 7.

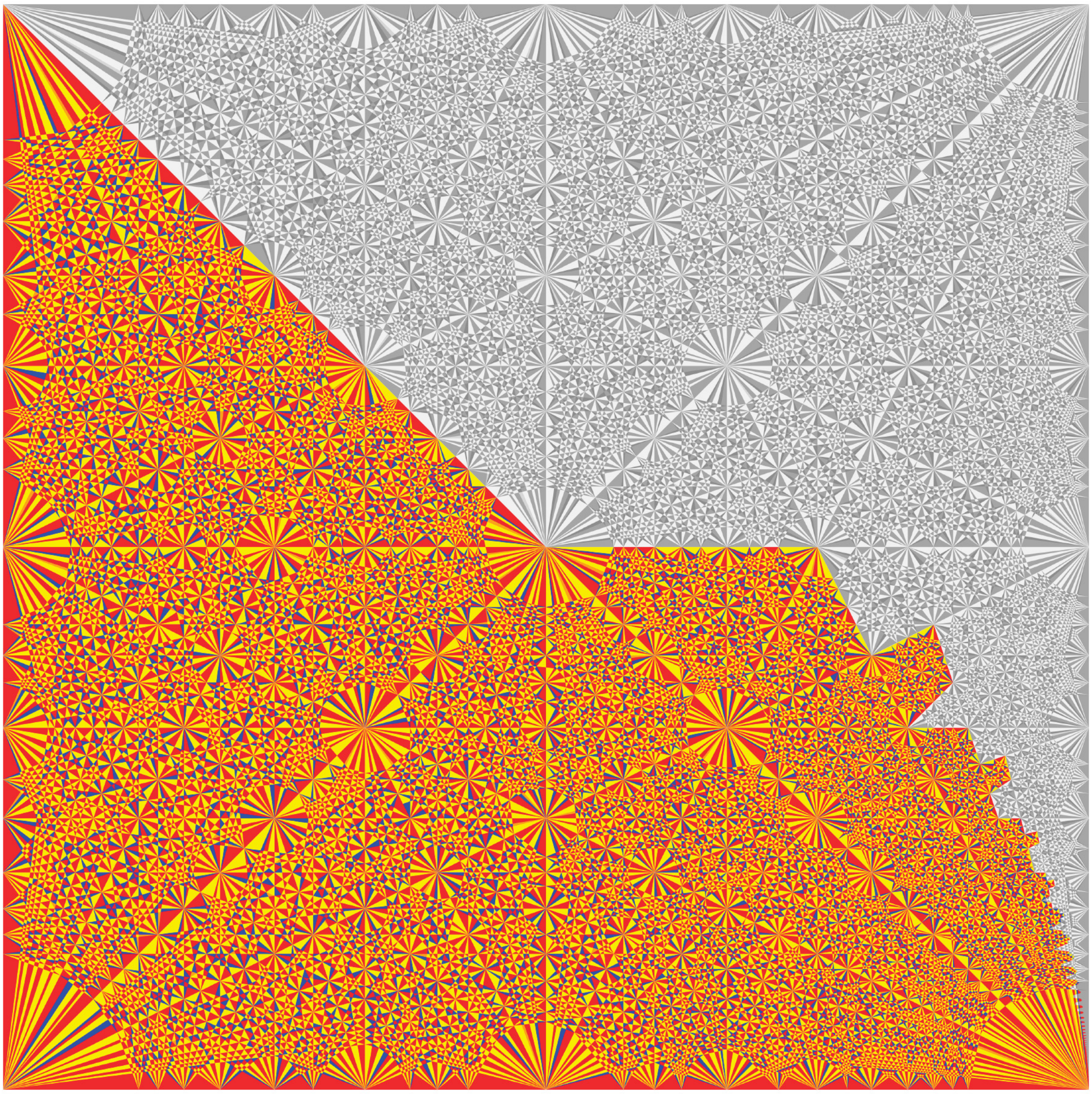


FIGURE 9. The Loudspeaker over the transformed tip of pike 7.

In the following Algorithm 1' and Algorithm 2' shall refer to the GSRS analogues of Algorithm 1 and Algorithm 2 from Chapter 4 respectively.

The main reason for the similarities of the patterns in these quadrilateral regions of the Loudspeaker is that the sets V_r and V_s for any parameters $r, s \in \mathbb{C}$ are likely to be of similar size and shape if the parameters are close to each other. Most sets of witnesses are more or less circular and the diameter of the smallest enclosing circle centered at the origin is strongly dependent on the absolute value of the corresponding parameter. The closer it gets to the unit circle, the larger the set of witnesses will be in general.

Now consider a quadrilateral $H = \operatorname{conv}(\{r_1, \ldots, r_4\}) \subseteq \operatorname{int}(\mathcal{G}_1)$ and assume that $r_1, \ldots, r_4$ are in either clockwise or counter-clockwise order and let

$$V := \bigcup_{r \in H} V_r.$$

If H is not too big (with respect to its position - the closer it is to the unit circle, the smaller it must be) then V won't be much different from any of the V_r, $r \in H$ and the decomposition of H

computed by Algorithm 2' won't be much different from the one computed by Algorithm 1'. In the case of the Loudspeaker the equivalence relation Algorithm 2' bases on (Definiton 5.1.25) is given by

$$\sim := \left\{(r,s) \in \mathbb{C}^2 \mid \forall\, i \in [\![1,4]\!] : \forall\, a \in V : \gamma_r^{(i)}(a) = \gamma_s^{(i)}(a)\right\}.$$

The equivalence classes are again either singletons, open line segments or nondegenerate and open polygons. In the following we will give another interpretation of these classes but before we can do so we need a few auxiliary definitions.

Definition 5.7.1. V_V *denotes the set of all singletons,* E_V *the set of all open line segments and* F_V *the set of all nondegenerate and open polygons in* $\mathbb{C}/_\sim$*.*

Definition 5.7.2. *For an embedding* $\mathcal{E} \subseteq \mathbb{C}$ *of some planar graph* $\mathrm{V}(\mathcal{E})$*,* $\mathrm{E}(\mathcal{E})$*,* $\mathrm{F}(\mathcal{E})$ *denote the sets of those subsets of* $\mathbb{C}$*, which correspond to the vertices (singletons), edges (simple arcs excluding end points) and faces (connected components of the complement of the union of all vertices and edges).*

Definition 5.7.3. *For* $(a,b) \in \mathbb{Z}[\mathrm{i}]$ *let*

$$\begin{aligned}\mathcal{E}_{(a,b)} &:= \{(x,y) \in \mathbb{C} \mid \exists\, c \in \mathbb{Z} : (a,b,c) \neq \mathbf{0} \wedge ax + by = c\}\\ \mathcal{E}_V &:= \bigcup_{(a,b)\in V} \mathcal{E}_{(a,b)} \cup \mathcal{E}_{(-b,a)} \cup \mathcal{E}_{(b,a)} \cup \mathcal{E}_{(-a,b)}.\end{aligned}$$

With the definitions above we get by Lemma 5.1.22 (v) (also cf. Theorem 4.3.3)

Lemma 5.7.4. (**Weitzer**) $\mathrm{V}_V = \mathrm{V}(\mathcal{E}_V)$*,* $\mathrm{E}_V = \mathrm{E}(\mathcal{E}_V)$*, and* $\mathrm{F}_V = \mathrm{F}(\mathcal{E}_V)$*.*

So the classes of $\sim$ are the vertices, edges, and faces of a planar embedding of a graph given by (at most) $4\,|V|$ infinite families of parallel and equidistant lines. This embedding is invariant under certain operations on V and also possesses several axes of symmetries. These and other properties are summarized in the next lemma.

Definition 5.7.5. *Let*

$$R(V) := \left\{\widehat{z} \mid z \in V \wedge \nexists Z \in V : \exists\, n \in \mathbb{N}_{\geq 2} : n\widehat{z} = \widehat{Z}\right\}$$

where $\widehat{z} := (\max\{|a|,|b|\}, \min\{|a|,|b|\})$ *for all* $z = (a,b) \in \mathbb{Z}[\mathrm{i}]$*.*

Lemma 5.7.6. (**Weitzer**)

(i) $\mathbb{Z}[\mathrm{i}] \subseteq \mathrm{V}(\mathcal{E}_V)$

(ii) $V \subseteq W \Rightarrow \mathcal{E}_V \subseteq \mathcal{E}_W$

In particular: $\mathrm{V}_V \subseteq \mathrm{V}_W$

$\forall\, e \in \mathrm{E}_V : \exists\, n \in \mathbb{N} : \exists\, e_1,\dots,e_n \in \mathrm{E}_W : \overline{e} = \bigcup_{i=1}^n \overline{e_i}$

$\forall\, f \in \mathrm{F}_V : \exists\, n \in \mathbb{N} : \exists\, f_1,\dots,f_n \in \mathrm{F}_W : \overline{f} = \bigcup_{i=1}^n \overline{f_i}$

(iii) $\mathcal{E}_V \cap ([0,1]^2 + z) = \mathcal{E}_V \cap ([0,1]^2) + z$ *for all* $z \in \mathbb{Z}[\mathrm{i}]$

$\mathcal{E}_V = \{(x,-y) \mid (x,y) \in \mathcal{E}_V\} = \{(y,x) \mid (x,y) \in \mathcal{E}_V\}$

In particular: $\mathcal{E}_V = \{(-x,y) \mid (x,y) \in \mathcal{E}_V\}$

$\mathcal{E}_V$ *possesses the following axes of symmetries for all* $c \in \mathbb{Z}$*:*

$$x = c/2, y = c/2, x + y = c, x - y = c$$

(iv) $\mathcal{E}_V = \mathcal{E}_{\{(a,-b)\mid(a,b)\in V\}} = \mathcal{E}_{\{(b,a)\mid(a,b)\in V\}} = \mathcal{E}_{\left\{z\in V \mid \nexists Z\in V : \exists\, n\in\mathbb{N}_{\geq 2} : nz = Z\right\}}$

In particular: $\mathcal{E}_V = \mathcal{E}_{\{(-a,b)\mid(a,b)\in V\}}$

$\mathcal{E}_V = \mathcal{E}_{V\setminus\{(0)\}}$

$\mathcal{E}_V = \mathcal{E}_{R(V)}$.

Proof. Follows directly from Lemma 5.1.22 (v). □

Figure 10 shows several examples of sets of witnesses and the corresponding partition of the unit square. The axes of symmetries given in (iii) are clearly visible. Also note that by (iv) it is sufficient to give only those elements of V which lie in the first quadrant and below or on the first median and do not "divide" any other element of V.

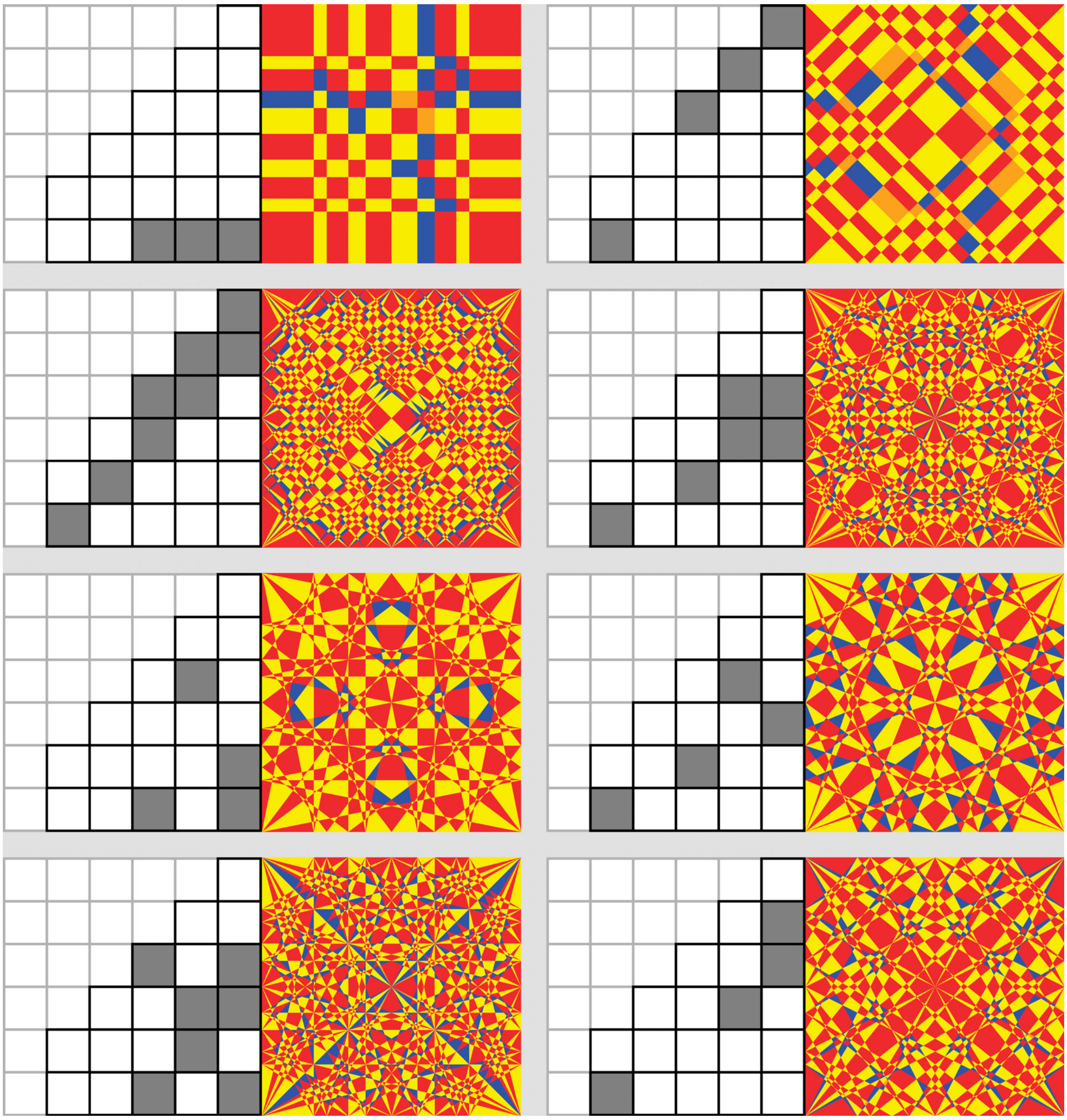


FIGURE 10. Several examples of sets of witnesses and the corresponding decomposition of the unit square computed by Algorithm 2'.

To see the reason behind the similarities of the patterns we require the following definitions.

DEFINITION 5.7.7.

$$h_1 : \mathbb{R}^2 \times (\mathbb{R} \setminus \{0\}) \to \mathbb{C} \qquad (x, y, z) \mapsto (x, y)\frac{1}{z}$$

$$h_2 : \mathbb{R}^3 \to \mathbb{C} \qquad (x, y, z) \mapsto (x, y)\frac{1}{\gcd(\{x, y, z\})}$$

Definition 5.7.8. *For* $R = \operatorname{conv}\left(\left\{(r_x^{(1)}, r_y^{(1)}), \ldots, (r_x^{(4)}, r_y^{(4)})\right\}\right)$, $S \subseteq \mathbb{C}$ *convex quadrilaterals where* $(r_x^{(1)}, r_y^{(1)}), \ldots, (r_x^{(4)}, r_y^{(4)})$ *are in either clockwise or counter-clockwise order let*

$$A_R := \begin{pmatrix} r_x^{(1)} & r_x^{(2)} & r_x^{(3)} \\ r_y^{(1)} & r_y^{(2)} & r_y^{(3)} \\ 1 & 1 & 1 \end{pmatrix} \cdot \operatorname{DiagM}\left(\operatorname{adj}\left(\begin{pmatrix} r_x^{(1)} & r_x^{(2)} & r_x^{(3)} \\ r_y^{(1)} & r_y^{(2)} & r_y^{(3)} \\ 1 & 1 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} r_x^{(4)} \\ r_y^{(4)} \\ 1 \end{pmatrix}\right)$$

where $\operatorname{DiagM}(\mathbf{v})$ *is the diagonal matrix with main diagonal* $\mathbf{v}$ *for any complex vector* $\mathbf{v}$*, and*

$$A_{R,S} := A_S \cdot \operatorname{adj}(A_R).$$

Furthermore let for all $r = (x, y) \in \mathbb{C}$

$$t_{R,S}(r) := h_1(A_{R,S} \cdot (x, y, 1)^T)$$

and for all $z \in \mathbb{Z}[\mathrm{i}]$

$$T_{R,S}(z) := \{h_2((a, b, -c) \cdot A_{S,R}) \mid c \in \mathbb{Z} \wedge \{(x, y) \in \mathbb{C} \mid ax + by = c\} \cap \operatorname{conv}(R) \neq \emptyset\}$$

$$T_{R,S}(V) := \bigcup_{(a,b) \in V} T_{R,S}((a, b)) \cup T_{R,S}((-b, a)) \cup T_{R,S}((b, a)) \cup T_{R,S}((-a, b)).$$

Note that $t_{R,S}$ is the unique affine transformation which maps the quadrilateral R to the quadrilateral S. With the above definitions we are able to formulate the following theorem which will finally explain the "self-similarities".

Theorem 5.7.9. (**Weitzer**) *Let* $R, S \subseteq \mathbb{C}$ *convex quadrilaterals. Then*

(i) $t_{R,S}(\mathcal{E}_V \cap \operatorname{conv}(R)) \subseteq \mathcal{E}_{T_{R,S}(V)} \cap \operatorname{conv}(S)$

(ii) $\nexists\, W \subseteq \mathbb{Z}[\mathrm{i}] : t_{R,S}(\mathcal{E}_V \cap \operatorname{conv}(R)) \subsetneq \mathcal{E}_W \cap \operatorname{conv}(S) \subsetneq \mathcal{E}_{T_{R,S}(V)} \cap \operatorname{conv}(S)$.

Proof. Since V is finite the set

$$N := \{(a, b), (-b, a), (b, a), (-a, b) \mid (a, b) \in V\}$$

contains a normal vector of any line $\{(x, y) \in \mathbb{C} \mid ax + by = c\} \subseteq \mathcal{E}_V$ $(a, b, c \in \mathbb{R})$. If

$$L = \{(x, y) \in \mathbb{C} \mid ax + by = c\}$$

is such a line with $(a, b) \in N$ and $c \in \mathbb{Z}$, the transformed line

$$L_t := t_{R,S}(L) = \left\{(x, y) \in \mathbb{C} \mid ((a, b, -c) \cdot A_{S,R}) \cdot (x, y, 1)^T = 0\right\})$$

is contained in $\mathcal{E}_{h_2((a,b,-c) \cdot A_{S,R})}$. It follows that

$$t_{R,S}(\mathcal{E}_V \cap \operatorname{conv}(R)) \subseteq \mathcal{E}_{T_{R,S}(V)} \cap \operatorname{conv}(S).$$

Among all representations of $L_t = \{(x, y) \in \mathbb{C} \mid Ax + By = C\}$ $(A, B, C \in \mathbb{Z})$, h_2 selects the one where $\gcd(\{A, B, C\}) = 1$ which implies the minimality of $\mathcal{E}_{T_{R,S}(V)}$ under all refinements $\mathcal{E}_W \cap \operatorname{conv}(S)$ $(W \subseteq \mathbb{Z}[\mathrm{i}])$ of $t_{R,S}(\mathcal{E}_V \cap \operatorname{conv}(R))$. □

The previous theorem explains how the given set $V \subseteq \mathbb{Z}[\mathrm{i}]$ can be transformed to another set $T_{R,S}(V) \subseteq \mathbb{Z}[\mathrm{i}]$ which is minimal under all subsets of $\mathbb{Z}[\mathrm{i}]$ which induce a refinement of the partition of the region $R \in \mathbb{C}$ inside the region $S \in \mathbb{C}$.

As an example consider R to be the tip of pike seven, so $R = \operatorname{conv}(\{P_8(7), P_7(7), P_6(7), P_5(7)\})$. The set of witnesses V_R for R is approximately disk-shaped and satisfies $\{z \in \mathbb{Z}[\mathrm{i}] \mid |z| \leq 82\} \subseteq V_R \subseteq \{z \in \mathbb{Z}[\mathrm{i}] \mid |z| < 85\}$. With $S = ((0, 0), (1, 0), (1, 1), (0, 1))$ the transformed set $R(T_{R,S}(V_R))$ is given by

$\{(5, 4), (5, 5), (6, 0), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6), (7, 0), (7, 1), (7, 2), (7, 3), (7, 4), (7, 5), (7, 6), (7, 7), (8, 0), (8, 1), (8, 2), (8, 3), (8, 4), (8, 5), (8, 6), (8, 7), (8, 8), (9, 0), (9, 1), (9, 2), (9, 3), (9, 4), (9, 5), (9, 6), (9, 7), (9, 8), (10, 0), (10, 1), (10, 2), (10, 3), (10, 4), (10, 5), (10, 6), (10, 7), (11, 0), (11, 1), (11, 2), (11, 3), (11, 4), (11, 5), (11, 6)\}$.

By Theorem 5.7.9 the partition of S corresponding to $R(T_{R,S}(V_R))$ is a refinement of the transformed (under $t_{R,S}$) partition of R corresponding to $R(V_R)$. But what is really surprising is that $R(T_{R,S}(V_R))$ seems to contain much less elements than the original $R(V_R)$ and the resulting partition is therefore - though slightly finer - almost identical to the original one. This is the very reason behind the "self-similar" patterns. Of course our choice for R was crucial. We took the tip of pike seven which we saw on the images to contain exactly the repeating pattern. But there are other good choices for R which all have in common that the corresponding transformation $T_{R,S}$ reduces the size (number of elements, but also maximum absolute value) of the corresponding set of witnesses drastically, as it did in the example above. It turns out that the quadrilaterals

$$\operatorname{conv}\left(\left\{\left(\frac{pm+n}{pm+n+1},\frac{m}{pm+n+1}\right),\left(\frac{p(m+1)+n}{p(m+1)+n+1},\frac{m+1}{p(m+1)+n+1}\right),\right.\right.$$
$$\left.\left.\left(\frac{p(m+1)+n-1}{p(m+1)+n},\frac{m+1}{p(m+1)+n}\right),\left(\frac{pm+n-1}{pm+n},\frac{m}{pm+n}\right)\right\}\right)$$

where $p,m,n\in\mathbb{N}$ work very well due to the following reason: The matrix

$$A=\begin{pmatrix} p & -1 & pm+n \\ 1 & 0 & m \\ p & -1 & pm+n+1 \end{pmatrix}$$

is a scalar multiple of $A_{S,R}$ where R is as above and $S=((0,0),(1,0),(1,1),(0,1))$. Therefore it can also be used to transform any given $(a,b)\in\mathbb{Z}[\mathrm{i}]$. Let $c\in\mathbb{Z}$ be such that

$$\{(x,y)\in\mathbb{C}\mid ax+by=c\}\cap\operatorname{conv}(R)\neq\emptyset$$

and assume w.l.o.g that $\gcd(a,b,c)=1$. Then one gets for the transformed element

$$h_2((a,b,-c)\cdot A)=(ap+b-cp,-a+c).$$

Now consider the following lemma:

Lemma 5.7.10. (**Weitzer**) *Let $P\subseteq\mathbb{C}$ finite,*

$$\begin{aligned} m &:= \lceil\min\{ap+bq\mid(p,q)\in P\}\rceil \\ M &:= \lfloor\max\{ap+bq\mid(p,q)\in P\}\rfloor, \end{aligned}$$

and $z\in\mathbb{Z}[\mathrm{i}]$. Then

(i) $\mathcal{E}_z\cap\operatorname{conv}(P)=\bigcup_{c\in[\![m,M]\!]}\{(x,y)\in\mathbb{C}\mid ax+by=c\}\cap\operatorname{conv}(P)$

(ii) $\forall\, c\in[\![m,M]\!]:\{(x,y)\in\mathbb{C}\mid ax+by=c\}\cap\operatorname{conv}(P)\neq\emptyset.$

In particular: $P=\{(0,0),(1,0),(1,1),(0,1)\}\Rightarrow\{ap+bq\mid(p,q)\in P\}=\{0,a,b,a+b\}$
$P=\left\{(0,0),(\frac{1}{2},0),(\frac{1}{2},\frac{1}{2})\right\}\Rightarrow\{ap+bq\mid(p,q)\in P\}=\frac{1}{2}\{0,a,a+b\}.$

Proof. Follows from simple geometric considerations. □

(The purpose of the second "In particular" statement gets clear when considering the axes of symmetries given in Lemma 5.7.6. To compute a partition of the unit square, it suffices to consider only the partition of the triangle $\left\{(x,y)\in\mathbb{C}\mid 0\leq x\leq\frac{1}{2}\wedge 0\leq y\leq x\right\}$.)

It follows now from the lemma that

$$\max\{|ap+b-cp|+|-a+c|-|a|-|b|\mid c\in\mathbb{Z}\wedge\{(x,y)\in\mathbb{C}\mid ax+by=c\}\cap\operatorname{conv}(R)\neq\emptyset\}\leq$$

$$\max\left\{|ap+b-cp|+|-a+c|-|a|-|b|\;\middle|\right.$$
$$c\in\left\{a\frac{pm+n}{pm+n+1}+b\frac{m}{pm+n+1},a\frac{p(m+1)+n}{p(m+1)+n+1}+b\frac{m+1}{p(m+1)+n+1},\right.$$
$$\left.\left.a\frac{p(m+1)+n-1}{p(m+1)+n}+b\frac{m+1}{p(m+1)+n},a\frac{pm+n-1}{pm+n}+b\frac{m}{pm+n}\right\}\right\}\leq 0$$

which means that the 1-norm of the transformed element is always less than or equal to the 1-norm of the original element. Thus the transformed set $R(T_{R,S}(V_R))$ is likely to be smaller than the original, more or less disk-shaped, $R(V_R)$ (based on experimental evidence it is even much smaller). So if R is a quadrilateral of the above form, the decomposition of R given by Algorithm 1' is very similar to the one given by Algorithm 2' for $V := V_R$ (or even better $V := \bigcup_{r\in\mathbb{R}} V_r$) which is again very similar to the decomposition of the unit square given by Algorithm 2' for $T_{R,S}(V_R)$. Altogether we get that the decompositions of two quadrilaterals R and S of the above form will be similar (i.e. one is - more or less - a refinement of the other) which explains the "self-similarities" in the pattern given by Algorithm 1'.

CHAPTER 6

Shift Radix Systems over imaginary quadratic Euclidean domains

6.1. Introduction and definitions

In the present chapter we will repeat basic definitions first introduced by Attila Pethő and Peter Varga which allow to define Shift Radix Systems in the sense of Chapter 3 and Chapter 5 on imaginary quadratic Euclidean domains. Subsequently we will prove first results. Most of the material presented in this chapter will be part of [**Pethő et al., IP**] which is currently in preparation. Since this is a joint work, contributions of Weitzer will be marked by (**Weitzer**) while original material of Pethő and Varga will be cited as [**Pethő et al., IP**] only.

It is well known that there are exactly five imaginary quadratic fields the ring of integers of which is Euclidean [**Motzkin, 1949**], that is $\mathbb{Q}(\sqrt{D})$ where $D \in \{-1,-2,-3,-7,-11\}$.

Definition 6.1.1. *For* $D \in \{-1,-2,-3,-7,-11\}$ *let*

$$\begin{aligned} \mathbb{E}_D &:= \mathcal{O}_{\mathbb{Q}(\sqrt{D})} \\ \omega_D &:= \begin{cases} \sqrt{D} & \text{if } D \in \{-1,-2\} \\ \frac{1+\sqrt{D}}{2} & \text{if } D \in \{-3,-7,-11\} \end{cases} \end{aligned}$$

and for all $r = x + \omega_D y \in \mathbb{C}$ *with* $x, y \in \mathbb{R}$ *let*

$$\begin{aligned} (x,y)_D &:= r \\ \Re_D(r) &:= x \\ \Im_D(r) &:= y \end{aligned}$$

the D*-real and* D*-imaginary parts of* r*.*

Note that we consider both $\mathbb{Q}(\sqrt{D})$ and $\mathbb{E}_D$ to be embedded in $\mathbb{C}$. $\mathbb{E}_D$ is a free $\mathbb{Z}$-module with integral basis $(1, \omega_D)$, i.e.

$$\mathbb{E}_D = \{(a,b)_D \mid a, b \in \mathbb{Z}\}.$$

Furthermore $(1, \omega_D)$ is a basis of $\mathbb{C}$ as a $\mathbb{R}$-vector space and thus $\Re_D(r)$ and $\Im_D(r)$ are well-defined. Indeed

$$\begin{aligned} \Re_D(r) &= \Re(r) - \Re(\omega_D)\frac{\Im(r)}{\Im(\omega_D)} \\ \Im_D(r) &= \frac{\Im(r)}{\Im(\omega_D)}. \end{aligned}$$

For the remaining part of the chapter D shall be a fixed element of $\{-1,-2,-3,-7,-11\}$ if not stated otherwise.

6.2. A floor function is needed

To be able to define Shift Radix Systems on $\mathbb{E}_D$ we need two ingredients. We recall the definition of Shift Radix Systems (or Gaussian Shift Radix System for that matter):

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_d, -\lfloor \mathbf{r}\mathbf{a} \rfloor)$$

for any $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ and $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$. What we need first is a generalization of the scalar product. This is the easy part - since we consider $\mathbb{Q}(\sqrt{D})$ and $\mathbb{E}_D$ to be embedded in $\mathbb{C}$ we simply use the same definition of $\mathbf{ra}$ that we used for Gaussian Shift Radix Systems. But what about the floor function? The purpose of the floor function is to map the outcome of the scalar product to a lattice point again while keeping the error small. There are of course many ways to achieve this, but we would also like our floor function to be more or less uniform for all possible values of D. A very general way to define a floor function is the following: Consider a subset T of $\mathbb{C}$ which induces a tiling on $\mathbb{C}$ (i.e. $\mathbb{C}$ is the disjoint union of translated copies of T) such that every copy of T contains exactly one lattice point. Then one can define the floor of any complex number to be the unique lattice point which is in the same translated copy of T. From this perspective, having more or less uniform floor functions for all possible values of D would mean that the different shapes T_D that are used to define the respective floor functions are very similar. A possible choice for the shape and the one that Pethő and Varga use is that of a rectangular "sail bent in the wind" which is why they refer to it as the sail set.

Definition 6.2.1. [**Pethő et al., IP**] *Let*

$$\begin{aligned} T_D &:= \left\{ r \in \mathbb{C} \mid |r+1| \geq 1 > |r| \wedge -\frac{1}{2} \leq \Im_D(r) < \frac{1}{2} \right\} \\ \sim_D &:= \left\{ (r,s) \in \mathbb{C}^2 \mid \exists\, t \in \mathbb{E}_D : \{r,s\} \subseteq T_D + t \right\} \end{aligned}$$

Furthermore let

$$\begin{aligned} \lfloor \cdot \rfloor_D : \mathbb{C} &\to \mathbb{E}_D \\ r &\mapsto \lfloor r \rfloor_D \end{aligned}$$

where $\lfloor r \rfloor_D \in \mathbb{E}_D$ *such that* $[r]_{\sim_D} \cap \mathbb{E}_D = \{\lfloor r \rfloor_D\}$.

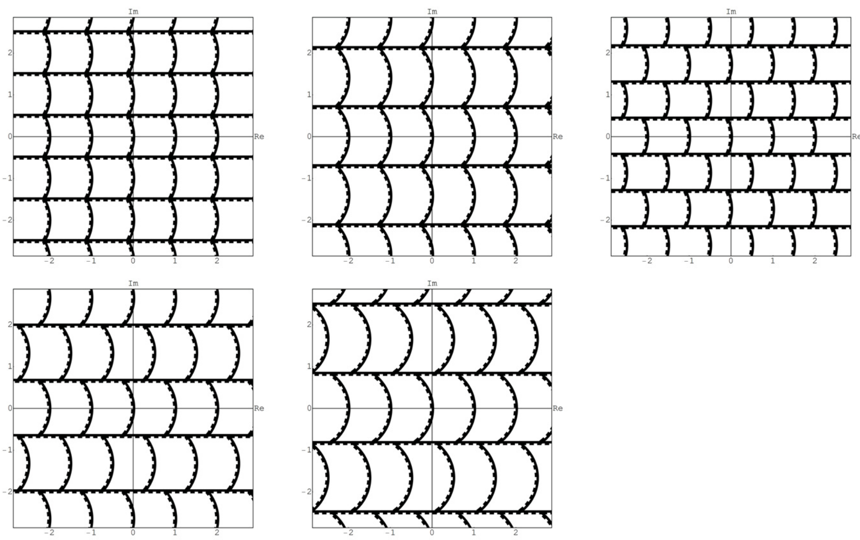


Figure 1. Tilings of $\mathbb{C}$ given by the sets T_D, $D \in \{-1, -2, -3, -7, -11\}$.

Lemma 6.2.2. [**Pethő et al., IP**](**Weitzer**) *Let* $(x, y) \in \mathbb{C}$. *Then*

$$\lfloor x + \mathrm{i}y \rfloor_D = \left\lfloor x - \left\lfloor \frac{y}{\Im(\omega_D)} + \frac{1}{2} \right\rfloor \Re(\omega_D) \right\rfloor + \omega_D \left\lfloor \frac{y}{\Im(\omega_D)} + \frac{1}{2} \right\rfloor$$

if $\left(x - \left\lfloor x - \left\lfloor \frac{y}{\Im(\omega_D)} + \frac{1}{2} \right\rfloor \Re(\omega_D) \right\rfloor - \left\lfloor \frac{y}{\Im(\omega_D)} + \frac{1}{2} \right\rfloor \Re(\omega_D)\right)^2 + \left(y - \left\lfloor \frac{y}{\Im(\omega_D)} + \frac{1}{2} \right\rfloor \Im(\omega_D)\right)^2 < 1,$

$$\lfloor x + \mathrm{i}y \rfloor_D = \left\lfloor x - \left\lfloor \frac{y}{\Im(\omega_D)} + \frac{1}{2} \right\rfloor \Re(\omega_D) \right\rfloor + 1 + \omega_D \left\lfloor \frac{y}{\Im(\omega_D)} + \frac{1}{2} \right\rfloor$$

otherwise.

In particular:
If $D \in \{-1,-2\}$*:*

$$\lfloor x+\mathrm{i}y \rfloor_D = \lfloor x \rfloor + \omega_D \left\lfloor \frac{y}{\sqrt{D}} + \frac{1}{2} \right\rfloor$$
$$\text{if } (x - \lfloor x \rfloor)^2 + \left(y - \left\lfloor \frac{y}{\sqrt{D}} + \frac{1}{2} \right\rfloor \sqrt{D}\right)^2 < 1,$$
$$\lfloor x+\mathrm{i}y \rfloor_D = \lfloor x \rfloor + 1 + \omega_D \left\lfloor \frac{y}{\sqrt{D}} + \frac{1}{2} \right\rfloor$$
otherwise.

If $D \in \{-3,-7,-11\}$*:*

$$\lfloor x+\mathrm{i}y \rfloor_D = \left\lfloor x - \left\lfloor \frac{2y}{\sqrt{D}} + \frac{1}{2} \right\rfloor \frac{1}{2} \right\rfloor + \omega_D \left\lfloor \frac{2y}{\sqrt{D}} + \frac{1}{2} \right\rfloor$$
$$\text{if } \left(x - \left\lfloor x - \left\lfloor \frac{2y}{\sqrt{D}} + \frac{1}{2} \right\rfloor \frac{1}{2} \right\rfloor - \left\lfloor \frac{2y}{\sqrt{D}} + \frac{1}{2} \right\rfloor \frac{1}{2}\right)^2 + \left(y - \left\lfloor \frac{2y}{\sqrt{D}} + \frac{1}{2} \right\rfloor \frac{\sqrt{D}}{2}\right)^2 < 1,$$
$$\lfloor x+\mathrm{i}y \rfloor_D = \left\lfloor x - \left\lfloor \frac{2y}{\sqrt{D}} + \frac{1}{2} \right\rfloor \frac{1}{2} \right\rfloor + 1 + \omega_D \left\lfloor \frac{2y}{\sqrt{D}} + \frac{1}{2} \right\rfloor$$
otherwise.

Proof. Follows from simple geometric considerations. If the sail set (for a given D) was replaced by a rectangle (straight lines instead of arcs) the floor function would be given by the respective (depending on D) first case. The case differentiation takes care of the arcs. □

Definition 6.2.3. [**Pethő et al., IP**] *For* $d \in \mathbb{N}$ *and* $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{C}^d$ *the mapping*

$$\varepsilon_{D,\mathbf{r}} : {\mathbb{E}_D}^d \to {\mathbb{E}_D}^d$$
$$\mathbf{a} = (a_1, \ldots, a_d) \mapsto (a_2, \ldots, a_d, -\lfloor \mathbf{ra} \rfloor_D)$$

where $\mathbf{ra} = \sum_{i=1}^{d} r_i a_i$*, is called the* d*-dimensional Shift Radix System on* $\mathbb{E}_D$ *associated with* $\mathbf{r}$*, and* $\mathbf{r}$ *is called the parameter of* $\varepsilon_{D,\mathbf{r}}$*. Furthermore we define*

$$\mathcal{F}_{D,d} := \left\{\mathbf{r} \in \mathbb{C}^d \mid \forall\, \mathbf{a} \in {\mathbb{E}_D}^d : \exists\, i,j \in \mathbb{N} : \varepsilon^i_{D,\mathbf{r}}(\mathbf{a}) = \varepsilon^{i+j}_{D,\mathbf{r}}(\mathbf{a})\right\}$$
$$\mathcal{F}^{(0)}_{D,d} := \left\{\mathbf{r} \in \mathbb{C}^d \mid \forall\, \mathbf{a} \in {\mathbb{E}_D}^d : \exists\, i \in \mathbb{N} : \varepsilon^i_{D,\mathbf{r}}(\mathbf{a}) = \mathbf{0}\right\}$$

where $\varepsilon^i_{D,\mathbf{r}}(\mathbf{a})$ *means* i*-fold application of* $\varepsilon_{D,\mathbf{r}}$ *to* $\mathbf{a}$*. Elements of* $\mathcal{F}^{(0)}_{D,d}$ *are said to have the finiteness property.*

Lemma 6.2.4. [**Pethő et al., IP**](**Weitzer**) *Let* $(x,y) \in \mathbb{C}$ *and* $(a,b)_D \in \mathbb{E}_D$*. Then*

$$\varepsilon_{D,x+\mathrm{i}y}(a+\omega_D b) = -\lfloor x(a+\Re(\omega_D)b) - y\Im(\omega_D)b + \mathrm{i}(x\Im(\omega_D)b + y(a+\Re(\omega_D)b)) \rfloor_D$$

In particular:
If $D \in \{-1,-2\}$*:*

$$\varepsilon_{D,x+\mathrm{i}y}(a+\omega_D b) = -\left\lfloor ax - by\sqrt{D} \right\rfloor - \omega_D \left\lfloor ay\frac{\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor$$
$$\text{if } \left(ax - by\sqrt{D} - \left\lfloor ax - by\sqrt{D} \right\rfloor\right)^2 + \left(ay + bx\sqrt{D} - \left\lfloor ay\frac{\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor \sqrt{D}\right)^2 < 1,$$
$$\varepsilon_{D,x+\mathrm{i}y}(a+\omega_D b) = -\left\lfloor ax - by\sqrt{D} \right\rfloor - 1 - \omega_D \left\lfloor ay\frac{\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor$$
otherwise.

If $D \in \{-3,-7,-11\}$*:*

$$\varepsilon_{D,x+\mathrm{i}y}(a+\omega_D b) = -\left\lfloor \left(a+\tfrac{b}{2}\right)x - by\frac{\sqrt{D}}{2} - \left\lfloor \left(a+\tfrac{b}{2}\right)y\frac{2\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor \frac{1}{2} \right\rfloor$$
$$-\omega_D \left\lfloor \left(a+\tfrac{b}{2}\right)y\frac{2\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor$$
$$\text{if } \Bigg(\left(a+\tfrac{b}{2}\right)x - by\frac{\sqrt{D}}{2} - \left\lfloor \left(a+\tfrac{b}{2}\right)y\frac{2\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor \frac{1}{2} -$$
$$\left\lfloor \left(a+\tfrac{b}{2}\right)x - by\frac{\sqrt{D}}{2} - \left\lfloor \left(a+\tfrac{b}{2}\right)y\frac{2\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor \frac{1}{2} \right\rfloor\Bigg)^2 +$$
$$\left(\left(a+\tfrac{b}{2}\right)y + bx\frac{\sqrt{D}}{2} - \left\lfloor \left(a+\tfrac{b}{2}\right)y\frac{2\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor \frac{\sqrt{D}}{2}\right)^2 < 1,$$
$$\varepsilon_{D,x+\mathrm{i}y}(a+\omega_D b) = -\left\lfloor \left(a+\tfrac{b}{2}\right)x - by\frac{\sqrt{D}}{2} - \left\lfloor \left(a+\tfrac{b}{2}\right)y\frac{2\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor \frac{1}{2} \right\rfloor - 1$$
$$-\omega_D \left\lfloor \left(a+\tfrac{b}{2}\right)y\frac{2\sqrt{D}}{D} + bx + \frac{1}{2} \right\rfloor$$
otherwise.

Proof. Follows directly from Lemma 6.2.2. □

In analogy to similar concepts for Shift Radix Systems in Chapter 3 and Gaussian Shift Radix Systems in Chapter 5 one gets the following definitions and results.

THEOREM 6.2.5. [**Pethő et al., IP**] *Let $d \in \mathbb{N}$. Then*

$$\mathcal{E}_d^{(\mathbb{C})} \subseteq \mathcal{F}_{D,d} \subseteq \overline{\mathcal{E}_d^{(\mathbb{C})}}.$$

COROLLARY 6.2.6. [**Pethő et al., IP**] *Let $d \in \mathbb{N}$. Then*

(i) $\mathcal{F}_{D,d} \subseteq \left\{\mathbf{r} \in \mathbb{C}^d \mid \rho(R(\mathbf{r})) \leq 1\right\}$

(ii) $\mathcal{F}_{D,d} \supseteq \left\{\mathbf{r} \in \mathbb{C}^d \mid \rho(R(\mathbf{r})) < 1\right\}$

(iii) $\partial\mathcal{F}_{D,d} = \left\{\mathbf{r} \in \mathbb{C}^d \mid \rho(R(\mathbf{r})) = 1\right\}.$

DEFINITION 6.2.7. *For $d \in \mathbb{N}$ let $\mathcal{C}_d^{(\mathbb{E}_D)} := \bigcup_{n\in\mathbb{N}_0} {\mathbb{E}_D}^n$ denote the set of (d-dimensional, imaginary quadratic) cycles.*

For a cycle $\pi = (\mathbf{a}_1,\dots,\mathbf{a}_k) \in \mathcal{C}_d^{(\mathbb{E}_D)}$ let $P_{\mathbb{C},D}(\pi) := \left\{\mathbf{r} \in \mathbb{C}^d \mid \forall\, i \in [\![1,k]\!] : \varepsilon_{D,\mathbf{r}}(\mathbf{a}_i) = \mathbf{a}_{i\%k+1}\right\}$, i.e. the set of those parameters $\mathbf{r}$ for which π is a cycle of the associated Shift Radix System on $\mathbb{E}_D$. $P_{\mathbb{C},D}(\pi)$ shall be referred to as the cutout set of π.

LEMMA 6.2.8. [**Pethő et al., IP**] *Let $d \in \mathbb{N}$. Then*

$$\mathcal{F}_{D,d}^{(0)} = \mathcal{F}_{D,d} \setminus \bigcup_{\pi\in\mathcal{C}_d^{(\mathbb{E}_D)}\setminus\{(\mathbf{0})\}} P_{\mathbb{C},D}(\pi)$$

LEMMA 6.2.9. [**Pethő et al., IP**](**Weitzer**) *Let $z = (a,b)_D, Z = (A,B)_D \in \mathbb{E}_D$, $u := \Re(\omega_D)$, and $v := \Im(\omega_D)$. Then*

$$\begin{aligned}\{r \in \mathbb{C} \mid \varepsilon_{D,r}(z) = Z\} = \Bigg\{(x,y) \in \mathbb{C} \mid\ & \Big(x((a+bu)^2+(bv)^2)+(a+bu)(A+Bu+1)+bBv^2\Big)^2 + \\ & \Big(y((a+bu)^2+(bv)^2)+(a+bu)Bv-(A+Bu+1)bv\Big)^2 \\ & \geq (a+bu)^2+(bv)^2 > \\ & \Big(x((a+bu)^2+(bv)^2)+(a+bu)(A+Bu)+bBv^2\Big)^2 + \\ & \Big(y((a+bu)^2+(bv)^2)+(a+bu)Bv-(A+Bu)bv\Big)^2 \quad \wedge \\ & -\frac{v}{2} \leq bvx+(a+bu)y+Bv < \frac{v}{2}\Bigg\}.\end{aligned}$$

PROOF. Follows from simple geometric considerations. □

LEMMA 6.2.10. [**Pethő et al., IP**] *Let $d \in \mathbb{N}$, $\mathbf{r} \in \operatorname{int}(\mathcal{F}_{D,d})$, $\rho \in (\rho(R(\mathbf{r})),1)$, $\|\cdot\|_\rho$ norm on $\mathbb{C}^d$ with $\|R(\mathbf{r})\mathbf{a}\|_\rho \leq \rho\,\|\mathbf{a}\|_\rho$ for all $\mathbf{a} \in \mathbb{C}^d$ (cf. proof of Theorem 3.4.2), and $\mathbf{a} \in {\mathbb{E}_D}^d$ such that $\varepsilon_{D,\mathbf{r}}^k(\mathbf{a}) = \mathbf{a}$ for some $k \in \mathbb{N}$. Then*

$$\|\mathbf{a}\|_\rho \leq \frac{\|(0,\dots,0,1)\|_\rho}{1-\rho}.$$

In particular: $\left\{\pi \in \mathcal{C}_d^{(\mathbb{E}_D)} \mid \mathbf{r} \in P_{\mathbb{C},D}(\pi)\right\}$ is a finite set.

LEMMA 6.2.11. [**Pethő et al., IP**](**Weitzer**) *Let $d \in \mathbb{N}$, $\mathbf{r} \in \mathbb{C}^d$, and $\mathbf{a},\mathbf{b} \in {\mathbb{E}_D}^d$. Then*

$$2\Im_D(\mathbf{ra}) \notin \mathbb{Z}_o \Leftrightarrow (\varepsilon_{D,\mathbf{r}}(\mathbf{a}) = \mathbf{b} \Leftrightarrow \varepsilon_{D,\overline{\mathbf{r}}}(\overline{\mathbf{a}}) = \overline{\mathbf{b}})$$
$$2\Im_D(\mathbf{ra}) \in \mathbb{Z}_o \Rightarrow (\varepsilon_{D,\mathbf{r}}(\mathbf{a}) = \mathbf{b} \Rightarrow \varepsilon_{D,\overline{\mathbf{r}}}(\overline{\mathbf{a}}) - \overline{\mathbf{b}} \in \{(0,-1)_D,(1,-1)_D\}).$$

In particular: If $\pi = (\mathbf{a}_1,\dots,\mathbf{a}_k) \in \mathcal{C}_d^{(\mathbb{E}_D)}$ and $\mathbf{r} \in \operatorname{int}(P_{\mathbb{C},D}(\pi))$ then

$$(\mathbf{a}_1,\dots,\mathbf{a}_k) \text{ cycle of } \varepsilon_{D,\mathbf{r}} \Leftrightarrow (\overline{\mathbf{a}_1},\dots,\overline{\mathbf{a}_k}) \text{ cycle of } \varepsilon_{D,\overline{\mathbf{r}}}.$$

PROOF. Follows directly from the definitions. □

The previous lemma and the "In particular" part of Lemma 6.2.10 imply that $\mathcal{F}_{D,1}^{(0)}$ and $\mathcal{F}_{D,1}^{(0)}$ reflected at the real axis coincide almost everywhere. Parts where the two sets might not coincide are contained in the union of their respective boundaries.

DEFINITION 6.2.12. [**Pethő et al., IP**] *Let $d \in \mathbb{N}$ and $\mathbf{r} \in \mathbb{C}^d$.*

- $\mathbf{r}$ *is called a regular point (for $\mathcal{F}_{D,d}^{(0)}$) iff there exists an open neighborhood of* $\mathbf{r}$ *which intersects with only finitely many cutout sets.*
- $\mathbf{r}$ *is called a weakly critical point (for $\mathcal{F}_{D,d}^{(0)}$) iff any open neighborhood of* $\mathbf{r}$ *intersects with infinitely many cutout sets.*
- $\mathbf{r}$ *is called a critical point (for $\mathcal{F}_{D,d}^{(0)}$) iff for every open neighborhood B of* $\mathbf{r}$ *the set $B \backslash \mathcal{F}_{D,d}^{(0)}$ cannot be covered by finitely many cutout sets.*

6.3. Main result: Critical points

Using the fact that all cycles for a given parameter $r \in \mathbb{C}$ are contained in a disk as stated in Lemma 6.2.10 the following approximative images of $\mathcal{F}_{D,1}^{(0)}$ have been computed. The orignal images have a resolution of 4096 times 4096 pixels each and every pixel has been colored white, gray or black depending on containment in $\mathcal{F}_{D,1}$ and $\mathcal{F}_{D,1}^{(0)}$. So despite the fact that these images show only approximations of the sets $\mathcal{F}_{D,1}^{(0)}$, every pixel - as a representation of a single point in a mathematical sense - does have the correct color. The very surprising observation one makes when looking on those images is that in the cases $D = -1$ or $D = -3$ the set $\mathcal{F}_{D,d}^{(0)}$ does seem to have critical points while in the case $D = -7$ it seems to have only weakly critical points and in the cases $D = -2$ and $D = -11$ even no critical or weakly critical points at all. In the present section we will prove the last observation for both cases by giving a list of cutout sets which separates $\mathcal{F}_{D,1}^{(0)}$ from the boundary of $\mathcal{F}_{D,1}$. The list has been found by manual search.

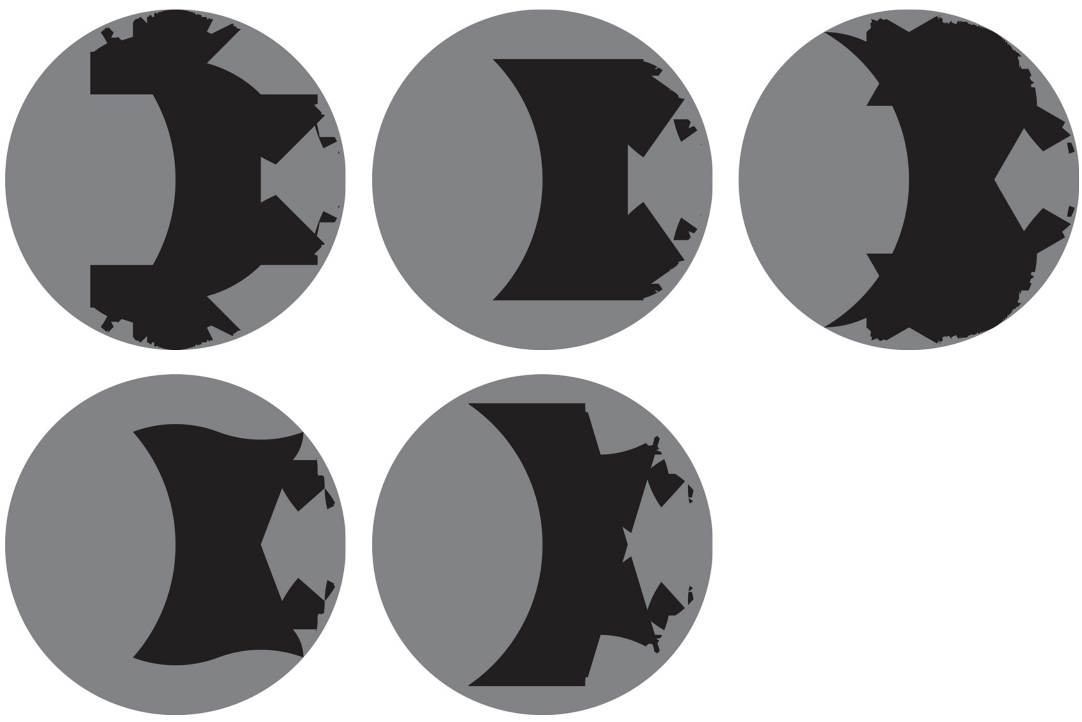


FIGURE 2. Approximations of $\mathcal{F}_{D,1}^{(0)}$ for $D \in \{-1, -2, -3, -7, -11\}$.

DEFINITION 6.3.1. [**Pethő et al., IP**](**Weitzer**) *Let*

$$\Big(\big((x_{2,1},y_{2,1}),(a_{2,1},b_{2,1})\big),\dots,\big((x_{2,45},y_{2,45}),(a_{2,45},b_{2,45})\big)\Big) := \Bigg($$

$$\big((1,0),\,(-2,0)\big),\ \big((-\tfrac{529}{4023},\tfrac{22378908}{45415717}),\,(0,1)\big),\ \big((-\tfrac{8413}{3862276},\tfrac{6385}{8993}),\,(0,1)\big),\ \big((-\tfrac{560}{3763},\tfrac{166}{229}),$$
$$(0,1)\big),\ \big((\tfrac{11051}{36427},\tfrac{12022}{16987}),\,(0,1)\big),\ \big((-\tfrac{39833}{139318},\tfrac{634841}{887952}),\,(0,1)\big),\ \big((-\tfrac{587}{32542},\tfrac{1260970}{1501501}),\,(0,1)\big),$$
$$\big((\tfrac{20911}{27059},\tfrac{183}{517}),\,(0,1)\big),\ \big((-\tfrac{3533}{7022},\tfrac{1411}{1988}),\,(0,1)\big),\ \big((\tfrac{645}{3757},\tfrac{1432877}{1660169}),\,(0,1)\big),\ \big((\tfrac{844688}{1266909},\tfrac{2031}{3445}),$$
$$(0,4)\big),\big((\tfrac{44399}{51256},\tfrac{4447}{14348}),\,(0,2)\big),\big((\tfrac{781981}{1137704},\tfrac{159}{260}),\,(0,4)\big),\big((\tfrac{3741}{6160},\tfrac{2237}{3237}),\,(0,2)\big),\big((\tfrac{18563}{132052},\tfrac{677269}{744909}),$$
$$(0,1)\big),\ \big((-\tfrac{273}{461},\tfrac{256}{357}),\,(0,1)\big),\ \big((-\tfrac{23531}{44649},\tfrac{2367}{3041}),\,(0,1)\big),\ \big((-\tfrac{2504}{4903},\tfrac{53361}{66614}),\,(0,1)\big),$$
$$\big((\tfrac{2295978}{14352937},\tfrac{128937}{134770}),\,(0,1)\big),\ \big((-\tfrac{22537}{155137},\tfrac{19631}{20469}),\,(0,1)\big),\ \big((-\tfrac{1324}{2503},\tfrac{85287}{104894}),\,(0,1)\big),$$
$$\big((\tfrac{186647}{247677},\tfrac{278}{433}),\,(0,2)\big),\ \big((\tfrac{81473}{111068},\tfrac{86419}{129984}),\,(0,2)\big),\ \big((-\tfrac{1087}{2004},\tfrac{670}{809}),\,(0,1)\big),\ \big((\tfrac{19}{25},\tfrac{16}{25}),\,(0,2)\big),$$
$$\big((\tfrac{27}{37},\tfrac{25}{37}),\,(0,2)\big),\big((\tfrac{13}{17},\tfrac{54}{85}),\,(0,2)\big),\big((\tfrac{7647}{10000},\tfrac{16}{25}),\,(0,2)\big),\big((\tfrac{7339}{10000},\tfrac{1347}{2000}),\,(0,2)\big),\big((\tfrac{1979}{20000},\tfrac{4961}{5000}),$$
$$(0,1)\big),\ \big((-\tfrac{1979}{20000},\tfrac{397}{400}),\,(0,1)\big),\ \big((-\tfrac{2701}{5000},\tfrac{8399}{10000}),\,(0,1)\big),\ \big((-\tfrac{1097}{2000},\tfrac{4169}{5000}),\,(0,1)\big),$$
$$\big((\tfrac{1527}{2000},\tfrac{6429}{10000}),\,(0,2)\big),\ \big((\tfrac{3831}{5000},\tfrac{6413}{10000}),\,(0,2)\big),\ \big((\tfrac{3699}{5000},\tfrac{6711}{10000}),\,(0,2)\big),\ \big((\tfrac{7321}{10000},\tfrac{6767}{10000}),\,(0,2)\big),$$
$$\big((\tfrac{7419}{10000},\tfrac{1339}{2000}),\,(0,2)\big),\big((\tfrac{3683}{5000},\tfrac{3377}{5000}),\,(0,2)\big),\big((-\tfrac{1087}{2000},\tfrac{4183}{5000}),\,(0,1)\big),\big((-\tfrac{1089}{2000},\tfrac{8387}{10000}),\,(0,1)\big),$$
$$\big((-\tfrac{1089}{2000},\tfrac{1677}{2000}),\,(0,1)\big),\ \big((\tfrac{1}{10},\tfrac{7}{5\sqrt{2}}),\,(0,1)\big),\ \big((\tfrac{1}{100}(50+\sqrt{1534}),-\tfrac{-100+\sqrt{1534}}{100\sqrt{2}}),\,(0,1)\big),$$
$$\big((\tfrac{9}{10},\tfrac{3}{5\sqrt{2}}),\,(0,1)\big)$$

$$\Bigg),$$

$$\Big(\big((x_{11,1},y_{11,1}),(a_{11,1},b_{11,1})\big),\dots,\big((x_{11,47},y_{11,47}),(a_{11,47},b_{11,47})\big)\Big) := \Bigg($$

$$((1,0),\,(-2,0))\ \big((-\tfrac{529}{4023},\tfrac{22378908}{45415717}),\,(0,1)\big),\big((\tfrac{25699}{75158},\tfrac{11951}{22586}),\,(2,0)\big),\big((\tfrac{122233}{192089},\tfrac{5593}{12399}),\,(0,1)\big),$$
$$\big((\tfrac{6229}{23994},\tfrac{22353}{28738}),\,(0,9)\big),\ \big((\tfrac{2039}{57213},\tfrac{17365}{20941}),\,(0,1)\big),\ \big((\tfrac{3099}{4183},\tfrac{442047}{1060847}),\,(0,1)\big),\ \big((-\tfrac{39923}{156499},\tfrac{22371}{26896}),$$
$$(0,1)\big),\ \big((\tfrac{4038}{5203},\tfrac{4722}{11383}),\,(0,1)\big),\ \big((\tfrac{285}{406},\tfrac{752}{1417}),\,(0,1)\big),\ \big((\tfrac{15765}{22453},\tfrac{431}{725}),\,(0,1)\big),\ \big((\tfrac{2023}{7895},\tfrac{2634}{2981}),$$
$$(0,1)\big),\ \big((-\tfrac{810241}{3496246},\tfrac{662044}{743591}),\,(0,1)\big),\ \big((\tfrac{127129}{185005},\tfrac{42539}{67882}),\,(0,4)\big),\ \big((-\tfrac{109151}{435226},\tfrac{1106}{1235}),\,(0,1)\big),$$
$$\big((\tfrac{1499}{5037},\tfrac{10953}{12284}),\,(0,1)\big),\ \big((-\tfrac{8495}{29356},\tfrac{259913}{290617}),\,(0,1)\big),\ \big((\tfrac{755}{851},\tfrac{3083}{7406}),\,(0,1)\big),\ \big((-\tfrac{15483}{32584},\tfrac{4513239}{5265740}),$$
$$(0,1)\big),\big((-\tfrac{39752315}{80135632},\tfrac{1130}{1337}),\,(0,1)\big),\big((-\tfrac{45318560}{90412991},\tfrac{235960}{280199}),\,(0,1)\big),\big((-\tfrac{422566}{838723},\tfrac{6443}{7665}),\,(0,1)\big),$$
$$\big((-\tfrac{7361}{14390},\tfrac{105082}{125711}),\,(0,1)\big),\ \big((-\tfrac{724614}{1438463},\tfrac{2019}{2369}),\,(0,1)\big),\ \big((-\tfrac{4861}{9600},\tfrac{1020}{1199}),\,(0,1)\big),$$
$$\big((-\tfrac{1064}{2059},\tfrac{166081}{196678}),\,(0,1)\big),\ \big((-\tfrac{545}{1034},\tfrac{168253}{200773}),\,(0,1)\big),\ \big((\tfrac{13}{50},\tfrac{24}{25}),\,(0,1)\big),\ \big((\tfrac{13}{51},\tfrac{49}{51}),\,(0,1)\big),$$
$$\big((-\tfrac{45}{82},\tfrac{34}{41}),\,(0,1)\big),\big((-\tfrac{1135}{2048},\tfrac{1699}{2048}),\,(0,1)\big),\big((-\tfrac{1125}{2048},\tfrac{851}{1024}),\,(0,1)\big),\big((-\tfrac{1123}{2048},\tfrac{1701}{2048}),\,(0,1)\big),$$
$$\big((-\tfrac{1083}{2048},\tfrac{869}{1024}),\,(0,1)\big),\ \big((-\tfrac{1075}{2048},\tfrac{433}{512}),\,(0,1)\big),\ \big((-\tfrac{1069}{2048},\tfrac{873}{1024}),\,(0,1)\big),\ \big((-\tfrac{531}{1024},\tfrac{1745}{2048}),$$
$$(0,1)\big),\ \big((-\tfrac{529}{1024},\tfrac{875}{1024}),\,(0,1)\big),\ \big((\tfrac{505}{2048},\tfrac{991}{1024}),\,(0,1)\big),\ \big((\tfrac{511}{2048},\tfrac{1983}{2048}),\,(0,1)\big),\ \big((\tfrac{513}{2048},\tfrac{991}{1024}),$$
$$(0,1)\big),\ \big((\tfrac{135}{512},\tfrac{987}{1024}),\,(0,1)\big),\ \big((\tfrac{129106}{516339},\tfrac{2147435}{2219844}),\,(0,1)\big),\ \big((\tfrac{1}{212}(-140+\sqrt{573}),\tfrac{\sqrt{11}}{4}),\,(0,3)\big),$$
$$\big((\tfrac{-550-\sqrt{42130}}{1500},\tfrac{\sqrt{11}\left(-25+2\sqrt{42130}\right)}{1500}),\,(0,1)\big),\quad \big((\tfrac{1}{48}(-33+\sqrt{93}),\tfrac{1}{48}\sqrt{11}(3+\sqrt{93})),\,(0,1)\big),$$
$$\big((\tfrac{1639+\sqrt{10021}}{6600},\tfrac{539+\sqrt{10021}}{200\sqrt{11}}),\,(0,1)\big)$$

$$\Bigg),$$

and let $C_0^{(2)}(n)$ *denote the ultimate period of the orbit of* $(a_{2,n},b_{2,n})_{-2}$ *under* $\varepsilon_{-2,(x_{2,n},y_{2,n})}$ *for all* $n \in [\![1,45]\!]$ *and* $C_0^{(11)}(n)$ *the ultimate period of the orbit of* $(a_{11,n},b_{11,n})_{-11}$ *under* $\varepsilon_{-11,(x_{11,n},y_{11,n})}$ *for all* $n \in [\![1,47]\!]$*. Furthermore let for all* $n \in \mathbb{Z}$*:*

$$C_1^{(-D)}(n) := ((-n,1)_D,(n,-1)_D)$$
$$C_2^{(-D)}(n) := ((-n,1)_D,(n+1,-1)_D).$$

THEOREM 6.3.2. [**Pethő et al., IP**](**Weitzer**)

(i) $\mathcal{F}_{-2,1}^{(0)}$ *has no weakly critical points (and thus no critical points)* r *satisfying* $r \in \overline{\mathcal{F}_{-2,1}^{(0)}}$

(ii) $\mathcal{F}_{-11,1}^{(0)}$ *has no weakly critical points (and thus no critical points)* r *satisfying* $r \in \overline{\mathcal{F}_{-11,1}^{(0)}}$.

PROOF. For any cycle $\pi \in \mathcal{C}_1^{(\mathbb{E}_D)}$ let $\overline{\pi}$ denote the cycle one gets if all elements of π are replaced by their complex conjugates. The cutout sets of the cycles $C_1(n)^{(2)}$ $(n \in \mathbb{Z})$, $C_0^{(2)}(1),\dots,C_0^{(2)}(45)$,

$\overline{C_0^{(2)}(1)}, \ldots, \overline{C_0^{(2)}(45)}$, and $C_1(n)^{(11)}$ $(n \in \mathbb{Z})$, $C_0^{(11)}(1), \ldots, C_0^{(11)}(47)$, $\overline{C_0^{(11)}(1)}, \ldots, \overline{C_0^{(11)}(47)}$ respectively, completely cover the ring centered at the origin in the complex plane with inner radius $\frac{99}{100}$ and outer radius 1. $\square$

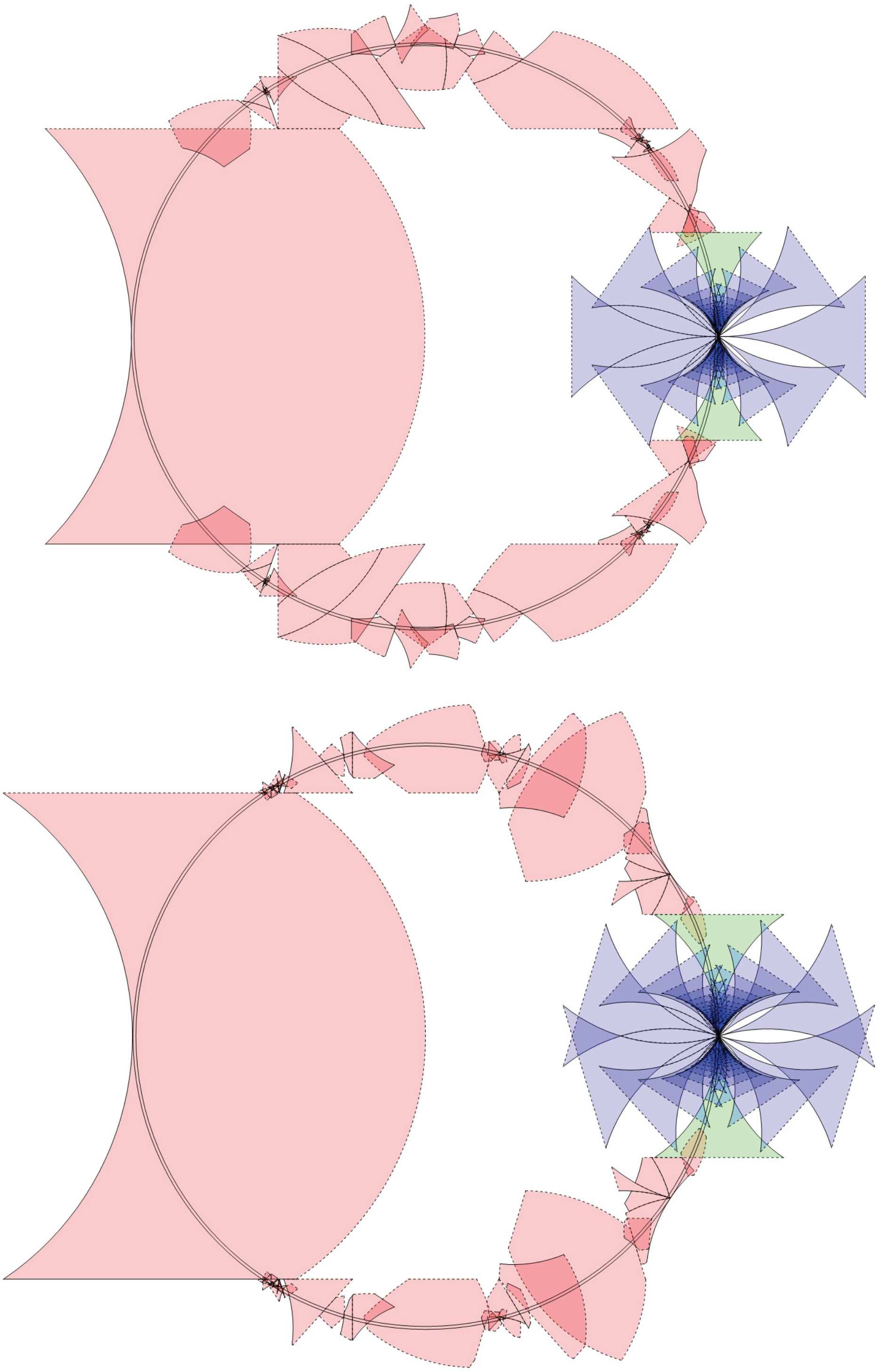


FIGURE 3. Cutout sets separating $\mathcal{F}^{(0)}_{-2,1}$ and $\mathcal{F}^{(0)}_{-11,1}$ from the unit circle (green: first singular cycle, red: remaining singular cycles, blue: cycles of infinite family).

6.4. Getting rid of Euclid

After successful definition of Shift Radix Systems on imaginary quadratic Euclidean domains we proved a first surprising result. In terms of weakly critical and critical points the five different Euclidean domains seem to admit three different types of behaviors. While two of them most probably have critical points and one has only weakly critical points, the other two have neither which we proved in the previous section. Looking at the definition of Shift Radix Systems one makes the following observation: Though our initial intention was to define Shift Radix Systems only for the Euclidean cases of imaginary quadratic extensions, in regard of the definition that we found the five specific values for D that we allowed are in no way special whatsoever. The definition of the floor function (in a certain sense there are actually two different definitions, cf. the "In particular" part of Lemma 6.2.2) works perfectly well for a continuum of values for D. The first definition of the floor function that is in use when D is ether -1 or -2 works if $D \in (-4, 0)$ as in this case the two lines that bound T_D still intersect with the two circles. For the same reason the other definition of the floor function, in use when D is -3, -7 or -11, works if $D \in (-16, 0)$. The figures below show approximations of the corresponding sets $\mathcal{F}_{D,1}^{(0)}$ in both cases and for several values of D. One can see that weakly critical and critical points seem to get into existence and disappear again when D changes continuously in the respective intervals.

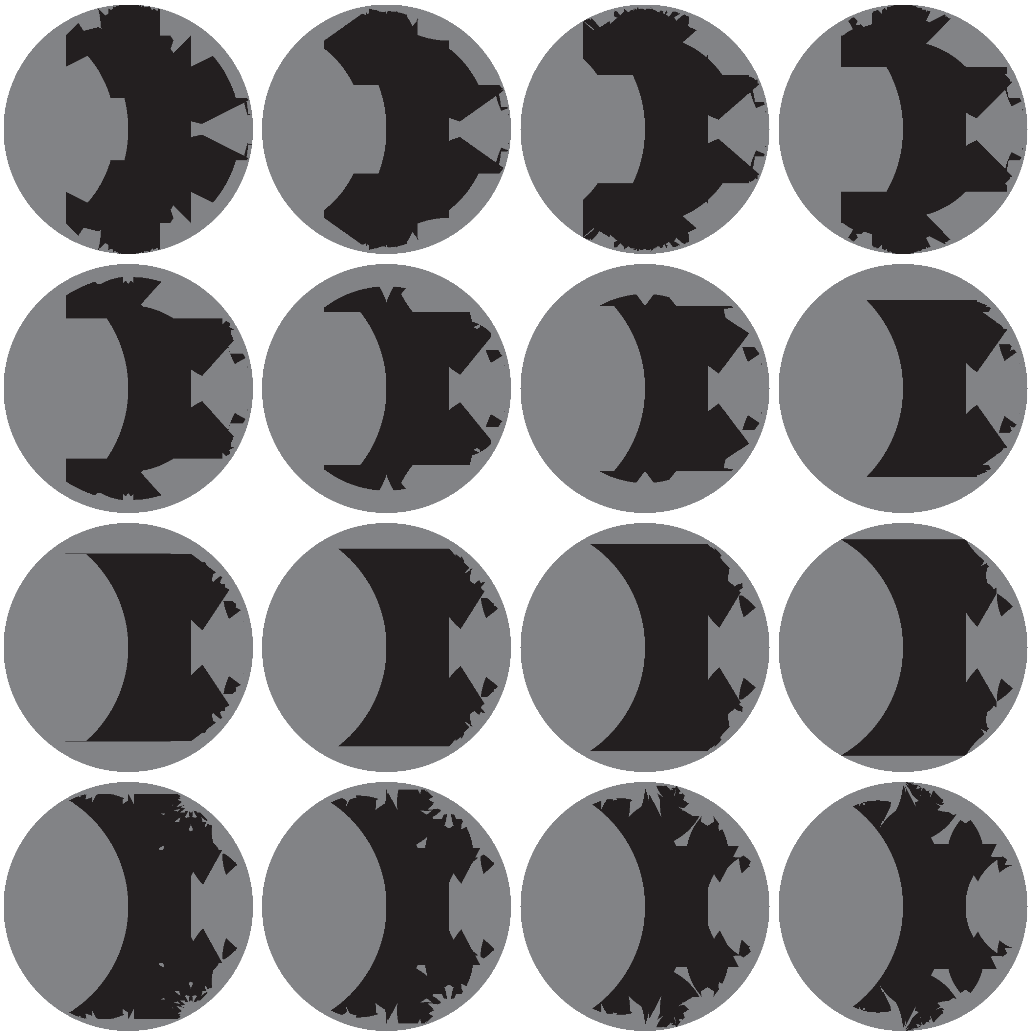


FIGURE 4. Approximation of $\mathcal{F}_{D,1}^{(0)}$ using the first version of the floor function (cf. Lemma 6.2.2, "In particular"), $D = -1/4, -2/4, \ldots, -15/4, -399/100$.

FIGURE 5. Approximation of $\mathcal{F}^{(0)}_{D,1}$ using the second version of the floor function (cf. Lemma 6.2.2, "In particular"), $D = -4/10, -11/10, \ldots, -158/10, -1599/100$.

6.5. On the boundary

In this last section of the chapter we will disprove the common believe that mathematics, though full of beauty, lacks romance. While computations fail when $D = 0$, for $D = -4$ respectively $D = -16$ the corresponding sets $\mathcal{F}_{D,1}^{(0)}$ can be computed. Whatever their mathematical interpretation may be, the human one is obvious and probably as universal as mathematics itself. The author would like to dedicate this last "result" to the mother of their wonderful children and the love of his life - Elisabeth.

FIGURE 6. Approximation of $\mathcal{F}_{D,1}^{(0)}$ inside of $\mathcal{F}_{D,1}$ for both versions of the floor function, $D = -4$ and $D = -16$ respectively. The image is rotated by $\pi/2$.

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