A second look at binary digits

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Binary digits

Everybody knows that $(10)_2 = 2$ and $(11011)_2 = 27$.

Also, $-27 = (-11011)_2$. Or is it?

Some computers know that $-27 = (1111111111100101)$ (signed word), and that $32767 + 1 = -32768$.

Some people know that $-1 = (11111\ldots)_2 \in \mathbb{Z}_2$ (start with LSD here), and

$$-27 = (101001111\ldots)_2.$$

Can we do better?

The expansion algorithm

Define the dynamic mapping $T : \mathbb{Z} \to \mathbb{Z} : a \mapsto \begin{cases} \frac{a}{2} & \text{if } a \text{ even;} \\ \frac{a-1}{2} & \text{if } a \text{ odd.} \end{cases}$

Now to expand a, write 0 if a even and 1 otherwise, and continue with $T(a)$. Done when $T^n(a) = 0$.

Example: $27 \xrightarrow{1} 13 \xrightarrow{1} 6 \xrightarrow{0} 3 \xrightarrow{1} 1 \xrightarrow{1} 0$.

However, $-1 \xrightarrow{1} -1 \ldots$

Try other digits: $\mathcal{D} = \{d_0, d_1\}$, with $d_i \equiv i \pmod 2$.

Criterion for the existence of a 1-cycle: $\frac{a-d}{2} = a \Leftrightarrow a = -d$.
So this is hopeless!

Negabinary expansions

Try other basis -2, with digits $\{0, 1\}$:

$-27 \xrightarrow{1} 14 \xrightarrow{0} -7 \xrightarrow{1} 4 \xrightarrow{0} -2 \xrightarrow{0} 1 \xrightarrow{1} 0$, so $-27 = (100101)_{-2}$.

Theorem (Grünwald 1885) All integers have a finite expansion on the integer basis $b \leq -2$ and digits $\{0, 1, \ldots, |b| - 1\}$.

Proof: there are no cycles except $0 \xrightarrow{0} 0$!

Excursion: the balanced ternary expansion uses basis $+3$ and digits $\{-1, 0, 1\}$, and expands all integers finitely. If only computers had three-way switches!

Theorem Let $a \in \mathbb{Z}_3$. Then $a \in \mathbb{Z}$ if and only if its balanced ternary expansion is finite.

A curious question

Definition A digit set $\mathcal{D}$ is valid for basis ± 2 if all integers have a finite representation

$$\sum_{i=0}^{\ell} d_i(\pm 2)^i \quad (d_i \in \mathcal{D}).$$

We know that no digit sets are valid for basis $+2$; for basis -2, we know the valid digit set $\{0, 1\}$, and thus also $\{0, -1\}$ by an automorphism of the additive group.

Question Are there any others?

Answer Yes, infinitely many!

Expansions of zero

Is it possible to have a digit set without zero? Yes!

The definition of the mapping T and of the stopping criterion is the same (if you formulate it like I do!).

Example: basis -2, digits $\{1, 4\}$. Expand -27:

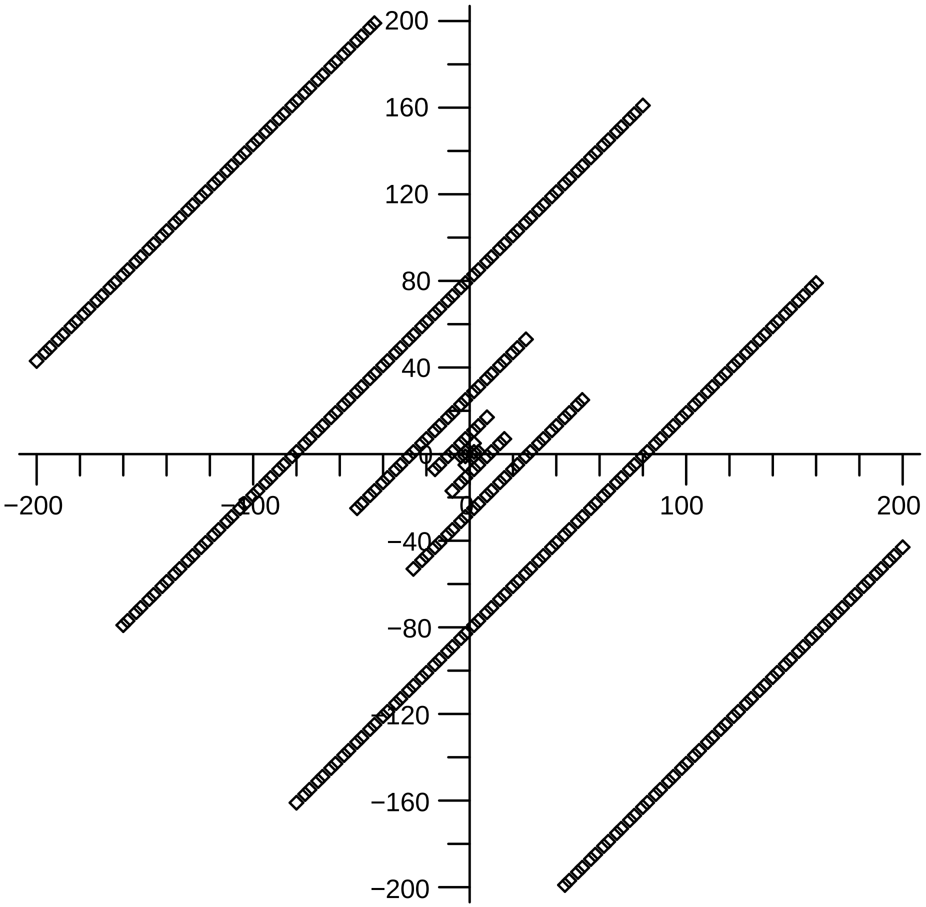
$-27 \xrightarrow{1} 14 \xrightarrow{4} -5 \xrightarrow{1} 3 \xrightarrow{1} -1 \xrightarrow{1} 1 \xrightarrow{1} 0$, so $-27 = (111141)_{-2}$.

Interesting: $0 \xrightarrow{4} 2 \xrightarrow{4} 1 \xrightarrow{1} 0$, a 3-cycle!

So, $0 = ()_{-2} = (144)_{-2} = (144144)_{-2} = \ldots$

Theorem Any valid digit set gives rise to a nontrivial expansion of zero.

Experiments



The figure plots all pairs of integers (x, y), with $|x|, |y| \leq 200$, that are valid digit sets for basis -2.

Results

Theorem (CvdW 2008) The digit set $\{d, D\}$ with $d < D$ is valid for basis -2 if and only if

(i) one of d, D is even and one is odd (trivial)

(ii) if $3 \mid dD$, then one of d, D is 0 and 3 does not divide the other (avoid 1-cycles except 0)

(iii) we have $2d \leq D$ and $2D \geq d$ (0 is expansible)

(iv) $D - d = 3^i$ for some $i \geq 0$ (the real stuff!)

For example, the only valid digit sets with 0 are $\{0, \pm 1\}$. On the other hand, the sets $\{1, 3^i + 1\}$ are valid for all $i \geq 0$.

Higher-dimensional analogues

There is no reason to limit the theory of number systems to $\mathbb{Z}$. Consider this setup:

- V is an abelian group.
- $\phi : V \to V$ is an endomorphism of V, with $[V : \phi(V)] < \infty$.
- $\mathcal{D}$ represents V modulo $\phi(V)$.

Then we can define $T : V \to V : a \mapsto \phi^{-1}(a - d_a)$, where $d_a \in \mathcal{D}$ has $a \equiv d_a \pmod{\phi(V)}$. We call $(V, \phi, \mathcal{D})$ a pre-number system, and additionally a number system when all elements of V are finitely expansible.

Theorem (Okazaki-CvdW) If $(V, \phi, \mathcal{D})$ is a number system, then V^{tor} is a direct summand of V and is bounded, and V/V^{tor} has finite rank.

Generalised binary systems

Here, we will consider generalised binary pre-number systems:

- α is an algebraic integer of norm ± 2.
- V is a fractional ideal of $\mathbb{Q}(\alpha)$.

If $V_2 = \beta V_1$ for some $\beta \in \mathbb{Q}(\alpha)$, then $(V_1, \alpha, \mathcal{D})$ and $(V_2, \alpha, \beta\mathcal{D})$ are isomorphic as pre-number systems.

Easy necessary conditions to have finite expansibility of all $a \in V$:

- α and $\alpha - 1$ must be non-units of $\mathbb{Z}[\alpha]$.
- α must be expanding: for all $\sigma : \mathbb{Z}[\alpha] \hookrightarrow \mathbb{C}$ we have $|\sigma(\alpha)| > 1$.

Note that any monic and expanding $f \in \mathbb{Z}[x]$ with $|f(0)|$ prime is automatically irreducible.

Expanding polynomials of given norm

On my website

`www.opt.math.tugraz.at/ cvdwoest/maths/expanding`

I collected some software and tables about enumeration of expanding and Pisot polynomials. Using a MAGMA implementation of ideas due to Schur, Dufresnoy-Pisot, Chamfy, and Kovács-Burcsi, I computed all monic expanding polynomials with integer coefficients and constant term ± 2 up to degree 20, as well as several cases with higher norm.

The computation for (e.g.) degree 13 and norm 2 takes less than 13 seconds on an Athlon.

The periodic set

Because α is expanding, the mapping T is almost a contraction on V, and the unique finite subset $\mathcal{P} \subset V$ that is invariant under T is called the periodic set of the pre-number system.

Lemma The periodic set of $(\mathbb{Z}, -2, \{d, D\})$ is the arithmetic progression $\left\{\left\lceil \frac{2d-D}{3} \right\rceil, \ldots, \left\lfloor \frac{2D-d}{3} \right\rfloor\right\}$.

In higher dimensions, the periodic set is usually quite irregular. Work of the Austro-Hungarian school has led to several (exponential) algorithms to compute the periodic set for any pre-number system.

Theorem $(V, \alpha, \mathcal{D})$ is a number system if and only if the action of T on $\mathcal{P}$ has exactly one cycle, which passes through 0.

The tile

There is a continuous variant of the (discrete) periodic set, called the tile of the pre-number system, because it usually tiles $V \otimes \mathbb{R}$.

For $(\mathbb{Z}, -2, \{d, D\})$, it is the interval $\left[\frac{2d-D}{3}, \frac{2D-d}{3}\right]$.

These tiles have the following properties:

- they are compact and the closure of their interior.
- they have fractal boundary.
- they may have infinitely many connected components, but they are connected when $|\mathcal{D}| = 2$.

To prove a higher-dimensional analogue of the main Theorem, we must characterise the lattice points in the tile, and describe the action of T on them.

Work in progress

More-or-less-theorem Let α be an expanding algebraic integer of norm ± 2. Then up to finitely many exceptions, a digit set $\mathcal{D} = \{d_0, d_0 + \delta\}$ makes $(\mathbb{Z}[\alpha], \alpha, \mathcal{D})$ into a number system if and only if:

(i) $(d_0, \alpha - 1) = (d_1, \alpha - 1) = (1)$

(ii) there is a nontrivial zero expansion

(iii) δ is a product of prime divisors of $\alpha - 1$ that are unramified, totally split and lie over different primes of $\mathbb{Z}$

Note that for a given degree d, there are only finitely many expanding α of degree d and norm ± 2. The smallest nonmaximal order among them is generated by $x^4 + x^2 + 4$ (Potiopa 1997). The smallest example with a nontrivial ideal class group is $x^8 - x^6 - x^2 + 2$ (CvdW 2009).

Technical assumptions

I need the following:

(i) $\alpha - 1$ is expanding;

(ii) the Hausdorff dimension of the tile is less than $\dim_{\mathbb{Z}} \mathbb{Z}[\alpha]$;

(iii) $\mathbb{Z}[\alpha]$ is a maximal order;

(iv) $(\mathbb{Z}[\alpha], \alpha, \{0, 1\})$ is a number system.

The last assumption says that the minimal polynomial of α is a CNS polynomial.

I hope to remove all of these assumptions.

Example

A famous example is $\tau = \frac{-1+\sqrt{-7}}{2}$ satisfying $x^2 + x + 2$. This basis has cryptographic significance because it can be used to speed up operations on Koblitz elliptic curves.

$x^2 + x + 2$ is a CNS polynomial, so (iv) is satisfied.

$\mathbb{Z}[\tau]$ is maximal, and $\tau - 1 = (\tau + 1)^2$, where $(\tau + 1)$ is an unramified prime of norm 2, and hence split.

All conjugates of τ have the same modulus, so the assumption on the Hausdorff dimension of the boundary follows from a theorem of Veerman.

So the Theorem holds for basis τ.

Experimental verification

This was verified experimentally for all pairs $\{a+b\tau, c+1+d\tau\}$ with $a, b, c, d \in \{-4, \ldots, 4\}$, a and c even. In all valid pairs, the difference is $\pm(\tau + 1)^e$, with $0 \leq e \leq 7$.

The attractors have the "right" number of elements, except (e.g.) for $\{\tau, \tau + 1\}$, where it has 3.