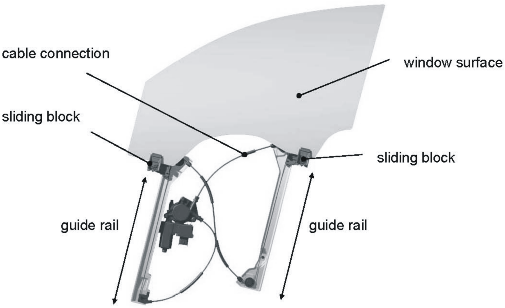
Motion of a Line Segment Whose Endpoint Paths Have Equal Arc Length

A. Gfrerrer

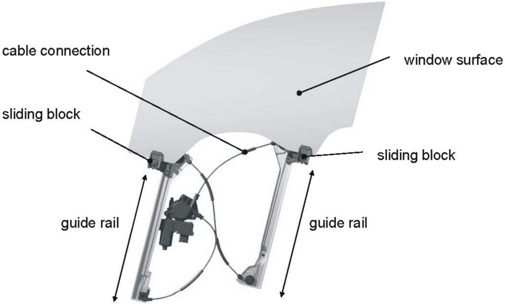
16th Scientific-Professional Colloquium on Geometry and Graphics

Baška, September 09 - 13, 2012

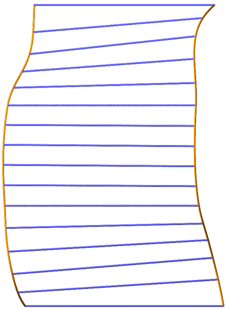
Problem Formulation

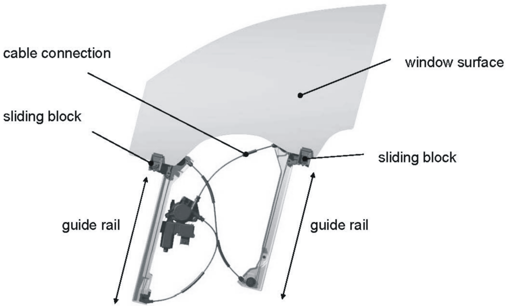

cable connection
window surface
sliding block
sliding block
guide rail
guide rail

How can you move a rod so that its endpoint paths have equal length?



How can you move a rod so that its endpoint paths have equal length?





$$A \ldots \vec{OA} = \mathbf{a}(t),\ B \ldots \vec{OB} = \mathbf{b}(t)$$

$$A \ldots \vec{OA} = \mathbf{a}(t),\ B \ldots \vec{OB} = \mathbf{b}(t)$$

$$\mathrm{dist}^2(A, B) = \langle \mathbf{b} - \mathbf{a}, \mathbf{b} - \mathbf{a} \rangle = d^2 = const.$$

$$A \ldots \vec{OA} = \mathbf{a}(t),\; B \ldots \vec{OB} = \mathbf{b}(t)$$

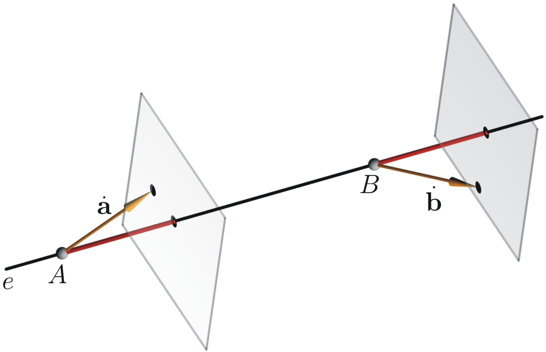
$$\operatorname{dist}^2(A, B) = \langle \mathbf{b} - \mathbf{a}, \mathbf{b} - \mathbf{a} \rangle = d^2 = const.$$

$$\langle \dot{\mathbf{a}}, \mathbf{b} - \mathbf{a} \rangle = \langle \dot{\mathbf{b}}, \mathbf{b} - \mathbf{a} \rangle, \text{ projection theorem}$$

$$A \ldots \vec{OA} = \mathbf{a}(t),\ B \ldots \vec{OB} = \mathbf{b}(t)$$

$$\mathrm{dist}^2(A, B) = \langle \mathbf{b} - \mathbf{a}, \mathbf{b} - \mathbf{a} \rangle = d^2 = const.$$

$$\langle \dot{\mathbf{a}}, \mathbf{b} - \mathbf{a} \rangle = \langle \dot{\mathbf{b}}, \mathbf{b} - \mathbf{a} \rangle, \text{ projection theorem}$$

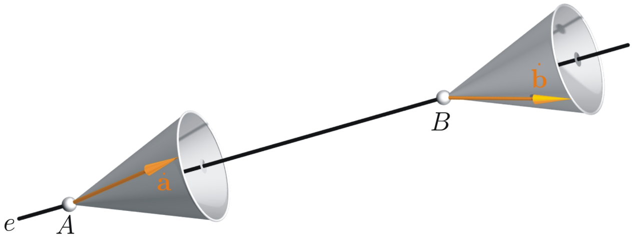


$A \ldots \vec{OA} = \mathbf{a}(t),\ B \ldots \vec{OB} = \mathbf{b}(t)$

$\mathrm{dist}^2(A, B) = \langle \mathbf{b} - \mathbf{a}, \mathbf{b} - \mathbf{a} \rangle = d^2 = const.$

$\langle \dot{\mathbf{a}}, \mathbf{b} - \mathbf{a} \rangle = \langle \dot{\mathbf{b}}, \mathbf{b} - \mathbf{a} \rangle$, projection theorem

$|\dot{\mathbf{a}}| = |\dot{\mathbf{b}}| \Longrightarrow \angle(\vec{AB}, \dot{\mathbf{a}}) = \angle(\vec{AB}, \dot{\mathbf{b}})$



Σ ... moving system

Σ ... moving system

Σ^* ... fixed system

Σ . . . moving system

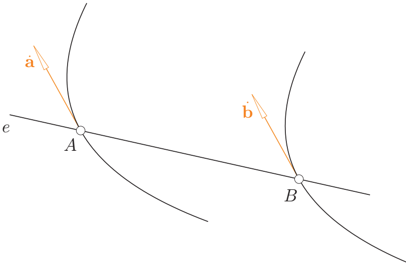
Σ^* . . . fixed system

Σ/Σ^* . . . motion

The Planar Case

planar case A

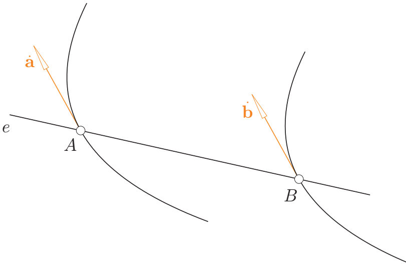
$\angle(\vec{AB}, \dot{\mathbf{a}}) = \angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$



planar case A

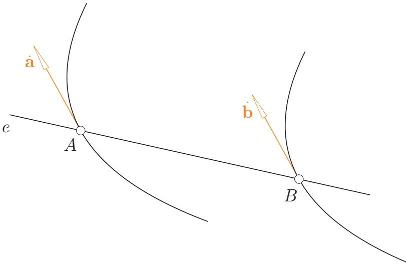
$\angle(\vec{AB}, \dot{\mathbf{a}}) = \angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$

$\dot{\mathbf{a}} = \dot{\mathbf{b}}$ for all $t \in [t_0, t_1]$



planar case A

$\angle(\vec{AB}, \dot{\mathbf{a}}) = \angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$

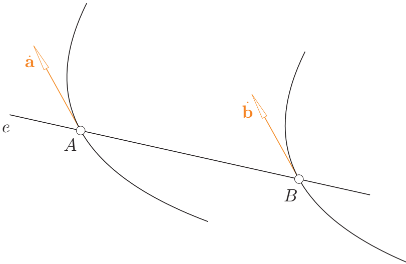


$\dot{\mathbf{a}} = \dot{\mathbf{b}}$ for all $t \in [t_0, t_1]$

curved translation

planar case A

$\angle(\vec{AB}, \dot{\mathbf{a}}) = \angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$

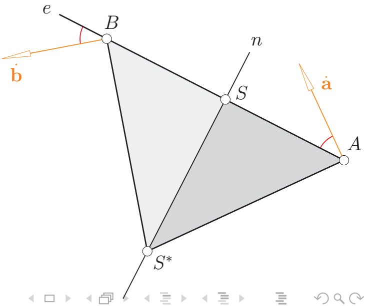


$\dot{\mathbf{a}} = \dot{\mathbf{b}}$ for all $t \in [t_0, t_1]$

curved translation

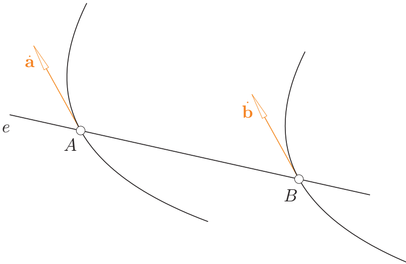
planar case B

$\angle(\vec{AB}, \dot{\mathbf{a}}) = -\angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$



planar case A

$\angle(\vec{AB}, \dot{\mathbf{a}}) = \angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$



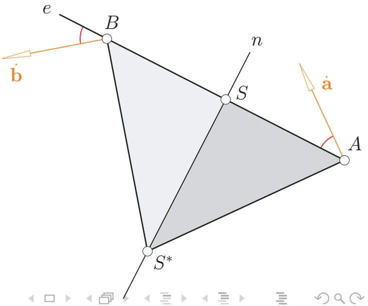
$\dot{\mathbf{a}} = \dot{\mathbf{b}}$ for all $t \in [t_0, t_1]$

curved translation

planar case B

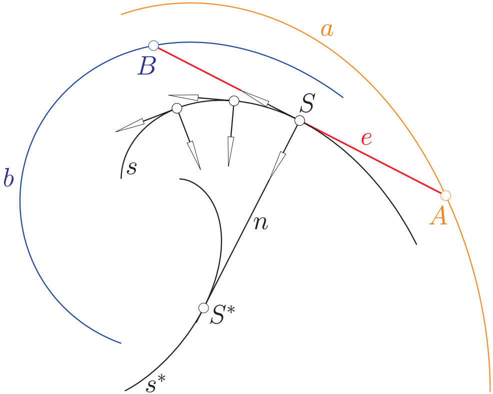
$\angle(\vec{AB}, \dot{\mathbf{a}}) = -\angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$

bisector n of AB = moving polhode



planar case B

Σ/Σ^* ... motion of a straight line n rolling on a curve s^*

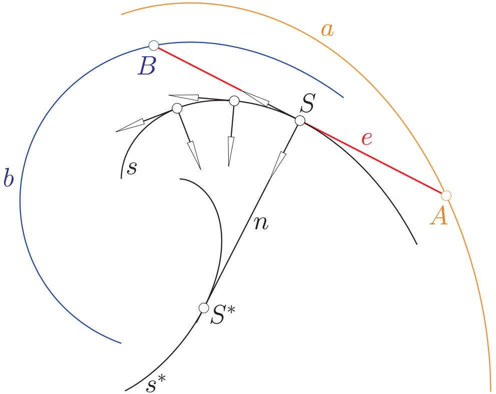


planar case B

Σ/Σ^* ... motion of a straight line n rolling on a curve s^*

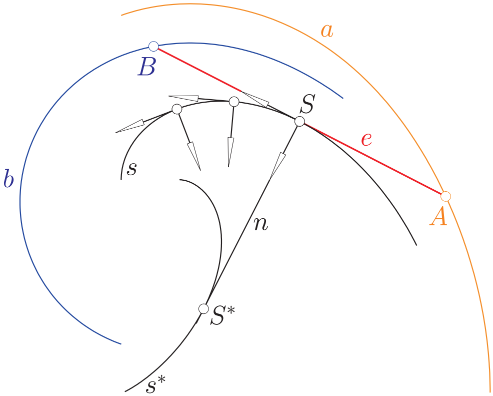
S ... midpoint of AB

Σ/Σ^* ... Frenet motion along the path s of S

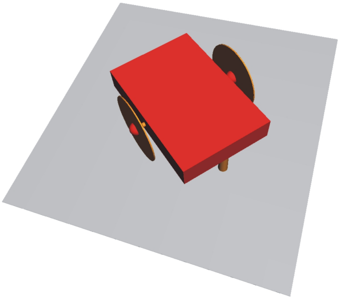


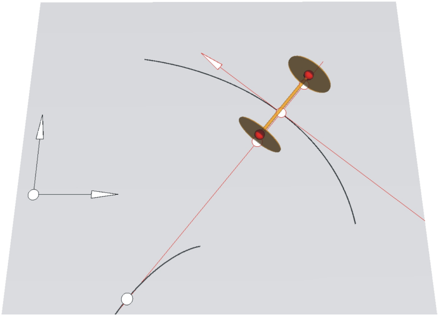
planar case B

s is a tractrix with respect to a and b



mobile robot with two wheels

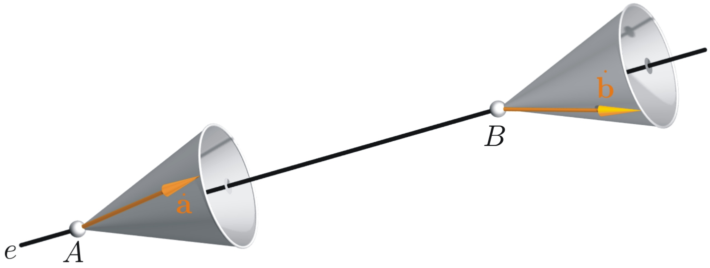




The Spatial Case

ruled surface Φ generated by the motion of $e = AB$:

$\mathbf{y}(t, u) = \mathbf{x}(t) + u\mathbf{e}(t)$ with $\langle \mathbf{e}, \mathbf{e} \rangle = 1$



ruled surface Φ generated by the motion of $e = AB$:

$\mathbf{y}(t, u) = \mathbf{x}(t) + u\mathbf{e}(t)$ with $\langle \mathbf{e}, \mathbf{e} \rangle = 1$

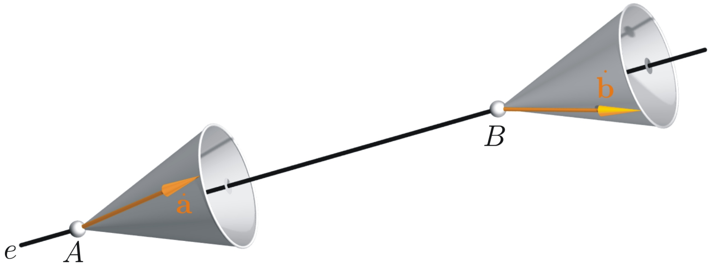
$A \ldots\ \mathbf{a}(t) = \mathbf{y}(t, a)$
$B \ldots\ \mathbf{b}(t) = \mathbf{y}(t, a + d)$

ruled surface Φ generated by the motion of $e = AB$:

$\mathbf{y}(t,u) = \mathbf{x}(t) + u\mathbf{e}(t)$ with $\langle \mathbf{e}, \mathbf{e} \rangle = 1$

$A \ldots \ \mathbf{a}(t) = \mathbf{y}(t,a)$
$B \ldots \ \mathbf{b}(t) = \mathbf{y}(t,a+d)$

$|\dot{\mathbf{a}}| = |\dot{\mathbf{b}}| \Longrightarrow (2a+d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$



$$(2a + d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

$$(2a+d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

spatial case A

$$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$$

$$(2a + d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

spatial case A

$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$

Φ is a cylinder (trivial case)

$$(2a + d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

spatial case A

$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$

Φ is a cylinder (trivial case)

spatial case B

$\dot{\mathbf{x}} = 0 \iff \mathbf{x} = \mathbf{c} = \mathbf{s}$

$$(2a + d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

spatial case A

$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$

Φ is a cylinder (trivial case)

spatial case B

$\dot{\mathbf{x}} = 0 \iff \mathbf{x} = \mathbf{c} = \mathbf{s}$

Φ is a cone and A, B are symmetric w.r.t. its vertex S (trivial case)

$$(2a+d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

spatial case A

$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$

Φ is a cylinder (trivial case)

spatial case B

$\dot{\mathbf{x}} = 0 \iff \mathbf{x} = \mathbf{c} = \mathbf{s}$

Φ is a cone and A, B are symmetric w.r.t. its vertex S (trivial case)

spatial case C

$\dot{\mathbf{e}}, \dot{\mathbf{x}} \neq 0$

$$(2a + d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

spatial case A

$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$

Φ is a cylinder (trivial case)

spatial case B

$\dot{\mathbf{x}} = 0 \iff \mathbf{x} = \mathbf{c} = \mathbf{s}$

Φ is a cone and A, B are symmetric w.r.t. its vertex S (trivial case)

spatial case C

$\dot{\mathbf{e}}, \dot{\mathbf{x}} \neq 0$

$$a + \frac{d}{2} = -\frac{\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle}{\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle}$$

$$(2a+d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

spatial case A

$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$

Φ is a cylinder (trivial case)

spatial case B

$\dot{\mathbf{x}} = 0 \iff \mathbf{x} = \mathbf{c} = \mathbf{s}$

Φ is a cone and A, B are symmetric w.r.t. its vertex S (trivial case)

spatial case C

$\dot{\mathbf{e}}, \dot{\mathbf{x}} \neq 0$

$$a + \frac{d}{2} = -\frac{\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle}{\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle}$$

The midpoint S of AB is the striction (cuspidal) point on $e = AB$.

An Example: screw motion

An Example: screw motion

all points on a common right cylinder around the screw axis have paths of equal length

An Example: screw motion

all points on a common right cylinder around the screw axis have paths of equal length

striction curve s is the helix generated by S

An interpolation problem

Given: series of positions A_iB_i, $i = 1, \ldots, n$ of the rod AB

Given: series of positions A_iB_i, $i = 1, \ldots, n$ of the rod AB

Wanted: motion Σ/Σ^* which

a) moves AB through the given positions and

b) guarantees equal path lengths of A and B

Given: series of positions A_iB_i, $i = 1, \dots, n$ of the rod AB

Wanted: motion Σ/Σ^* which

a) moves AB through the given positions and

b) guarantees equal path lengths of A and B

Construction:

Given: series of positions A_iB_i, $i = 1, \dots, n$ of the rod AB

Wanted: motion Σ/Σ^* which

a) moves AB through the given positions and

b) guarantees equal path lengths of A and B

Construction:

Step 1: Find a suitable curve s interpolating the midpoints S_i of A_iB_i

Given: series of positions A_iB_i, $i = 1, \ldots, n$ of the rod AB

Wanted: motion Σ/Σ^* which

a) moves AB through the given positions and

b) guarantees equal path lengths of A and B

Construction:

Step 1: Find a suitable curve s interpolating the midpoints S_i of A_iB_i

Step 2: Find a ruled surface Φ that interpolates $e_i = A_iB_i$ and whose striction curve is s

Step 1: curve $s \ldots \mathbf{s}(t)$ with $\mathbf{s}(t_i) = \dfrac{\mathbf{a}_i + \mathbf{b}_i}{2} =: \mathbf{s}_i,\ i = 1, \ldots, n$

Step 1: curve $s \ldots \mathbf{s}(t)$ with $\mathbf{s}(t_i) = \dfrac{\mathbf{a}_i + \mathbf{b}_i}{2} =: \mathbf{s}_i, \; i = 1, \ldots, n$

$\tau \ldots$ arclength on s: $\mathbf{s} = \mathbf{s}(\tau)$;

$\langle \mathbf{s}', \mathbf{s}' \rangle = 1$;

$\mathbf{s}(\tau_i) = \mathbf{s}_i$

Step 1: curve $s \dots \mathbf{s}(t)$ with $\mathbf{s}(t_i) = \dfrac{\mathbf{a}_i + \mathbf{b}_i}{2} =: \mathbf{s}_i,\ i = 1, \dots, n$

$\tau \dots$ arclength on s: $\mathbf{s} = \mathbf{s}(\tau)$;
$\langle \mathbf{s}', \mathbf{s}' \rangle = 1$;
$\mathbf{s}(\tau_i) = \mathbf{s}_i$

Step 2: Find a vector function $\mathbf{e} = \mathbf{e}(\tau)$ with

Step 1: curve $s \ldots \mathbf{s}(t)$ with $\mathbf{s}(t_i) = \dfrac{\mathbf{a}_i + \mathbf{b}_i}{2} =: \mathbf{s}_i$, $i = 1, \ldots, n$

$\tau \ldots$ arclength on s: $\mathbf{s} = \mathbf{s}(\tau)$;
$\langle \mathbf{s}', \mathbf{s}' \rangle = 1$;
$\mathbf{s}(\tau_i) = \mathbf{s}_i$

Step 2: Find a vector function $\mathbf{e} = \mathbf{e}(\tau)$ with

$$\mathbf{e}(\tau_i) = \mathbf{e}_i = \frac{\mathbf{b}_i - \mathbf{a}_i}{|\mathbf{b}_i - \mathbf{a}_i|} \quad i = 1, \ldots, n \qquad (\mathrm{I})$$

Step 1: curve $s \ldots \mathbf{s}(t)$ with $\mathbf{s}(t_i) = \dfrac{\mathbf{a}_i + \mathbf{b}_i}{2} =: \mathbf{s}_i$, $i = 1, \ldots, n$

$\tau \ldots$ arclength on s: $\mathbf{s} = \mathbf{s}(\tau)$;
$\langle \mathbf{s}', \mathbf{s}' \rangle = 1$;
$\mathbf{s}(\tau_i) = \mathbf{s}_i$

Step 2: Find a vector function $\mathbf{e} = \mathbf{e}(\tau)$ with

$$\mathbf{e}(\tau_i) = \mathbf{e}_i = \frac{\mathbf{b}_i - \mathbf{a}_i}{|\mathbf{b}_i - \mathbf{a}_i|} \quad i = 1, \ldots, n \qquad \text{(I)}$$

$$\langle \mathbf{s}', \mathbf{e}' \rangle = 0 \qquad \text{(S)}$$

Step 1: curve $s \dots \mathbf{s}(t)$ with $\mathbf{s}(t_i) = \dfrac{\mathbf{a}_i + \mathbf{b}_i}{2} =: \mathbf{s}_i$, $i = 1, \dots, n$

$\tau \dots$ arclength on s: $\mathbf{s} = \mathbf{s}(\tau)$;

$\langle \mathbf{s}', \mathbf{s}' \rangle = 1$;

$\mathbf{s}(\tau_i) = \mathbf{s}_i$

Step 2: Find a vector function $\mathbf{e} = \mathbf{e}(\tau)$ with

$$\mathbf{e}(\tau_i) = \mathbf{e}_i = \frac{\mathbf{b}_i - \mathbf{a}_i}{|\mathbf{b}_i - \mathbf{a}_i|} \quad i = 1, \dots, n \qquad \text{(I)}$$

$$\langle \mathbf{s}', \mathbf{e}' \rangle = 0 \qquad \text{(S)}$$

$$\langle \mathbf{e}, \mathbf{e} \rangle = 1 \qquad \text{(U)}$$

$\sigma := \angle(\mathbf{e}, \mathbf{s}') \ldots$ striction of Φ

$\sigma := \angle(\mathbf{e}, \mathbf{s}')$ ... striction of Φ

$\kappa := |\mathbf{s}''|$ ... curvature of s

$\sigma := \angle(\mathbf{e}, \mathbf{s}')$ ... striction of Φ

$\kappa := |\mathbf{s}''|$ ... curvature of s

$\mathbf{t} = \mathbf{s}'$, $\mathbf{h} = \frac{1}{|\mathbf{s}''|}\mathbf{s}''$, $\mathbf{b} = \mathbf{t} x \mathbf{h}$ ... Frenet frame of s

$\sigma := \angle(\mathbf{e}, \mathbf{s}')$ ... striction of Φ

$\kappa := |\mathbf{s}''|$ ... curvature of s

$\mathbf{t} = \mathbf{s}'$, $\mathbf{h} = \frac{1}{|\mathbf{s}''|}\mathbf{s}''$, $\mathbf{b} = \mathbf{t} x \mathbf{h}$ ... Frenet frame of s

$\langle \mathbf{s}', \mathbf{e} \rangle = \cos\sigma$

$\sigma := \angle(\mathbf{e}, \mathbf{s}')$ ... striction of Φ

$\kappa := |\mathbf{s}''|$ ... curvature of s

$\mathbf{t} = \mathbf{s}'$, $\mathbf{h} = \frac{1}{|\mathbf{s}''|}\mathbf{s}''$, $\mathbf{b} = \mathbf{t} x \mathbf{h}$ ... Frenet frame of s

$$\langle \mathbf{s}', \mathbf{e} \rangle = \cos\sigma$$

$$\langle \mathbf{s}'', \mathbf{e} \rangle + \underbrace{\langle \mathbf{s}', \mathbf{e}' \rangle}_{=\ 0} = -\sigma' \sin\sigma$$

$\sigma := \angle(\mathbf{e}, \mathbf{s}')$ ... striction of Φ

$\kappa := |\mathbf{s}''|$ ... curvature of s

$\mathbf{t} = \mathbf{s}'$, $\mathbf{h} = \frac{1}{|\mathbf{s}''|}\mathbf{s}''$, $\mathbf{b} = \mathbf{t} x \mathbf{h}$ ... Frenet frame of s

$$\langle \mathbf{s}', \mathbf{e} \rangle = \cos\sigma$$

$$\langle \mathbf{s}'', \mathbf{e} \rangle + \underbrace{\langle \mathbf{s}', \mathbf{e}' \rangle}_{=\ 0} = -\sigma' \sin\sigma$$

$$\begin{array}{lllllll} \varepsilon & \ldots & \langle \mathbf{s}', \mathbf{e} \rangle & - & \cos\sigma & = & 0 \\ \varepsilon_1 & \ldots & \langle \mathbf{s}'', \mathbf{e} \rangle & + & \sigma' \sin\sigma & = & 0 \end{array}$$

$\sigma := \angle(\mathbf{e}, \mathbf{s}')$ ... striction of Φ

$\kappa := |\mathbf{s}''|$ ... curvature of s

$\mathbf{t} = \mathbf{s}'$, $\mathbf{h} = \frac{1}{|\mathbf{s}''|}\mathbf{s}''$, $\mathbf{b} = \mathbf{t} x \mathbf{h}$ ... Frenet frame of s

$$\langle \mathbf{s}', \mathbf{e} \rangle = \cos\sigma$$

$$\langle \mathbf{s}'', \mathbf{e} \rangle + \underbrace{\langle \mathbf{s}', \mathbf{e}' \rangle}_{=\ 0} = -\sigma' \sin\sigma$$

$$\begin{array}{ccccccc} \varepsilon & \dots & \langle \mathbf{s}', \mathbf{e} \rangle & - & \cos\sigma & = & 0 \\ \varepsilon_1 & \dots & \langle \mathbf{s}'', \mathbf{e} \rangle & + & \sigma' \sin\sigma & = & 0 \end{array}$$

developable surface Γ with generators g parallel to $\mathbf{b}$

$\sigma := \angle(\mathbf{e}, \mathbf{s}') \ldots$ striction of Φ

$\kappa := |\mathbf{s}''| \ldots$ curvature of s

$\mathbf{t} = \mathbf{s}'$, $\mathbf{h} = \frac{1}{|\mathbf{s}''|}\mathbf{s}''$, $\mathbf{b} = \mathbf{t}x\mathbf{h} \ldots$ Frenet frame of s

$$\langle \mathbf{s}', \mathbf{e} \rangle = \cos\sigma$$

$$\langle \mathbf{s}'', \mathbf{e} \rangle + \underbrace{\langle \mathbf{s}', \mathbf{e}' \rangle}_{=\ 0} = -\sigma' \sin\sigma$$

$$\begin{array}{lllllll} \varepsilon & \ldots & \langle \mathbf{s}', \mathbf{e} \rangle & - & \cos\sigma & = & 0 \\ \varepsilon_1 & \ldots & \langle \mathbf{s}'', \mathbf{e} \rangle & + & \sigma' \sin\sigma & = & 0 \end{array}$$

developable surface Γ with generators g parallel to $\mathbf{b}$

$$\varepsilon_1 \quad \ldots \quad \langle \mathbf{h}, \mathbf{e} \rangle \quad + \quad \frac{\sigma' \sin\sigma}{\kappa} \quad = \quad 0$$

$\sigma := \angle(\mathbf{e}, \mathbf{s}')$ ... striction of Φ

$\kappa := |\mathbf{s}''|$ ... curvature of s

$\mathbf{t} = \mathbf{s}'$, $\mathbf{h} = \frac{1}{|\mathbf{s}''|}\mathbf{s}''$, $\mathbf{b} = \mathbf{t} x \mathbf{h}$ ... Frenet frame of s

$$\langle \mathbf{s}', \mathbf{e} \rangle = \cos\sigma$$

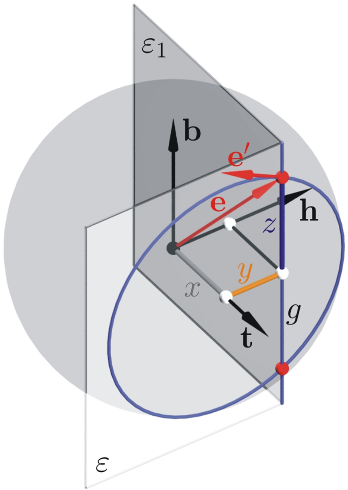
$$\langle \mathbf{s}'', \mathbf{e} \rangle + \underbrace{\langle \mathbf{s}', \mathbf{e}' \rangle}_{=\ 0} = -\sigma' \sin\sigma$$

$$\begin{array}{lllllll} \varepsilon & \ldots & \langle \mathbf{s}', \mathbf{e} \rangle & - & \cos\sigma & = & 0 \\ \varepsilon_1 & \ldots & \langle \mathbf{s}'', \mathbf{e} \rangle & + & \sigma' \sin\sigma & = & 0 \end{array}$$

developable surface Γ with generators g parallel to $\mathbf{b}$

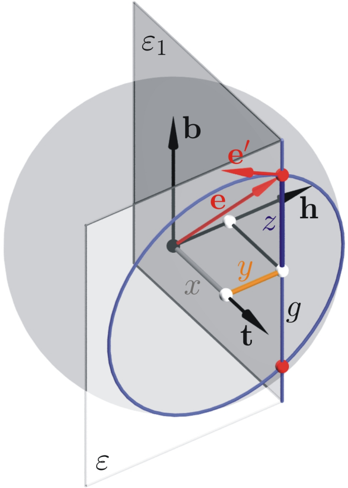
$$\varepsilon_1 \quad \ldots \quad \langle \mathbf{h}, \mathbf{e} \rangle \quad + \quad \frac{\sigma' \sin\sigma}{\kappa} \quad = \quad 0$$

The intersection of Γ with the unit sphere contains the spherical generator image $\mathbf{e} = \mathbf{e}(s)$ of Φ

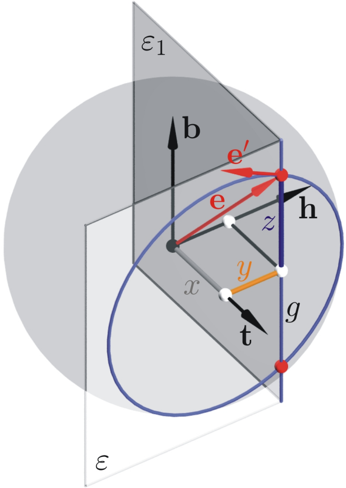


ε_1
b
e′
e
h
z
y
x
g
t
ε

$$
\begin{aligned}
x &= \cos\sigma \\
y &= -\frac{\sigma' \sin\sigma}{\kappa} \\
z &= \pm \sin\sigma \sqrt{1 - \frac{\sigma'^2}{\kappa^2}}
\end{aligned}
$$



$$x = \cos\sigma$$
$$y = -\frac{\sigma' \sin\sigma}{\kappa}$$
$$z = \pm\sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}$$



$$\mathbf{e} = \cos\sigma\cdot\mathbf{t} - \sigma'\sin\sigma\cdot\mathbf{h} \pm \sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

$$\mathbf{e} \;=\; \cos\sigma\cdot\mathbf{t} \;-\; \sigma'\sin\sigma\cdot\mathbf{h} \;\pm\; \sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

$$\mathbf{e} \quad = \quad \cos\sigma \cdot \mathbf{t} \quad - \quad \sigma' \sin\sigma \cdot \mathbf{h} \quad \pm \quad \sin\sigma \sqrt{1 - \tfrac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

Special Cases

$$\mathbf{e} \quad = \quad \cos\sigma \cdot \mathbf{t} \quad - \quad \sigma' \sin\sigma \cdot \mathbf{h} \;\pm\; \sin\sigma \sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

Special Cases

- $\sigma' = 0$: Φ is a ruled surface of constant striction; $\mathbf{e} \in [\mathbf{t}, \mathbf{b}]$

$$\mathbf{e} \;=\; \cos\sigma\cdot\mathbf{t} \;-\; \sigma'\sin\sigma\cdot\mathbf{h} \;\pm\; \sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

Special Cases

- $\sigma' = 0$: Φ is a ruled surface of constant striction; $\mathbf{e} \in [\mathbf{t}, \mathbf{b}]$
- $\sigma = 0$: Φ is the tangent surface of s

$$\mathbf{e} \;=\; \cos\sigma\cdot\mathbf{t} \;-\; \sigma'\sin\sigma\cdot\mathbf{h} \;\pm\; \sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

Special Cases

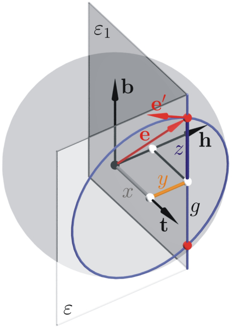
- $\sigma' = 0$: Φ is a ruled surface of constant striction;
  $\mathbf{e} \in [\mathbf{t}, \mathbf{b}]$
- $\sigma = 0$: Φ is the tangent surface of s
- $\kappa = 0$ (striction curve s is a straight line):
  a solution is possible only if $\sigma' = 0$, i.e.; $\sigma = const$.
  Φ is a ruled surface of constant slope and a straight line as striction curve.

$$\mathbf{e} \quad = \quad \cos\sigma \cdot \mathbf{t} \;\; - \;\; \sigma' \sin\sigma \cdot \mathbf{h} \;\; \pm \;\; \sin\sigma \sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

$$\mathbf{e} = \cos\sigma \cdot \mathbf{t} - \sigma' \sin\sigma \cdot \mathbf{h} \pm \sin\sigma\sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

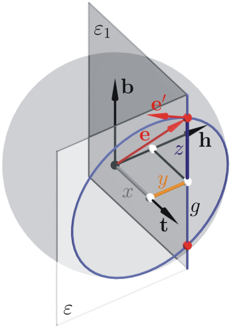
$$\varepsilon \quad \ldots \quad \langle \mathbf{t}, \mathbf{e} \rangle - \cos\sigma = 0$$

$$\varepsilon_1 \quad \ldots \quad \langle \mathbf{h}, \mathbf{e} \rangle + \frac{\sigma' \sin\sigma}{\kappa} = 0$$



$$\mathbf{e} = \cos\sigma \cdot \mathbf{t} - \sigma' \sin\sigma \cdot \mathbf{h} \pm \sin\sigma\sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

$$\begin{array}{llllll} \varepsilon & \ldots & \langle \mathbf{t}, \mathbf{e} \rangle & - & \cos\sigma & = 0 \\ \varepsilon_1 & \ldots & \langle \mathbf{h}, \mathbf{e} \rangle & + & \dfrac{\sigma' \sin\sigma}{\kappa} & = 0 \end{array}$$

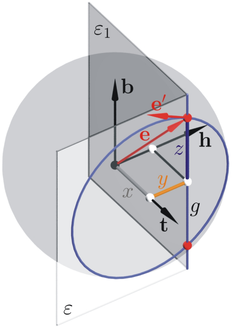


Construct a striction function σ with

$$\mathbf{e} \quad = \quad \cos\sigma \cdot \mathbf{t} \quad - \quad \sigma' \sin\sigma \cdot \mathbf{h} \quad \pm \quad \sin\sigma \sqrt{1 - \tfrac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

$$\varepsilon \quad \ldots \quad \langle \mathbf{t}, \mathbf{e} \rangle \quad - \quad \cos\sigma \quad = \quad 0$$

$$\varepsilon_1 \quad \ldots \quad \langle \mathbf{h}, \mathbf{e} \rangle \quad + \quad \frac{\sigma' \sin\sigma}{\kappa} \quad = \quad 0$$

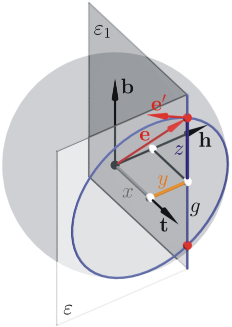


Construct a striction function σ with

$$\sigma'(\tau)^2 \quad \leq \quad \kappa^2(\tau)$$

$$\mathbf{e} = \cos\sigma \cdot \mathbf{t} - \sigma' \sin\sigma \cdot \mathbf{h} \pm \sin\sigma \sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

$$\begin{aligned} \varepsilon \quad \ldots \quad & \langle \mathbf{t}, \mathbf{e} \rangle - \cos\sigma = 0 \\ \varepsilon_1 \quad \ldots \quad & \langle \mathbf{h}, \mathbf{e} \rangle + \frac{\sigma' \sin\sigma}{\kappa} = 0 \end{aligned}$$



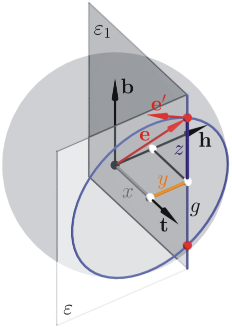
Construct a striction function σ with

$$\sigma'(\tau)^2 \leq \kappa^2(\tau)$$

$$\left.\begin{aligned} \sigma(\tau_i) &= \arccos\langle \mathbf{s}'(\tau_i), \mathbf{e}_i \rangle \\ \sigma'(\tau_i) &= -\frac{\langle \mathbf{s}''(\tau_i), \mathbf{e}_i \rangle}{\sin\sigma(\tau_i)} \end{aligned}\right\}, \quad i = 1, \ldots, n$$

$$\mathbf{e} = \cos\sigma \cdot \mathbf{t} - \sigma' \sin\sigma \cdot \mathbf{h} \pm \sin\sigma \sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

$$\begin{aligned} \varepsilon \quad \ldots \quad & \langle \mathbf{t}, \mathbf{e} \rangle - \cos\sigma = 0 \\ \varepsilon_1 \quad \ldots \quad & \langle \mathbf{h}, \mathbf{e} \rangle + \frac{\sigma' \sin\sigma}{\kappa} = 0 \end{aligned}$$



Construct a striction function σ with

$$\sigma'(\tau)^2 \leq \kappa^2(\tau)$$

$$\left.\begin{aligned} \sigma(\tau_i) &= \arccos\langle \mathbf{s}'(\tau_i), \mathbf{e}_i \rangle \\ \sigma'(\tau_i) &= -\frac{\langle \mathbf{s}''(\tau_i), \mathbf{e}_i \rangle}{\sin\sigma(\tau_i)} \end{aligned}\right\}, \quad i = 1, \ldots, n$$

Construct σ as a Hermite interpolant.

- H. Beck, Über Striktionsgebilde, *Jahresbericht D. M. V.* 37, 91–106, 1928.
- R. Behari, Some properties of the line of striction of a ruled surface, *Math. Notes Edingburgh math. Soc.* 31, 12–13, 1939.
- O. Bottema, G. R. Veldkamp, On the Lines in Space With Equal Distances to n Given Points, *Geometriae Dedicata* 6, 121-129, 1977.
- K. Brauner, H. R. Müller, Über Kurven, welche von den Endpunkten einer bewegten Strecke mit konstanter Geschwindigkeit durchlaufen werden, *Math. Z.* 47, 291–317, 1941.
- A. Gfrerrer, J. Lang, A. Harrich, M. Hirz, J. Mayr, Car Side Window Kinematics, *Computer Aided Design* 43, 2011, pp. 410–416.
- A. Gfrerrer, Motion of a Line Segment Whose Endpoint Paths Have Equal Arc Length, *Proc. of the IW-LGK-11*, 2011, pp. 21–26.

Thanks for your attention!