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ABSOLUTE CONTINUITY OF A LAW OF AN ITÔ PROCESS DRIVEN BY A LÉVY PROCESS TO ANOTHER ITÔ PROCESS

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Abstract: Let ξ_1 and ξ_2 be two solutions of two stochastic differential equations with respect to Lévy noise taking values in a certain type of Banach space. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the probability measures on the corresponding Skorohod space induced by ξ_1 and ξ_2, respectively. In the paper we are interested under which conditions $\mathcal{Q}_1$ is absolute continuous with respect to $\mathcal{Q}_2$. Moreover, we give an explicit formula for the Radon Nikodym derivative of $\mathcal{Q}_1$ with respect to $\mathcal{Q}_2$.

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1. Introduction

Let ξ_1 and ξ_2 be two $\mathbb{R}^d$-valued Lévy processes in $\mathbb{R}^d$ with characteristics (A_1, γ_1, ν_1) and (A_2, γ_2, ν_2). Sato gave in his book (see [14, Theorem 33.1]) exact conditions, under which the probability measures on the Skorohod space $\mathbb{D}([0,T];\mathbb{R}^d)$ of the two Lévy processes ξ_1 and ξ_2 are equivalent and gave an explicit formula of the Radon Nikodym derivative (see also the book of Gihman and Skorohod [9]). Kuzinski [13] generalized this result to Hilbert spaces.

However, often one is only interested in the absolute continuity of one process, e.g. ξ_1, with respect to the other process, e.g. ξ_2, which in fact is a weaker property as equivalence. In this note we consider only absolute continuity, which results in weaker conditions on ξ_1 and ξ_2 as if equivalence of ξ_1 and ξ_2

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would have been considered. Also, the proof of Kuzinski are based on Hilbert space theory. Here, in this note, we use another method which can also be applied to Banach spaces. Therefore, our result can be applied to investigate absolute continuity of two solution processes to infinite dimensional stochastic differential equations driven by Lévy processes, which we will illustrate in Example 2.9. In case of stochastic differential equations, absolute continuity is investigated recently by Fournier and Printems [8]

The change of measure formula (or the Girsanov Theorem) is a useful tool in stochastic analysis. For example, if ξ is a solution to a stochastic (partial) differential equation driven by a Wiener process, Maslowski and Seidler [15] have shown that under certain conditions on ξ the corresponding Markovian semigroup enjoys the strong Feller property by changing the underlying probability measure and applying a change of measure formula. In large deviation the change of measure formula is used to find the appropriate skeleton for solutions of stochastic (partial) differential equations (see for a review, e.g. [6]). In these works the underlying space was a Hilbert space. However, in recent years stochastic partial differential equation with respect to a Wiener process in Banach spaces have been considered by several authors (Brzeźniak [3], van Neerven, Veraar and Weiss [17], van Neerven [16]). Similarly, there exists several advantages considering stochastic partial differential equations driven by a Lévy process in Banach spaces (see e.g. [4, 10, 5]). For example in [10] we were able to weaken the conditions of the diffusion coefficients. These facts were the motivation to establish the change of measure formula in Banach spaces for solutions of infinite dimensional stochastic differential processes driven by Lévy processes.

In the first part of this note we give a short account of Lévy processes and Itô processes in Banach spaces. Then, we present the main result. Finally we give an example where we apply our result to two stochastic partial differential equations driven by a Lévy process and give an explicit formula of the density process.

2. Preliminares

To start, let us recall shortly the definition of a Lévy process, the definition of Lévy measures and the Lévy-Khintchine formula which determines the law of a Lévy process in a unique way.

Definition 2.1. (see Definition 1.6 [14, p. 3]) Let E be a Banach space. A stochastic process $L = \{L(t) : 0 \leq t < \infty\}$ is an E-valued Lévy process over a probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a right continuous filtration, if the following conditions are satisfied:

- For any choice $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots t_n$, the random variables $L(t_0)$, $L(t_1) - L(t_0)$, ..., $L(t_n) - L(t_{n-1})$ are independent.
- $L(0) = 0$ a.s.
- For all $0 \leq s < t$, the distribution of $L(t+s) - L(s)$ does not depend on s.
- L is stochastically continuous.
- The trajectories of L are a.s. cádlág on E.
- L is $\mathbb{F}$–adapted.

Let E be a Banach space. For any E-valued Lévy process $\{L(t) : t \geq 0\}$ there exists a trace class operator $Q : E \to E'$, a non negative Lévy measure ν concentrated on $E \setminus \{0\}$, and an element $m \in E'$ such that

$$\mathbb{E}e^{i\langle L(1),x\rangle} = \exp\Big(i\langle m, x\rangle - \frac{1}{2}\langle Qx, x\rangle \\ + \int_E \Big(1 - e^{i\langle y,x\rangle} + 1_{(-1,1)}(|y|_E) i\langle y, x\rangle\Big)\,\nu(dy)\Big), \quad x \in E'.$$

We call the measure ν *characteristic measure* of the Lévy process $\{L(t) : t \geq 0\}$. Moreover, the triplet (Q, m, ν) uniquely determines the law of the Lévy process.

Now, starting with an E–valued Lévy process over a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, one can construct an integer valued random measure by

$$\eta_L : \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}_+) \ni (B \times I) \mapsto \#\{s \in I \mid \Delta_s L \in B\} \in \mathbb{N}_0 \cup \{\infty\}.^1.$$

The random measure η_L is a so-called Poisson random measure, whose definition we give below.

Definition 2.2. (see [11], Definition I.8.1) Let $(S, \mathcal{S})$ be a measurable space and let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. A *Poisson random measure* η on $(S, \mathcal{S})$ with intensity $\nu \in M_+(S)$ over $\mathfrak{A}$ is a measurable function $\eta : (\Omega, \mathcal{F}) \to (M_I(S \times \mathbb{R}_+), \mathcal{M}_I(S \times \mathbb{R}_+))$, such that

[1]The *jump process* $\Delta X = \{\Delta_t X : 0 \leq t < \infty\}$ of a process X is given by $\Delta_t X(t) := X(t) - X(t-), t \geq 0$ and $\Delta_0 = 0$.

(i) for each $B \times I \in \mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+)$, $\eta(B \times I) := i_{B\times I} \circ \eta : \Omega \to \bar{\mathbb{N}}$ is a Poisson random variable with parameter[2] $\nu(B)\lambda(I)$;

(ii) η is independently scattered, i.e. if the sets $B_j \times I_j \in \mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+)$, $j = 1, \cdots, n$, are pair-wise disjoint, then the random variables $\eta(B_j \times I_j)$, $j = 1, \cdots, n$, are pair-wise independent;

(iii) for each $U \in \mathcal{S}$, the $\bar{\mathbb{N}}$-valued process $(N(t, U))_{t\geq 0}$ defined by

$$N(t, U) := \eta(U \times (0, t]), \quad t \geq 0$$

is $\mathbb{F}$-adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta(U \times (s, t])$ is independent of $\mathcal{F}_s$.

Remark 1. In the framework of Definition 2.2 the assignment

$$\nu : \mathcal{S} \ni A \mapsto \mathbb{E}\big[\eta(A \times (0, 1))\big]$$

defines a uniquely determined measure. We will denote the difference $\eta - \nu$ by $\tilde{\eta}$.

Remark 2. Assume that E is of martingale type p, $p \in [1, 2]$ (for the definition of martingale type p we refer to [4]). Let η be a time homogeneous Poisson random measure on E with an intensity measure ν which is a Lévy measure. Then, the process $L^\eta = \{L^\eta(t) : 0 \leq t < \infty\}$ defined by

$$L(t) := \int_0^t \int_E z\, \tilde{\eta}(dz, ds), \quad t \geq 0,$$

is an E–valued Lévy process with characteristic $\hat{\nu}$, such that $\hat{\nu} = \nu$ on the unit ball. For more details about the connection of a Lévy process and a Poisson random measure we refer to Applebaum [1].

Assume, that $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete probability space with right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $(Z, \mathcal{Z})$ be a measurable space, ν be a σ–finite positive measure on Z and η be a time homogeneous Poisson random measure over $\mathfrak{A}$ with compensator $\gamma = \nu \times \lambda_1$. Finally, fix $p \in (1, 2]$ and let E be a Banach space of martingale type p. Let

$$b_i : [0, \infty) \times E \to E, \quad i = 1, 2,$$

and

$$c_i : [0, \infty) \times E \times Z \to E, \quad i = 1, 2,$$

some mappings satisfying the following hypotheses.

[2] If $\nu(B)\lambda(I) = \infty$, then obviously $\eta(B) = \infty$ a.s..

H 2.3. The functions b_i, $i = 1, 2$, are uniformly bounded in the first variable and uniformly Lipschitz continuous in the second variable. In particular, there exists a constant $K = K_b > 0$ such that

$$|b_i(t, x) - b_i(t, y)| \le K_b|x - y|, \quad x, y \in E, \quad t \ge 0, \quad i = 1, 2.$$

H 2.4. The functions $c_i : [0, \infty) \times E \to L^p(Z, \nu; E)$, $i = 1, 2$, are uniformly bounded in the first variable and Lipschitz continuous in the second variable. In particular, there exists a constant $K = K_c > 0$ such that

$$\int_0^t \int_Z |c_i(s, x, z) - c_i(s, y, z)|^p \, \nu(dz) \, ds \le K|x - y|^p, \quad x, y \in E, \, i = 1, 2, \quad t \ge 0.$$

H 2.5. The function c_1 is $\sigma(c_2)$–measurable and we have for all $t \ge 0$ and $x \in E$

$$\mathrm{Rg}(c_1(t, x, \cdot)) \subset \mathrm{Rg}(c_2(t, x, \cdot))$$

(here, we denoted by Rg the range of a function).

H 2.6. The functions c_i, $i = 1, 2$, map E into a compact subspace of E. In particular there exists a Banach space E_1, $E_1 \hookrightarrow E$ compactly, and a constant $K = K_{E_1} > 0$ such that

$$\int_0^t \int_Z |c_i(s, x, z)|^p \, \nu(dz) \, ds \le K(1 + |x|^p_{E_1}), \quad x \in E \text{ and } i = 1, 2, \quad t \ge 0.$$

Remark 3. From Hypothesis 2.5 it follows by the theorem of Doob (see [12, Chapter 1, Lemma 1.13]) that there exists a measurable mapping $f : [0, \infty) \times E \to E$ such that $c_1 = c_2 \circ f$, i.e. $c_1(s, x, z) = f(s, x, c_2(s, x, z))$ for all $(s, x, z) \in [0, \infty) \times E \times Z$.

Let $x_0 \in E_1$ and let $\xi_i = \{\xi_i(t) : 0 \le t < \infty\}$, $i = 1, 2$, be two solutions to the two stochastic differential equations

$$\left\{ \begin{array}{lcl} \mathrm{d}\xi_i(t) & = & \int_Z c_i(t, \xi_i(t-), z) \, (\eta - \gamma)(dz, dt) + b_i(t, \xi_i(t-)) \, \mathrm{d}t, \\ \xi_i(0) & = & x_0, \qquad\qquad i = 1, 2\,. \end{array} \right. \tag{1}$$

By the assumptions on b_i and c_i, $i = 1, 2$, we can suppose that there exists a unique E-valued cádlág process satisfying the SDE given in (1) such that

$$\xi_i(t) = x_0 + \int_0^t \int_{\mathbb{R}^d} c_i(s, \xi_i(s-), z) \, (\eta - \gamma)(dz, ds) + \int_0^t b_i(s, \xi_i(s)) \, ds, \quad t \ge 0.$$

We denote the space of all cádlág function endowed by the Skorohod topology by $\mathbb{D}([0, \infty); E)$. For shortness we denote the Borel σ-field of $\mathbb{D}([0, \infty); E)$

by $\mathcal{A}$. Let us introduce on $\mathcal{A}$ a filtration $\mathbb{A} = (\mathcal{A}_t)_{t\geq 0}$ such that all admissible mappings[3] $X : \Omega \times [0,\infty) \to E$ are adapted with respect to $\mathbb{A}$.

Before presenting our result we have two introduce the following notation.

The processes ξ_1 and ξ_2 induce two probability measures $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on $\mathbb{D}([0,\infty); E)$. In particular,

$$\mathcal{Q}_i : \mathcal{A} \ni A \mapsto \mathbb{P}(\xi_i \in A). \tag{2}$$

For $t \geq 0$, let $\mathcal{Q}_i(t)$ be the restriction of $\mathcal{Q}_i$ on $\mathcal{A}_t$. We are interested in the Radon Nikodym derivative of $Q_1(t)$ with respect to $Q_2(t)$, $t \geq 0$.

For $i = 1, 2$, let $\nu_i = \{\nu_i(t) : 0 \leq t < \infty\}$ be the unique predictable measure valued processes given by

$$[0,\infty) \times \mathcal{B}(E) \ni (t, A) \mapsto \nu_i(t, A) := \int_Z 1_A(c_i(t, \xi_i(t-), z))\, \nu(dz). \tag{3}$$

In fact, since for $i = 1, 2$, the mapping $c_i : [0,\infty) \times E \times Z$ are measurable, the processes ξ_i^- are predictable, and it follows that for $i = 1, 2$, the measure-valued processes ν_i, $i = 1, 2$, are indeed predictable.

Since for any $(s, x) \in [0,\infty) \times E$ we have $\mathrm{Rg}(c_1(s, x, \cdot)) \subset \mathrm{Rg}(c_2(s, x, \cdot))$, the measure $\nu_{c,1}(s, x, \cdot)$ is absolutely continuous for any $(s, x) \in [0,\infty) \times E$ and with respect to the measure $\nu_{c,2}(s, x, \cdot)$, where

$$\nu_{c,i}(s, x, A) = \int_E 1_A(c_i(s, x, z))\, \nu(dz), \quad i = 1, 2.$$

Therefore, there exists a positive function $g_c : [0,\infty) \times E \times \mathcal{B}(E) \to \mathbb{R}$ such that

$$\nu_{c,1}(s, x, A) = \int_A g_c(s, x, y)\, \nu_{c,2}(s, x, dy). \tag{4}$$

Defining

$$g : [0,\infty) \times E \ni (s, x) \mapsto g(s, \xi_i(s-), x) \in \mathbb{R}^0_+, \tag{5}$$

[3] Here, we call a process X admissible, if there exist two predictable process $c : \Omega \times \mathbb{R}_0^+ \times Z \to E$ and $b : \Omega \times \mathbb{R}_0^+ \to E$ such that for any $I \in \mathcal{B}(\mathbb{R}_0^+)$ we have

$$\mathbb{E} \int_I \int_E |c(s, z)|^2 \nu(dz)\, ds < \infty,$$

and

$$X(t) = \int_0^t \int_E c(s, z)\, (\eta - \gamma)\, (dz, ds) + \int_0^t b(s)\, ds, \quad t \geq 0.$$

it follows that g is a predictable function-valued process such that we have for all $t \geq 0$ and $A \in \mathcal{B}(E)$

$$\nu_1(t, A) = \int_A g(t, x)\, \nu_2(t, dx). \tag{6}$$

Now, with these definitions we can present our main result.

Theorem 2.7. *Let $(Z, \mathcal{Z})$ be a measurable space and η be a time homogenous Poisson random measure on $(Z, \mathcal{Z})$ over a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\mathcal{F} = \sigma(\eta)$. Let E be of martingale type p, $1 < p \leq 2$, and let $c_i : [0, \infty) \times E \times Z \to E$, $b_i : [0, \infty) \times E \to E$, $i = 1, 2$, be measurable functions, satisfying the Hypothesis* **H 2.3-H 2.6**.

Suppose that for all $t \geq 0$ and $x \in E$ we have

$$\int_0^t (b_1(s, x) - b_2(s, x))\; ds = \int_0^t \int_Z (c_1(s, x, z) - c_2(s, x, z))\; \nu(dz)\, ds. \tag{7}$$

Then, the Radon Nikodym derivative of the probability measures $\mathcal{Q}_1$ and $\mathcal{Q}_2$ defined in (2) *is given by*

$$\frac{d\mathcal{Q}_1(t)}{d\mathcal{Q}_2(t)} \;=\; \mathcal{G}(t), \quad t \geq 0,$$

where $\mathcal{G} = \{\mathcal{G}(t) : 0 \leq t < \infty\}$ solves

$$\begin{cases} d\mathcal{G}(t) &= \displaystyle\int_Z (g(t, z)) - 1)\; \mathcal{G}(t-)(\eta - \gamma)(dz, dt), \\ \mathcal{G}(0) &= 1, \end{cases} \tag{8}$$

and $g = \{g(t) : 0 \leq t < \infty\}$ is the density process defined in (6).

Remark 4. Using the definition of g_c in (4) we can write instead of (8) the following

$$\begin{cases} d\mathcal{G}(t) &= \displaystyle\int_Z (g_c(t, \xi_2(t-), z)) - 1)\; \mathcal{G}(t-)(\eta - \gamma)(dz, dt), \\ \mathcal{G}(0) &= 1. \end{cases}$$

Remark 5. In case ξ_1 and ξ_2 are two Lévy processes, Theorem 2.7 leads to the same result as Theorem 33.1 in [14]. To illustrate this fact, we assume in the following that $E = \mathbb{R}^d$ and c_1 and c_2 are constant in time and space. Let ν_1 and ν_2 be two Lévy measures on $\mathbb{R}^d$ such that there exists a function $\rho : \mathbb{R}^d \to \mathbb{R}$ with

$\nu_2(A) = \int_A e^{\rho(s,x)}\, d\nu_1(t,dx)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \left(e^{2\rho(s,x)} - 1\right)^2 \nu(dz) < \infty$. Let $U = \{U(t) : t \geq 0\}$ be given by

$$U(t) \quad := \quad \int_0^t \int_{\mathbb{R}^d} \rho(s,z)\eta(dz,ds) - \int_0^t \int_{\mathbb{R}^d} (e^{\rho(s,z)} - 1)\nu_1(dz)ds, \quad t \geq 0.$$

Then, an application of the Itô formula shows that

$$\mathcal{G}(t) \quad = \quad \exp(U(t)), \quad t \geq 0.$$

Example 2.8. Let E be a Banach spaces of martingale type p. Let Z and Z_1 be two measurable spaces, ν and ν_1 be two σ–finite measures defined on Z and Z_1, respectively, and η and η_1 be two time homogeneous Poisson random measure on Z and Z_1 with intensity measure ν and ν_1, respectively. Let $c : E \times \mathbb{R}_+ \times Z \to B_1(E)$[4] and $c_1 : E \times \mathbb{R}_+ \times Z_1 \to E \setminus B_1(E)$ be two mappings such that

$$\int_0^t \int_Z |c(s,x,z)|^p \nu(dz)\, ds < \infty, \quad \text{and} \quad \int_0^t \int_{Z_1} |c_1(s,x,z)|^p\, \nu_1(dz)\, ds < \infty.$$

Let ξ_1 and ξ_2 be the solutions for $t \geq 0$ to

$$\xi_1(t) = x_0 + \int_0^t b(s,\xi_1(s-))\, ds$$
$$+ \int_0^t \int_Z c(s,\xi_1(s-),z)\, (\eta - \nu)(dz,ds) + \int_0^t \int_{Z_1} c_1(s,\xi_1(s-),z)\, \nu_1(dz)\, ds,$$

$$\xi_2(t) = x_0 + \int_0^t b(s,\xi_2(s-))\, ds$$
$$+ \int_0^t \int_Z c(s,\xi_2(s-),z)(\eta-\nu)(dz,ds) + \int_0^t \int_{Z_1} c_1(s,\xi_2(s-),z)\, (\eta_1 - \nu_1)(dz,ds).$$

Then, g defined in (5) is independent of time and given by

$$g(t,z) = \begin{cases} 1, & z \in Z, \\ 0, & z \in Z_1. \end{cases}$$

If ν_1 is a finite measure, it follows that the density process is given by

$$\mathcal{G}(t) = 1 + \int_0^t G(s-)\, \nu_1(E \setminus B_1(E))\, ds - \int_0^t G(s-)\, \eta_1(E \setminus B_1(E), ds), \quad t \geq 0.$$

[4] $B_1(E) = \{x \in E : |x| \leq 1\}$.

Example 2.9. Let E be a Banach spaces of martingale type p, $1 < p \le 2$, and A be an analytic operator on E with discrete spectrum $\{e_n : n \in \mathbb{N}\}$. Let U be the unit sphere in E, i.e. $U = \{x \in E : |x| = 1\}$ and $\sigma : \mathcal{B}(\partial U) \to \mathbb{R}_+$ be a finite measure defined by

$$B \in \mathcal{B}(\partial U) \quad B \text{ open} \quad \sigma(B) := \sum_{n\in\mathbb{N}} \lambda_n^{-\alpha} \chi_{e_n}(B).$$

Let ν be given by

$$\nu : \mathcal{B}(E) \ni B \mapsto \int_0^\infty 1_B(rx)\, k(r,x)\, dr\, \sigma(dx),$$

where $k(r,x)/r^{\alpha+1} \to 0$ as $r \to \infty$. Let η be a time homogenouse Poisson random measure with intensity measure $\nu|_U$, and η_1 be a time homogenouse Poisson random measure with intensity measure $\nu|_{E\setminus U}$. Let ξ_1 be the Ornstein Uhlenbeck process given by

$$\begin{cases} d\xi_1(t) &= A\xi_1(t) + \int_U z\, \tilde{\eta}(dz, ds), \\ \xi_1(0) &= x_0, \end{cases}$$

and ξ_2 be the Ornstein Uhlenbeck process given by

$$\begin{cases} d\xi_2(t) &= A\xi_2(t) + \int_U z\, \tilde{\eta}(dz, ds) + \int_{E\setminus U} z\, \tilde{\eta}_1(dz, ds), \\ \xi_2(0) &= x_0. \end{cases}$$

Then, at time $t \ge 0$ the probability measure $\mathcal{Q}_1(t)$ is absolute continuous with respect to $\mathcal{Q}_2(t)$ with Radon Nikodym derivative $\mathcal{G}$ given by

$$\mathcal{G}(t) = 1 + \int_0^t G(s-)\, \nu_1(E \setminus B_1(E))\, ds - \int_0^t G(s-)\, \eta_1(E \setminus B_1(E), ds).$$

Example 2.10. Also the change of measure formula in Lemma 6.16 of [2] follows from Theorem 2.7. Let $E = Z = \mathbb{R}^d$ and let η be a time homogenous Poisson random measure with compensator $\gamma = \lambda_d \times \lambda$. Let $v : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be a predictable mapping such that θ, given by $\theta := Id + v$, is invertible. Put

$$\eta_\theta : \mathcal{B}(Z) \times \mathcal{B}([0,\infty)) \ni (I \times A) \mapsto \int_Z 1_A(\theta(z))\, \eta(dz, ds).$$

Then the two Poisson random measure η and η_θ, where η_θ is defined by

$$\eta_\theta : \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}_+) \ni A \times I := \int_I \int_{\mathbb{R}^d} 1_A(\theta(z))\, \eta(dz, ds).$$

Fix $A \in \mathcal{B}(\mathbb{R}^d)$ and let the processes $N^A = \{N^A(t) : 0 \le t < \infty\}$ and $N^A = \{N^A_\theta(t) : 0 \le t < \infty\}$ defined for $t \ge 0$ by $N^A(t) = \eta(A \times [0,t])$ and

$$N^A_\theta(t) = \int_0^t \int_{\mathbb{R}^d} 1_A(\theta(z))\eta_\theta(dz, ds),$$

respectively. Let $\mathcal{Q}^A$ be the probability measure on $\mathbb{D}([0,\infty);\mathbb{R}^d)$ induced by N^A and let $\mathcal{Q}^A_\theta$ be the probability measure on $\mathbb{D}([0,\infty);\mathbb{R}^d)$ induced by N^A_θ. Then, it follows by Theorem 2.7 that the Radon Nikodym derivative $\mathcal{G}$ of N^A with respect to N^A_θ is given by

$$\begin{cases} d\mathcal{G}^A(t) &= \int_A \left[\det(J_\theta(t,z)) - 1\right] \mathcal{G}^A(t-)\,(\eta_\theta - \gamma)(dz, dt) \\ &= \int_A \det(J_v(t,z))\, \mathcal{G}^A(t-)\,(\eta_\theta - \gamma)(dz, dt), \\ \mathcal{G}^A(0) &= 1. \end{cases}$$

Here, J_θ and J_v denotes the Jacobian matrix of the function θ and v, respectively. In particular, we have

$$\frac{\mathrm{d}\mathcal{Q}^A(t)}{\mathrm{d}\mathcal{Q}^A_\theta(t)} = \mathcal{G}^A(t).$$

Let us assume that v is chosen in such a way, that for any $T \ge 0$

$$\int_0^T \int_{\mathbb{R}^d} |\det(J_v(s,z))|^p \;\lambda_d(dz)\,\lambda(ds) < \infty$$

and let $\mathcal{Q}$ the probability measure on $M((0,\infty] \times \mathbb{R}^d)$ induced by η and $\mathcal{Q}_\theta$ the probability measure on $M((0,\infty] \times \mathbb{R}^d)$ induced by η_θ. Let $c : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function satisfying the hypothesis of Theorem 2.7. Let ζ be a solution to

$$\begin{cases} \mathrm{d}\zeta(t) &= \int_{\mathbb{R}^d} c(t, \zeta(t-), z)\,(\eta_\theta - \gamma)(dz, dt) \\ \zeta(0) &= x_0. \end{cases}$$

Note, under $\mathcal{Q}$ the process ζ is a martingale. Under $\mathcal{Q}_\theta$ the process ζ can be written as

$$\begin{cases} \mathrm{d}\zeta(t) = \displaystyle\int_{\mathbb{R}^d} c(t, \zeta(t-), z)\,(\eta - \gamma)(dz, dt) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^d} c(t, \zeta(t-), z)\,(\eta_\theta - \eta)(dz, dt) + b(t, \zeta(t-))\,dt\;, \\ \zeta(0) = x_0. \end{cases}$$

where

$$b(s,x)=\int_{\mathbb{R}^d}(c(s,x,z)-c(s,x,\theta(z)))\,\lambda_d(dz),\quad s\geq 0.$$

It follows from Theorem 2.7 that for all $\phi\in C(\mathbb{R}^d)$ we have $\mathbb{E}^{\mathfrak{Q}_\theta}\mathcal{G}\phi(\zeta)=\mathbb{E}^{\mathfrak{Q}}\phi(\zeta)$.

Proof of Theorem 2.7. In the first step we construct a probability measure $\mathbb{P}_0$ on $\mathfrak{A}$ such that $\mathbb{P}_0\gg\mathbb{P}$ and

$$\mathbb{P}\big(\xi_1\in A\big)=\mathbb{P}_0\big(\xi_2\in A\big),\quad A\in\mathbb{D}([0,\infty);E). \tag{9}$$

For $i=1,2$, let us define the random measure η_i

$$\eta_i:\mathcal{B}(E)\times\mathcal{B}(\mathbb{R}_+)\ni(B\times I)\mapsto\int_I\int_E 1_B(c_i(s,\xi_i(s-),z))\,\eta(dz,ds),$$

and the corresponding compensators γ_i over $\mathfrak{A}$ by

$$\gamma_i:\mathcal{B}(E)\times\mathcal{B}(\mathbb{R}_+)\ni(B\times I)\mapsto\int_I\int_E 1_B(c_i(s,\xi_i(s-),z))\,\nu(dz,ds),$$

Let us now define a new probability measure $\mathbb{P}_0$ on $\mathfrak{A}$ in the following way. For any $A\in\mathcal{B}(E)$, $I\in\mathcal{B}([0,\infty))$ and $k\in\mathbb{N}$ we put

$$\mathbb{P}_0(\eta_2(A\times I)=k):=\mathbb{P}(\eta_1(A\times I)=k).$$

Therefore, η_2 has compensator γ_1 under $\mathbb{P}_0$. Note that the process ξ_2 can also be written as follows

$$\xi_2(t)=x_0+\int_0^t\int_Z z\,(\eta_2-\gamma_1)(dz,ds)+\int_0^t b_2(s,\xi_2(s-))\,ds+\Gamma(t),\quad t\geq 0,$$

where

$$\Gamma(t):=\int_0^t\int_Z(c_1(s,\xi_2(s-),z)-c_2(s,\xi_2(s-),z))\,dz\,ds.$$

Condition (7) implies that

$$\xi_2(t)=x_0+\int_0^t\int_Z z\,(\eta_2-\gamma_1)(dz,ds)+\int_0^t b_1(s,\xi_2(s-))\,ds,\quad t\geq 0.$$

Hence, ξ_2 solves a SDE under $\mathbb{P}_0$ given by

$$\begin{cases}\mathrm{d}\xi(t) &= \int_Z z\,(\eta_2-\gamma_1)(dz,dt)+b_1(s,\xi(t-))\,dt,\\ \xi(0) &= x_0,\end{cases}$$

and, therefore, we can conclude that

$$\mathbb{P}(\xi_1 \in A) = \mathbb{P}_0(\xi_2 \in A), \quad t \geq 0.$$

Since

$$\frac{d\mathcal{Q}_1(A)}{d\mathcal{Q}_2(A)} = \frac{\mathrm{d}\mathbb{P}(\xi_1 \in A)}{\mathrm{d}\mathbb{P}(\xi_2 \in A)} = \frac{\mathrm{d}\mathbb{P}_0(\xi_2 \in A)}{\mathrm{d}\mathbb{P}(\xi_2 \in A)},$$

we will calculate in the second step the Radon Nikodym derivative of $\mathbb{P}_0 \circ \xi_2$ with respect to $\mathbb{P} \circ \xi_2$. In fact, we will show

$$\mathbb{E}^{\mathbb{P}_0} e^{-\lambda \xi_2(t)} = \mathbb{E}^{\mathbb{P}} \mathcal{G}(t) e^{-\lambda \xi_2(t)}. \tag{10}$$

Assume for the time being that (10) is valid. Then it follows from Identity (7) that, if ξ is the canonical process on $\mathbb{D}([0,\infty);E)$ (i.e. for $\omega \in \mathbb{D}([0,\infty);E)$, $\xi(\omega) = \omega$)

$$\mathbb{E}^{\mathcal{Q}_1} e^{-\lambda \xi(t)} = \mathbb{E}^{\mathbb{P}} e^{-\lambda \xi_1(t)} = \mathbb{E}^{\mathbb{P}_0} e^{-\lambda \xi_2(t)} = \mathbb{E}^{\mathbb{P}} (\mathcal{G} \circ \xi_2)(t)\, e^{-\lambda \xi_2(t)}$$
$$= \mathbb{E}^{\mathcal{Q}_2} \mathcal{G}(t)\, e^{-\lambda \xi(t)}. \tag{11}$$

By the definition of the Radon Nikodym derivative, the assertion follows. Put

$$Z(t) := \exp\left(-\lambda \xi_2(t) + \lambda \int_0^t \int_E \left[e^{-\lambda z} - 1 + \lambda z \right] \gamma_1(dz, ds) \right.$$
$$\left. + \lambda \int_0^t b_1(s, \xi_2(s-))\, ds \right).$$

An application of the Itô formula and taking into account that

$$\int_0^t b_1(s, \xi_2(s-))\, ds = \int_0^t b_2(s, \xi_2(s-))\, ds + \Gamma(t)$$

give for $t > 0$ and $\mathbb{P}_0$-a.s.

$$dZ(t) = -\lambda \int_0^t \int_E Z(s-) z (\eta_2 - \gamma_1)(dz, ds)$$
$$+ \int_0^t \int_E Z(s-) \left[e^{-\lambda z} - 1 + \lambda z \right] (\eta_2 - \gamma_1)(dz, ds).$$

Since the compensator of η_2 is γ_1 over $(\Omega, \mathcal{F}, \mathbb{P}_0)$, it follows that $\mathbb{E}^{\mathbb{P}_0} Z(t) = 1$. Hence, our aim is to show that equation (10) is valid. Applying the Itô formula we obtain

$$\mathbb{E}^{\mathbb{P}} \left[\mathcal{G}(t) Z(t) \right] = 1 - \lambda \mathbb{E}^{\mathbb{P}} \int_0^t \int_E \mathcal{G}(s-) Z(s-) z (\eta_2 - \gamma_1)(dz, ds)$$

$$
\begin{aligned}
&+\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)\left[e^{-\lambda z}-1+\lambda z\right](\eta_2-\gamma_1)(dz,ds)\\
&+\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)\left[g(s,z)-1\right](\eta_2-\gamma_1)(dz,ds)\\
&+\mathbb{E}^{\mathbb{P}}\int_0^t\Big[\int_E\Big[\mathcal{G}(s)Z(s)-\mathcal{G}(s-)Z(s-)\\
&-Z(s-)\mathcal{G}(s-)\left(e^{-\lambda z}-1\right)-Z(s-)\mathcal{G}_i(s-)\left(g(s,z)-1\right)\Big]\eta_2(dz,ds)\Big].
\end{aligned}
$$

Since η_2 has compensator γ_1 under $\mathbb{P}$ we get

$$
\begin{aligned}
= \quad & 1-\lambda\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)z(\gamma_1-\gamma_1)(dz,ds)\\
&+\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)\left[e^{-\lambda z}-1+\lambda z\right](\gamma_1-\gamma_1)(dz,ds)\\
&+\mathbb{E}^{\mathbb{P}}\int_0^t\Big[\int_E \mathcal{G}(s-)Z(s-)\big[e^{-\lambda z}g(s,z)-1-\left(e^{-\lambda z}-1\right)\\
&-\left(g(s,z)-1\right)\big]\gamma_1(dz,ds)\Big].
\end{aligned}
$$

By identity (6) we get

$$
\begin{aligned}
= \quad & 1-\lambda\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)z(1-g(s,z))\nu_1(s,dz)\,ds\\
&+\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)\left[e^{-\lambda z}-1+\lambda z\right](1-g(s,z))\nu_1(s,dz)\,ds\\
&+\mathbb{E}^{\mathbb{P}}\int_0^t\Big[\int_E \mathcal{G}(s-)Z(s-)\big[e^{-\lambda z}g(s,z)-e^{-\lambda z}-g(s,z)+1\big]\nu_1(s,dz)\,ds\Big].
\end{aligned}
$$

Some calculations lead to

$$
\begin{aligned}
= \quad & 1-\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)\lambda z(1-g(s,z))\nu_1(s,dz)\\
&+\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)\Big[\big[e^{-\lambda z}-1+\lambda z\big](1-g(s,z))\\
&+e^{-\lambda z}g(s,z)-e^{-\lambda z}-g(s,z)-1\Big]\nu_1(s,dz)\,ds\\
= \quad & 1-\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)\lambda z(1-g(s,z))\nu_1(s,dz)\,ds\\
&+\mathbb{E}^{\mathbb{P}}\int_0^t\int_E \mathcal{G}(s-)Z(s-)\big[\lambda z-\lambda z g(s,z)\big]\nu_1(s,dz)\,ds=1.
\end{aligned}
$$

Therefore, $\mathbb{E}^{\mathbb{P}_0} Z(t) = \mathbb{E}^{\mathbb{P}} \mathcal{G}(t) Z(t)$ and (10) is valid, hence, by (11) the assertion follows. □

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