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Maximal Inequalities of the Itô Integral with Respect to Poisson Random Measures or Lévy Processes on Banach Spaces

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Abstract We are interested in maximal inequalities satisfied by a stochastic integral driven by a Poisson random measure in a general Banach space.

Keywords Stochastic integral of jump type · Poisson random measures · Lévy process · Inequalities

Mathematics Subject Classifications (2010) Primary 60H15 · Secondary 60G57

1 Introduction

Stochastic integration with respect to a Wiener process in Banach spaces is well established, see e.g. Brzeźniak [7], Dettweiler [13], Neidhart [23] and van Neerven et al. [29]. Stochastic integration in Banach spaces with respect to Lévy processes is not that well established and of increasing interest. For example, the works [2, 4, 8, 24, 28] are about stochastic integration in Banach spaces. In this paper, our focus will be on maximal inequalities satisfied by the stochastic integral driven by Lévy processes respective Poisson random measures.

Let us assume that $(Z, \mathcal{Z})$ is a measurable space and $\tilde{\eta}$ is a time homogeneous compensated Poisson random measure defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t<\infty}, \mathbb{P})$ with right continuous filtration and with an intensity measure ν on Z, to be specified later. Let us assume that $1 < p \le 2$ is fixed and that E is a separable Banach space of martingale type p, for a Definition on Banach

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spaces of martingale p see Definition 2.10. Let $I = \{I(t) : 0 \le t < \infty\}$ be the Itô integral driven by the compensated Poisson random measure $\tilde{\eta}$, i.e.

$$I(t) = \int_0^t \int_Z \xi(s;z)\tilde{\eta}(dz;ds), \quad t \ge 0,$$

where $\xi : [0,\infty) \times \Omega \to L^p(Z,\nu;E)$ is a progressively measurable process satisfying certain integrability conditions which will be specified later.

We are interested in maximal inequalities satisfied by the process I. To be more precise, the main result of this article, i.e. Theorem 2.13 is that for any continuous and convex function Φ, $\Phi(0) = 0$, satisfying a certain growth condition and having a strictly increasing and non negative derivative, there exists a constant $C > 0$, independent of η and ξ, such that

$$\mathbb{E}\Phi\left(\sup_{0\le s\le t}|I(s)|\right) \le C\,\mathbb{E}\Phi\left(\left(\int_0^t \int_Z |\xi(s;z)|^p\,\eta(dz;ds)\right)^{\frac{1}{p}}\right), \tag{1.1}$$

where η is the time homogeneous Poisson random measure without compensator. Note, that if $p = 2$ and E is a Hilbert space, the inequality follows from the classical Burkholder Davis Gundy inequality. Furthermore, from inequality 1.1 another useful inequality can be derived, which is stated in Corollary 2.14. To be more precise, it will shown that for every $p \in (1,2]$ and $n \in \mathbb{N}$ there exist constants $C > 0$ and $\bar{C} > 0$, depending only on E, p and n, such that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|I(s)|^q &\le C\,\mathbb{E}\left(\int_0^t \int_Z |\xi(s;z)|^p\,\eta(dz;ds)\right)^{\frac{q}{p}} \\ &\le \bar{C}\left(\mathbb{E}\int_0^t \int_Z |\xi(s;z)|^q\,\nu(dz)ds + \mathbb{E}\left(\int_0^t \int_Z |\xi(s;z)|^p\,\nu(dz)\,ds\right)^{\frac{q}{p}}\right),\end{aligned} \tag{1.2}$$

where $q = p^n$. Note, that inequalities 1.1 and 1.2 are written in terms of Poisson random measures. However, it is straightforward to show that these two inequalities imply two inequalities which are valid for martingales driven by Lévy processes of pure jump type.

Related inequalities, are proven in Marinelli et al., Lemma 2.2 [21], Bass and Cranston, Lemma 5.2 [6], and Protter and Talay, Lemma 4.1 [26]. However, the inequalities 1.1 and 1.2 are new in this general form.

The inequalities 1.1 and 1.2 can be used to show for instance the well-posedness of SPDEs as it has been used in Marinelli et al., Lemma 2.2 [21] and in Brzezniak and Hausenblas [9]. Here, the inequality is used to show that the trajectories of the solution of an SPDE belongs to the Skorohod space.

The organization of the paper is the following. In Section 2 we introduce some preliminaries necessary for our work and present our main result, i.e. Theorem 2.13. The proof of the Theorem 2.13 is given in Section 3. Then in Section 4 we prove Corollary 2.14. In Appendices A and B we give a short account about the original Burkholder-Davis-Gundy-inequality and convex functions.

Notation *Let* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ *and* $\bar{\mathbb{N}} := \mathbb{N}_0 \cup \{\infty\}$. *Let* $(Z,\mathcal{Z})$ *be a measurable space. By* $M_+(Z)$ *we denote the family of all positive measures on* Z*, by* $\mathcal{M}_+(Z)$ *we* *denote the σ-field on* $M_+(Z)$ *generated by functions* $i_B : M_+(Z) \ni \mu \mapsto \mu(B) \in \mathbb{R}_+$, $B \in \mathcal{Z}$. *By* $M_I(Z)$ *we denote the family of all σ–finite integer valued measures on* Z, *by* $\mathcal{M}_I(Z)$ *we denote the σ-field on* $M_I(Z)$ *generated by functions* $i_B : M_I(Z) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{Z}$. *By* $M^+_\sigma(Z)$ *we denote the set of all σ–finite and positive measures on* Z, *by* $\mathcal{M}^+_\sigma(Z)$ *we denote the σ-field on* $M^+_\sigma(Z)$ *generated by functions* $i_B : M^+_\sigma(Z) \ni \mu \mapsto \mu(B) \in \mathbb{R}$, $B \in \mathcal{Z}$.

For any Banach space Y *and number* $q \in [1, \infty)$, *we denote by* $\mathcal{N}(\mathbb{R}_+; Y)$ *the space (of equivalence classes) of progressively-measurable processes* $\xi : \mathbb{R}_+ \times \Omega \to Y$ *and by* $\mathcal{M}^q(\mathbb{R}_+; Y)$ *the Banach space consisting of those* $\xi \in \mathcal{N}(\mathbb{R}_+; Y)$ *for which* $\mathbb{E} \int_0^\infty |\xi(t)|^q_Y \, dt < \infty$.

2 Main Results

Let us first introduce the notation of time homogeneous Poisson random measures over a filtered probability space.

Definition 2.1 Let $(Z, \mathcal{Z})$ be a measurable space and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space.

A *time homogeneous Poisson random measure* η on $(Z, \mathcal{Z})$ over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, is a measurable function $\eta : (\Omega, \mathcal{F}) \to (M_I(Z \times \mathbb{R}_+), \mathcal{M}_I(Z \times \mathbb{R}_+))$, such that

- (i) for each $B \times I \in \mathcal{Z} \times \mathcal{B}(\mathbb{R}_+)$, $\eta(B \times I) := i_{B\times I} \circ \eta : \Omega \to \bar{\mathbb{N}}$ is a Poisson random variable with parameter[1] $\nu(B)\lambda(I)$.
- (ii) η is independently scattered, i.e. if the sets $B_j \times I_j \in \mathcal{Z} \times \mathcal{B}(\mathbb{R}_+)$, $j = 1, \cdots, n$, are pairwise disjoint, then the random variables $\eta(B_j \times I_j)$, $j = 1, \cdots, n$ are mutually independent.
- (iii) for each $U \in \mathcal{Z}$, the $\bar{\mathbb{N}}$-valued process $(N(t, U))_{t>0}$ defined by

$$N(t, U) := \eta((0, t] \times U), \quad t > 0$$

is $(\mathcal{F}_t)_{t\geq 0}$-adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta((s, t] \times U)$ is independent of $\mathcal{F}_s$.

Remark 2.2 A time homogeneous Poisson random measure is not the most general Poisson random measure. For more details we refer to Jacod and Shiryaev [17].

Remark 2.3 In the framework of Definition 2.1 the map

$$\nu : \mathcal{Z} \ni A \mapsto \mathbb{E}\big[\eta(A \times (0, 1))\big]$$

defines a uniquely determined measure. This measure ν is called *the intensity measure* of the time homogeneous Poisson random measure.

Given a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with right continuous filtration, the predictable random field $\mathcal{P}$ on $\mathbb{R}_+ \times \Omega$ is the σ–field generated

[1] If $\nu(B)\lambda(I) = \infty$, then obviously $\eta(B \times I) = \infty$ a.s..

by all continuous $(\mathcal{F}_t)_{t\geq 0}$–adapted processes (see e.g. Kallenberg [18, Chapter 25]). A real valued stochastic process $\{x(t) : 0 \leq t < \infty\}$ defined over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called predictable, if the mapping $x : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is $\mathcal{P}/\mathcal{B}(\mathbb{R})$-measurable. A random measure γ on $\mathcal{Z} \times \mathcal{B}(\mathbb{R}_+)$ over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called predictable, iff for each $U \in \mathcal{Z}$, the $\mathbb{R}$–valued process $\mathbb{R}_+ \ni t \mapsto \gamma(U \times (0, t])$ is predictable.

Definition 2.4 Assume that η is a time homogeneous Poisson random measure over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The *compensator* of η is the unique predictable random measure, denoted by γ, on $\mathcal{Z} \times \mathcal{B}(\mathbb{R}_+)$ over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that for each $T < \infty$ and $A \in \mathcal{Z}$ with $\mathbb{E}\eta(A \times (0, T]) < \infty$, the $\mathbb{R}$-valued processes $\{\tilde{N}(t, A) : 0 \leq t \leq T\}$ defined by

$$\tilde{N}(t, A) := \eta(A \times (0, t]) - \gamma(A \times (0, t]), \quad 0 \leq t \leq T,$$

is a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Remark 2.5 Assume that η is a time homogeneous Poisson random measure with intensity $\nu \in M^+_\sigma(Z)$ over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. It turns out that the compensator γ of η is uniquely determined and moreover

$$\gamma : \mathcal{Z} \times \mathcal{B}(\mathbb{R}_+) \ni (A, I) \mapsto \nu(A) \times \lambda(I).$$

The difference between a time homogeneous Poisson random measure η and its compensator γ, i.e. $\tilde{\eta} = \eta - \gamma$, is called a *time homogeneous compensated Poisson random measure*.

Poisson random measures arise in a natural way by means of Lévy processes.

Definition 2.6 Let E be a Banach space. A stochastic process $\{L(t) : 0 \leq t < \infty\}$ is a Lévy process if the following conditions are satisfied.

- for any choice $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the random variables $L(t_0)$, $L(t_1) - L(t_0), \ldots, L(t_n) - L(t_{n-1})$ are independent;
- $L_0 = 0$ a.s.;
- For all $0 \leq s < t$, the distribution of $L(t + s) - L(s)$ does not depend on s;
- L is stochastically continuous;
- the trajectories of L are a.s. càdlàg on E.

The characteristic function of a Lévy process is uniquely defined and is given by the Lévy-Khinchin formula. For simplicity we restrict ourselves to a separable p-stable Banach space E, $p \in [1, 2]$ (for a Definition on p–stable Banach spaces we refer to Linde [20]). For any E-valued Lévy process $L = \{L(t) : 0 \leq t < \infty\}$ there exist a positive operator $Q : E' \to E$, a non negative measure ν concentrated on $E \setminus \{0\}$ such that $\int_E (1 \wedge |z|^p)\, \nu(dz) < \infty$, and an element $m \in E$ such that (see e.g. the articles [1, 4, 5] or Theorem 5.7.3 [20])

$$\mathbb{E}e^{i\langle L(1),x\rangle} = \exp\Big(i\langle m, x\rangle - \frac{1}{2}\langle Qx, x\rangle + \int_E \big(1 - e^{i\langle y,x\rangle} + 1_{(-1,1)}(|y|_E)i\langle y, x\rangle\big)\, \nu(dy)\Big), \quad x \in E'.$$

We call the measure ν characteristic measure of the Lévy process L. Moreover, the triplet (Q, m, ν) uniquely determines the law of the Lévy process. Now, one can construct a Poisson random measure with an intensity measure which is related to ν.

Example 2.7 Given a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with right continuous filtration and a Banach space of stable type p, we can associate to each E-valued Lévy process $L = \{L(t) : 0 \leq t < \infty\}$ over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ of pure jump type[2] with characteristic measure ν, a counting measure, being denoted by η_L and being defined over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, by

$$\mathcal{B}(E) \times \mathcal{B}(\mathbb{R}_+) \ni (B, I) \mapsto \eta_L(B \times I) := \#\{s \in I \mid \Delta_s L \in B\} \in \bar{\mathbb{N}}_0.$$

Here, the jump process $\Delta L = \{\Delta_t L : 0 \leq t < \infty\}$ of a process L is given by $\Delta_t L(t) := L(t) - L(t-) = L(t) - \lim_{\epsilon\to 0} L(t-\epsilon)$, $t > 0$, $\Delta_0 L = 0$. If ν is symmetric and supported by the unit ball, then the counting measure is a time homogeneous Poisson random measure with intensity measure ν. Moreover,

$$L(t) = \int_0^t \int_Z z\, \tilde{\eta}_L(dz, ds), \quad t \geq 0.$$

Vice versa, given a time homogeneous Poisson random measure one can get a Lévy process. For this propose we define now the notion of Lévy measures.

Definition 2.8 (see Linde, Chapter 5.4 [20]) Let E be a separable Banach space and let E' be its dual. A symmetric[3] σ-finite Borel measure λ on E is called a Lévy measure if and only if

(i) $\lambda(\{0\}) = 0$, and
(ii) the function[4]

$$E' \ni a \mapsto \exp\left(\int_E (\cos\langle x, a\rangle - 1)\, \lambda(dx)\right)$$

is a characteristic function of a Radon measure on E.

An arbitrary σ-finite Borel measure λ on E is called a Lévy measure, provided its symmetric part $\frac{1}{2}(\lambda + \lambda^-)$, where $\lambda^-(A) := \lambda(-A)$, $A \in \mathcal{B}(E)$, is a Lévy measure. The class of all Lévy measures on $(E, \mathcal{B}(E))$ will be denoted by $\mathcal{L}(E)$.

Remark 2.9 (Dettweiler [14]) If E is a Banach space of type p, then $\nu : \mathcal{B}(E) \to \mathbb{R}_+$ is a Lévy measure iff $\nu(\{0\}) = 0$, ν is σ-finite and

$$\int_E |z|^p \nu(dz) < \infty.$$

[2] We call a Lévy process of pure jump type, iff $Q = 0$.

[3] i.e. $\lambda(A) = \lambda(-A)$ for all $A \in \mathcal{B}(E)$.

[4] As remarked in Linde, Chapter 5.4 [20] we do not need to suppose that the integral $\int_E (\cos\langle x, a\rangle - 1)\, \lambda(dx)$ is finite. However, see Corollary 5.4.2 in ibidem, if λ is a symmetric Lévy measure, then, for each $a \in E'$, the integral in question is finite.

Moreover, given an arbitrary time homogeneous Poisson random measure η on a Banach space E of type p, then, iff the intensity measure is a Lévy measure, the integral $\int_I \int_Z z\,\tilde\eta(dz, ds)$ is well defined for any $I \in \mathcal{B}(\mathbb{R}_+)$ and the process $L = \{L(t) : 0 \le t < \infty\}$ defined for $t \ge 0$ by

$$L(t) = \int_0^t \int_Z z\,\tilde\eta(dz, ds)$$

is a Lévy process.

For more details about the relationship between Poisson random measures and Lévy processes we refer e.g. to Applebaum [3], Ikeda and Watanabe [16] or Peszat and Zabczyk [25].

In general, it is difficult to define a stochastic integral in an arbitrary Banach space. Here we restrict ourselves to Banach spaces of martingale type p, whose definition is given below.

Definition 2.10 Let $0 < p \le 2$. A Banach space E is of martingale type p iff there exists a constant $C = C(E, p) > 0$ such that for all E-valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds

$$\sup_{0\le n\le N} \mathbb{E}|M_n|_E^p \le C\,\mathbb{E}\sum_{n=0}^N |M_n - M_{n-1}|_E^p, \tag{2.1}$$

where as usually, we put $M_{-1} = 0$. We denote the smallest constant satisfying Eq. 2.1 by $C_p(E)$.

Remark 2.11 Let $1 < p \le 2$. There are many Banach spaces which are of martingale type p. To give an idea we itemize the examples below:

(a) If a Banach space is of martingale type p, then it is of martingale type q, $q \le p$;
(b) Every Banach space is of martingale type 1;
(c) Every Hilbert space is of martingale type 2, and $C_2(E) = 1$;
(d) If $(S, \mathcal{S}, \sigma)$ is a probability space, then $\mathrm{L}^p(S, \mathcal{S}, \sigma)$ is of martingale type p;
(e) Let E be of martingale type p and $A : E \to E$ an operator with domain $\mathrm{dom}(A)$. If A^{-1} is bounded, then $\mathrm{dom}(A)$ is isomorphic to E and therefore of martingale type p;
(f) Assume E_1 and E_2 are a Banach space of martingale type p, where E_2 is continuously and densely embedded in E_1. Then for any $\vartheta \in (0, 1)$ the complex interpolation space $[E_1, E_2]_\vartheta$ and the real interpolation space $(E_1, E_2)_{\vartheta,p}$ are of martingale type p;
(g) Let $(S, \mathcal{S})$ be a measurable space. Then $L^\infty(S, \mathcal{S}, \sigma)$, $L^1(S, \mathcal{S}, \sigma)$ are often not of martingale type p for any $p > 1$. The space $\mathcal{C}([0, 1]; \mathbb{R})$ is not of martingale type p for any $p > 1$.

Let $1 < p \le 2$ and E be a separable Banach space of martingale type p. Let $(Z, \mathcal{Z})$ be a measurable space and $\nu \in M_\sigma^+(Z)$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ be a filtered probability space with right continuous filtration, let η be a time homogeneous Poisson random measure on Z over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with intensity measure ν. We will denote by $\tilde{\eta} = \eta - \gamma$ the compensated Poisson random measure associated to η.

We have proved in [8] that there exists a unique continuous linear operator which associates with each progressively measurable process $\xi \in \mathcal{M}^p(\mathbb{R}_+; L^p(Z, \nu; E))$ an adapted càdlàg E-valued process, denoted by $\int_0^t \int_Z \xi(r, x)\tilde{\eta}(dx, dr)$, $t \geq 0$, such that if the process $\xi \in \mathcal{M}^p(\mathbb{R}_+, L^p(Z, \nu; E))$ is a random step process with representation

$$\xi(r) = \sum_{j=1}^{n} 1_{(t_{j-1}, t_j]}(r)\, \xi_j, \quad r \geq 0,$$

where $\{t_0 = 0 < t_1 < \ldots < t_n < \infty\}$ is a finite partition of $[0, \infty)$ and for all j, ξ_j is an E-valued $\mathcal{F}_{t_{j-1}}$ measurable, p-summable random variable, then

$$\int_0^t \int_Z \xi(r, z)\, \tilde{\eta}(dz, dr) = \sum_{j=1}^{n} \int_Z \xi_j(z)\, \tilde{\eta}\left(dz, (t_{j-1} \wedge t, t_j \wedge t]\right). \tag{2.2}$$

The continuity mentioned above means that there exists a constant $C = C(E) > 0$ independent of ξ and η such that

$$\mathbb{E}\Big|\int_0^t \int_Z \xi(r, z)\, \tilde{\eta}(dz, dr)\Big|^p \leq C\,\mathbb{E} \int_0^t \int_Z |\xi(r, z)|^p\, \nu(dz)\, dr, \;\; t \geq 0. \tag{2.3}$$

As mentioned in the beginning we are interested in inequalities satisfied by the stochastic processes $I = \{I(t) : 0 \leq t < \infty\}$ given by

$$\mathbb{R}_+ \ni t \mapsto I(t) = \int_0^t \int_Z \xi(s, z)\, \tilde{\eta}(dz, ds).$$

First we introduce the following definition.

Definition 2.12 Let $\Phi : [0, \infty) \to \mathbb{R}$ be a strictly increasing convex function. We say Φ satisfies *the growth condition*, iff there exists a constant $c > 0$ with

$$\Phi(2\lambda) \leq 2^c\, \Phi(\lambda), \quad \lambda \in [0, \infty). \tag{2.4}$$

We denote the smallest constant which satisfies Eq. 2.4 by c_Φ (for details see Appendix B).

Theorem 2.13 *Let $1 < p \leq 2$ and let E be a separable Banach space of martingale type p, $(Z, \mathcal{Z})$ be a measurable space and $\nu \in M_\sigma^+(Z)$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space with right continuous filtration. Assume that $\tilde{\eta}$ is a compensated time homogeneous Poisson random measure on Z over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with intensity ν.*

Let $\Phi : [0, \infty) \to \mathbb{R}_+$ be a function such that

- *$\Phi : [0, \infty) \to \mathbb{R}$ is a convex function, $\Phi(0) = 0$, and has a non–negative and strictly increasing derivative;*

- *Φ satisfies the growth condition with constant c_Φ;*
- *the function $[0,\infty) \ni x \mapsto \Phi(x^{\frac{1}{p}}) \in \mathbb{R}$ is convex.*

Then there exists a constant $C > 0$, only depending on E, Φ and p, such that for any $\xi \in \mathcal{M}^p(\mathbb{R}_+; L^p(Z,\nu;E)) \cap \mathcal{M}^{c_\Phi}(\mathbb{R}_+; L^{c_\Phi}(Z,\nu;E))$ we have

$$\mathbb{E} \sup_{0\le s\le t} \Phi\left(\left| \int_0^s \int_Z \xi(r;z)\,\tilde{\eta}(dz;dr) \right| \right) \le C\,\mathbb{E}\Phi\left(\left(\int_0^t \int_Z |\xi(s;z)|^p\,\eta(dz;ds) \right)^{\frac{1}{p}} \right). \tag{2.5}$$

In particular, $C = C_p(E)\,c_\Phi (m_0)^{c_\Phi}$, where $m_0 = \inf\{n \ge 1 : p - nc_\Phi \le 1\} \vee 5$.

By similar means as Bass and Cranston [6] have shown in Lemma 5.2, or Protter and Talay [26] have shown in Lemma 4.1, we will prove the following corollary.

Corollary 2.14 *Under the assumption of Proposition 2.13 the following holds. For all $n \in \mathbb{N}$ there exists a constant $C > 0$, only depending on E, p, and n, such that for any $\xi \in \mathcal{M}^p(\mathbb{R}_+; L^p(Z,\nu;E)) \cap \mathcal{M}^q(\mathbb{R}_+; L^q(Z,\nu;E))$ we have*

$$\mathbb{E} \sup_{0<s\le t} \left| \int_0^t \int_Z \xi(s,z)\tilde{\eta}(dz;ds) \right|^q \le 2^{2-p} \sum_{l=1}^n \bar{C}(l)\mathbb{E}\left(\int_0^t \int_Z |\xi(s,z)|^{p^l}\,\nu(dz)\,ds \right)^{p^{n-l}},$$

where $q = p^n$, $\bar{C}(0) = C_p(E)$ and $\bar{C}(i) = \bar{C}(i-1)2^{p^{n-i+1}+2p^{n-1}}\,(m(n-i)-1)^{(p-1)\frac{r}{p}}$, $m(i) = [p^{i-1}(p-1)] + 1$.

Assume in the next paragraph that Z and E are separable Banach spaces, E of martingale type p. Let $X = \{X(t) : 0 \le t < \infty\}$ be a process arising by stochastic integration of a Lévy process of pure jump type. In particular, we assume that there exists a Z-valued Lévy process L with characteristic ν, where $\int |z|^p \nu(dz) < \infty$, and a predictable and càglàd process $h \in \mathcal{M}^p(\mathbb{R}_+, L(Z,E))$ such that

$$X(t) := \int_0^t h(s)\,dL(s), \quad t \ge 0. \tag{2.6}$$

Then, $\Delta_t X = h(t)\Delta_t L$, $t \ge 0$ and Proposition 2.13 reads as follows.

Corollary 2.15 *Let $\Phi : [0,\infty) \to \mathbb{R}_+$ be a function such that*

- *$\Phi : [0,\infty) \to \mathbb{R}$ is a convex function, $\Phi(0) = 0$, and Φ has a non–negative and strictly increasing derivative ϕ;*
- *Φ satisfies the growth condition with constant c_Φ;*
- *the function $[0,\infty) \ni x \mapsto \Phi(x^{\frac{1}{p}}) \in \mathbb{R}$ is convex.*

There exists a constant $C > 0$, only depending on E, p, and Φ, such that for any X having representation 2.6 with $h \in \mathcal{M}^p(\mathbb{R}_+, L(Z,E))$, h being predictable and càglàd and L being a p–integrable and Z–valued Lévy process with

$$\mathbb{E} \sum_{0\le s\le t} |\Delta_s X|^{c_\Phi} < \infty, \quad t \ge 0,$$

we have

$$\mathbb{E}\Phi\left(\sup_{0\le s\le t}|X(s)|\right)\le C\,\mathbb{E}\Phi\left(\left(\sum_{0\le s\le t}|\Delta_s X|^p\right)^{\frac{1}{p}}\right).$$

In particular, $C = C_p(E)\,c_\Phi(m_0)^{(p-1)c_\Phi}$, $m_0 = \inf\{n\ge 1 : p - nc_\Phi \le 1\}$.

Furthermore, Corollary 2.14 reads as follows.

Corollary 2.16 *Under the assumption of Corollary* 2.15 *the following holds. For all* $n\in\mathbb{N}$ *there exists a constant* $\bar{C}>0$*, only depending on* E*,* p*, and* n*, such that for any* X *having representation* 2.6 *with* $h\in\mathcal{M}^p(\mathbb{R}_+, L(Z, E))$ *being predictable and càglàd and* L *being a* p*-integrable and* Z*–valued Lévy process with*

$$\mathbb{E}\sum_{0\le s\le t}|\Delta_s X|^q<\infty,$$

we have

$$\mathbb{E}\sup_{0<s\le t}|X(s)|^q\le 2^{2-p}\sum_{l=1}^{n}\bar{C}(l)\,\mathbb{E}\left(\sum_{0\le s\le t}\mathbb{E}\left[|\Delta_s X|^{p^l}\mid\mathcal{F}_{s-}\right]\right)^{p^{n-l}},$$

where $q=p^n$, $\bar{C}(0)=\bar{C}$ *and* $\bar{C}(i)=\bar{C}(i-1)2^{p^{n-i+1}+2p^{n-1}}(m(n-i)-1)^{(p-1)\frac{r}{p}}$, $m(i) = [p^{i-1}(p-1)]+1$.

3 Proof of the Theorem 2.13

In Appendix A we verified in Theorem A.3 that the Burkholder Gundy inequality formulated for a discrete martingale also holds in Banach spaces of martingale type p. With this starting point we show Theorem 2.13 by the following steps. First we give in Lemma 3.1 an estimate of

$$\mathbb{E}\sup_{0\le s\le T}\Phi\left\{\left|\int_0^T\int_Z\xi(s,x)\,\tilde{\eta}(dx,ds)\right|\right\}\tag{3.1}$$

for a certain class of simple processes. After this preparation it remains to show first that there exists a sequence $\{\xi_n : n\in\mathbb{N}\}$ converging to ξ in $\mathcal{M}(\mathbb{R}^+; L^p(Z,\nu; E))$ and consisting of processes satisfying the assumptions of Lemma 3.1, and, then to show secondly that the right hand side of the estimate for Eq. 3.1 converges to the right hand side of Eq. 2.5, i.e. to $\mathbb{E}\Phi\left\{\left(\int_0^T\int_Z|\xi(s,x)|^p\,\eta(dx,ds)\right)^{\frac{1}{p}}\right\}$.

Before starting with the actual proof of Proposition 2.13 we introduce some notation and state Lemma 3.1, which is essential for the proof.

We call a process $\xi\in\mathcal{M}^p(\mathbb{R}_+;\ L^P(Z,\nu;E))$ simple, if it has the following representation

$$\xi(s,z,\omega)=\sum_{k=1}^{K}1_{(s_{k-1},s_k]}(s)\sum_{j=1}^{J}\sum_{i=1}^{I}\xi_{ji}^{k}1_{A_{ji}^k\times B_j}(\omega,z),\quad (s,z,\omega)\in[0,T]\times Z\times\Omega\tag{3.2}$$

with $s_k - s_{k-1} = \tau > 0$ for $k = 1, \dots, K$, $\xi_i \in E$ for $i = 1, \dots, I$, $A^k_{ji} \in \mathcal{F}_{s_{k-1}}$, $k = 1, \dots, K$, and $B_j \in \mathcal{B}(Z)$ and $\nu(B_j) \le 1$ for $j = 1, \dots, J$. In addition, for fixed $k = 1, \dots, K$, the sets $\{A^k_{ji} \times B_j : j = 1, \dots, J$ and $i = 1, \dots, I\}$ are disjoint. Moreover, the sets $\{B_j : j = 1, \dots, J\}$ are disjoint.

Let $\mathfrak{A}$ be a family of simple processes having representation of the form 3.2 and fix a constant $\bar{C} > 0$. We say that the family $\mathfrak{A}$ is stable with respect to $\bar{C}$, if for each $\xi \in \mathfrak{A}$ with representation 3.2 we have $\tau J \le \bar{C}$ and $\tau^{\frac{p-1}{2p}} \nu(Z_J) \le \bar{C}$, where $Z_J = \cup_{j=1}^J B_j$.

Lemma 3.1 *Assume that* $\Phi : [0, \infty) \to \mathbb{R}_+$ *is a function such that*

- $\Phi : [0, \infty) \to \mathbb{R}$ *is a convex function,* $\Phi(0) = 0$, *and has a non–negative and strictly increasing derivative;*
- Φ *satisfies the growth condition;*
- *the function* $[0, \infty) \ni x \mapsto \Phi(x^{\frac{1}{p}}) \in \mathbb{R}$ *is convex.*

Assume that $\mathfrak{A}$ *is a family of processes having representation of the form 3.2 and being stable for a constant* $\bar{C} > 0$. *Moreover, let us assume that for all* $\xi \in \mathfrak{A}$ *we have*

$$\int_0^t \int_Z \mathbb{E}|\xi(s, z)|^{c_\Phi} \nu(dz)\, ds < \infty.$$

Then there exist two constants $C_1 > 0$ *and* $C_2 > 0$, *only depending on* E, p, Φ *and* $\bar{C}$, *such that for any function* $\xi \in \mathfrak{A}$ *the following inequality holds*

$$\mathbb{E} \sup_{0 \le t \le T} \Phi \left\{ \left| \int_0^T \int_Z \xi(s, x) \tilde{\eta}(dx, ds) \right| \right\} \le C_1 (1 + \tau) \mathbb{E}\Phi \left\{ \left(\int_0^T \int_Z |\xi(s, x)|^p \, \eta(dx, ds) \right)^{\frac{1}{p}} \right\}$$
$$+ C_2 \tau^{\frac{p-1}{2p}} \int_0^T \int_Z \mathbb{E}\Phi \{|\xi(s, z)|\} \nu(dz)\, ds. \quad (3.3)$$

By means of Lemma 3.1, Proposition 2.13 follows. Therefore, we assume for the time being that Lemma 3.1 is valid, postpone the proof and present first the proof of Proposition 2.13.

Proof of Proposition 2.13 By standard arguments one can find a sequence $\{\xi_n : n \in \mathbb{N}\}$ such that for any $n \in \mathbb{N}$, ξ_n has representation given in Eq. 3.2 and $\xi_n \to \xi$ in $\mathcal{M}^p(\mathbb{R}_+; L^p(Z, \nu; E))$ and $\mathcal{M}^{c_\Phi}(\mathbb{R}_+; L^{c_\Phi}(Z, \nu; E))$. In particular,

$$\xi_n(s, z, \omega) = \sum_{k=1}^{K_n} 1_{(s^n_{k-1}, s^n_k]}(s) \sum_{j=1}^{J_n} \sum_{i=1}^{I_n} \xi^{kn}_{ji} 1_{A^{kn}_{ji} \times B^n_j}(\omega, z), \quad (s, z, \omega) \in [0, T] \times Z \times \Omega,$$

with $s^n_k - s^n_{k-1} = \tau_n > 0$ for $k = 1, \dots, K_n$, $\xi^{kn}_{ji} \in E$ for $i = 1, \dots, I$, $A^{kn}_{ji} \in \mathcal{F}_{s^n_{k-1}}$, $k = 1, \dots, K_n$, and $B^n_j \in \mathcal{B}(Z)$ and $\nu(B^n_j) \le 1$ for $j = 1, \dots, J_n$. Moreover, it follows that

$$\int_0^T \int_Z \mathbb{E}\Phi \{|\xi_n(s, x) - \xi(s, x)|\} \nu(dx)\, ds \to 0 \quad \text{as} \quad n \to \infty. \quad (3.4)$$

Next, we can suppose that the family $\{\xi_n : n \in \mathbb{N}\}$ is stable with respect to $\bar{C} > 0$. In fact, if the sequence is not stable with respect to $\bar{C}$, one can construct a new sequence, which is stable by refinement of the time partition.

Having such a sequence, by the definition of the Itô integral and by an application of Fatou's Lemma we get

$$\mathbb{E} \sup_{0\le t\le T} \Phi \left\{ \left| \int_0^t \int_Z \xi(s,x)\tilde{\eta}(dx,ds) \right| \right\} = \mathbb{E} \sup_{0\le t\le T} \Phi \left\{ \left| \lim_{n\to\infty} \int_0^t \int_Z \xi_n(s,x)\tilde{\eta}(dx,ds) \right| \right\}$$
$$\le \lim_{n\to\infty} \mathbb{E} \sup_{0\le t\le T} \Phi \left\{ \left| \int_0^t \int_Z \xi_n(s,x)\tilde{\eta}(dx,ds) \right| \right\}.$$

Applying Lemma 3.1 we get

$$\mathbb{E} \sup_{0\le t\le T} \Phi \left\{ \left| \int_0^t \int_Z \xi(s,x)\tilde{\eta}(dx,ds) \right| \right\} = C \lim_{n\to\infty} (1+\tau_n) \mathbb{E}\Phi \left\{ \left(\int_0^T \int_Z |\xi_n(s,x)|^p \, \eta(dx,ds) \right)^{\frac{1}{p}} \right\}$$
$$+ C_2\, \tau_n^{\frac{p-1}{p}} \nu(Z_J^n)^{\frac{c_\Phi}{p}-1} \int_0^T \int_Z \mathbb{E}\Phi \{ |\xi_n(s,z)| \} \, \nu(dz)\, ds.$$

The assumption

$$\int_0^T \int_Z \mathbb{E} \, |\xi(s,z)|^{c_\Phi} \, \nu(dz)\, ds < \infty$$

and the stability of $\{\xi_n : n \in \mathbb{N}\}$ imply that there exists a constant $C > 0$ such that

$$\nu(Z_J^n)\tau_n^{\frac{p-1}{p}} \int_0^T \int_Z \mathbb{E}\Phi \{ |\xi_n(s,z)| \} \, \nu(dz)\, ds \le C.$$

Now, since Eq. 3.4 holds, the definition of the Itô integral implies that

$$\mathbb{E} \sup_{0\le t\le T} \Phi \left\{ \left| \int_0^t \int_Z \xi(s,x)\tilde{\eta}(dx,ds) \right| \right\} \le C \, \mathbb{E}\Phi \left\{ \left(\int_0^T \int_Z |\xi(s,x)|^p \, \eta(dx,ds) \right)^{\frac{1}{p}} \right\},$$

which is the assertion. □

Before starting with the proof of Lemma 3.1 let us state the following proposition.

Proposition 3.2 *Let X be a Poisson distributed random variable with parameter λ. Then*

$$\mathbb{E} 1_{\{X \ge 5\}} |X|^p \le C \lambda^p.$$

Proof Proposition 3.2 is shown by the following lines of calculations.

$$\begin{aligned}\mathbb{E}1_{\{X\geq 5\}}|X|^p &\leq \left(\mathbb{E}1_{\{X\geq 5\}}|X|^2\right)^{\frac{p}{2}} = \left(e^{-\lambda}\sum_{l=5}^{\infty} l^2\frac{\lambda^l}{l!}\right)^{\frac{p}{2}}\\ &\leq C\lambda^p\left(e^{-\lambda}\sum_{l=5}^{\infty}\frac{\lambda^{l-2}}{(l-2)!}\right)^{\frac{p}{2}} \leq C\lambda^p\left(e^{-\lambda}\sum_{l=5}^{\infty}\frac{\lambda^{l-2}}{(l-2)!}\right)^{\frac{p}{2}}\\ &\leq C\lambda^p\left(e^{-\lambda}\sum_{l=3}^{\infty}\frac{\lambda^l}{l!}\right)^{\frac{p}{2}} \leq C\lambda^p\left(e^{-\lambda}\sum_{l=0}^{\infty}\frac{\lambda^l}{l!}\right)^{\frac{p}{2}} \leq C\lambda^p.\end{aligned}$$

□

Proof of Lemma 3.1 Let ξ be a simple function with representation given in Eq. 3.2. By the definition of the stochastic integral we have

$$\int_0^T\int_Z \xi(s,x)\,\tilde{\eta}(dx,ds) = \sum_{k=1}^{K}\sum_{j=1}^{J}\left(\sum_{i=1}^{I} 1_{A_{ji}^k}\xi_{ji}^k\right)\tilde{\eta}(B_j\times(s_{k-1},s_k]).$$

Moreover, by the definition of the compensated Poisson random measure the sequence

$$\left\{\sum_{i=1}^{I} 1_{A_{ji}^k}\xi_{ji}^k\tilde{\eta}(B_j\times(s_{k-1},s_k]),\ m=(J-1)\,k+j\right\}_{m=1}^{JK}$$

is a sequence of martingale differences with values in E. Hence, the discrete Burkholder Davis Gundy inequality, i.e. Theorem A.3, gives

$$\begin{aligned}&\mathbb{E}\sup_{0\leq t\leq T}\Phi\left\{\left|\int_0^t\int_Z \xi(s,x)\tilde{\eta}(dx,ds)\right|\right\}\\ &\quad\leq C_p(E)\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\left|\sum_{i=1}^{I} 1_{A_{ji}^k}\xi_{ji}^k\tilde{\eta}(B_j\times(s_{k-1},s_k])\right|^p\right)^{\frac{1}{p}}\right\}.\end{aligned}$$

Recall that for fixed k and j, A_{ji}^k, $i=1,\ldots,I$ are disjoint sets. This implies that for each $\omega\in\Omega$ only one term of the inner sum will not be equal to zero. Therefore, we can write

$$\begin{aligned}&\mathbb{E}\sup_{0\leq t\leq T}\Phi\left\{\left|\int_0^t\int_Z \xi(s,x)\tilde{\eta}(dx,ds)\right|\right\}\\ &\quad\leq C_p(E)\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\left|1_{A_{ji}^k}\xi_i\tilde{\eta}(B_j\times(s_{k-1},s_k])\right|^p\right)^{\frac{1}{p}}\right\}. \qquad (3.5)\end{aligned}$$

Plugging in Eq. 3.5 the definition of $\tilde{\eta}(B_j\times(s_{k-1},s_k])$, the RHS of Eq. 3.5 reads

$$C_p(E)\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\left|1_{A_{ji}^k}\,\xi_{ji}^k\left(\sum_{l=1}^{\infty}1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\,l-\nu(B_j)\tau\right)\right|^p\right)^{\frac{1}{p}}\right\}.$$

Using $|x-y|^p \le 2^p(|x|^p+|y|^p)$ and Eq. B.7 we obtain

$$\ldots \le C_p(E)\,\mathbb{E}\Phi\left\{2\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\left|\sum_{l=1}^\infty 1_{A_{ji}^k}1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}l\xi_{ji}^k\right|\right.\right.$$
$$\left.\left.+2^p\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\left|1_{A_{ji}^k}\xi_{ji}^k\nu(B_j)\tau\right|^p\right)^{\frac{1}{p}}\right\}$$

Using the convexity of Φ and Eq. B.7 we get for $C_\Phi = 2^{c_\Phi}$

$$\ldots \le C_p(E)\,C_\Phi\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\sum_{l=1}^\infty 1_{A_{ji}^k}1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\,l^p\,\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}$$
$$+C_p(E)\,C_\Phi\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\left|1_{A_{ji}^k}\xi_{ji}^k\nu(B_j)\tau\right|^p\right)^{\frac{1}{p}}\right\}.$$

Let m_0 be chosen according to $p-(m_0-1)/c_\Phi \le 1$ and $m_0 > 5$. We split again the inner term in the inner sum. Doing so we arrive at

$$\ldots \le C_p(E)\,C_\Phi\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\sum_{l=1}^{m_0-1} 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\,1_{A_{ji}^k}\,l^p\,\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}$$
$$+C_p(E)\,C_\Phi\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\sum_{l=m_0}^\infty 1_{A_{ji}^k}1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}l^p\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}$$
$$+C_p(E)\,C_\Phi\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\left|1_{A_{ji}^k}\xi_{ji}^k\nu(B_j)\tau\right|^p\right)^{\frac{1}{p}}\right\}$$
$$=: I+II+III. \tag{3.6}$$

The first term in Eq. 3.6 can be estimated in the following way. Since $l \le m_0-1$, we can estimate l^{p-1} by $(m_0-1)^{p-1}$. By property B.7 we can put $(m_0-1)^{p-1}$ in front of the bracket and obtain

$$I \le \mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I 1_{A_{ji}^k}\sum_{l=1}^{m_0-1}1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\,l\,(m_0-1)^{p-1}\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}$$
$$\le (m_0-1)^{c_\Phi(p-1)/p}\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\sum_{l=1}^{m_0-1}1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\,1_{A_{ji}^k}l\,\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}$$

$$
\begin{aligned}
&\le (m_0-1)^{c_\Phi(p-1)/p}\, \mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\sum_{l=0}^\infty 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\, 1_{A^k_{ji}} l\, \left|\xi^k_{ji}\right|^p\right)^{\frac1p}\right\}\\
&= (m_0-1)^{c_\Phi(p-1)/p}\, \mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^p \eta(B_j\times(s_{k-1},s_k])\right)^{\frac1p}\right\}\\
&= (m_0)^{c_\Phi(p-1)/p}\, \mathbb{E}\Phi\left\{\left(\int_0^T\int_Z |\xi(s,z)|^p\,\eta(dz,ds)\right)^{\frac1p}\right\}.
\end{aligned}
$$

Here we used the fact that $m_0-1\le C\,l$, $l\ge 1$, and the definition of the stochastic integral. Now we consider the second term of Eq. 3.6. In order to be able to apply Theorem 2.2 of [22] we first split the second term into two terms. Moreover, we use the inequality $(x+y)^{\frac1p}\le x^{\frac1p}+y^{\frac1p}$, $x,y\ge 0$, to separate these two summands and obtain

$$
\begin{aligned}
II &= \mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I\sum_{l=m_0}^\infty 1_{A^k_{ji}} 1_{\{\eta(B_j\times(s_{k-1},s_k])=l\}}\, l^p \left|\xi^k_{ji}\right|^p\right)^{\frac1p}\right\}\\
&= \mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^p 1_{\{\eta(B_j\times(s_{k-1},s_k])\ge m_0\}}\, |\eta(B_j\times(s_{k-1},s_k])|^p\right)^{\frac1p}\right\}\\
&\le \mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^p \left(1_{\{\eta(B_j\times(s_{k-1},s_k])\ge m_0\}}\, |\eta(B_j\times(s_{k-1},s_k])|^p\right.\right.\right.\\
&\qquad\left.\left.\left. -\mathbb{E}\left[1_{\{\eta(B_j\times(s_{k-1},s_k])\ge m_0\}}\, |\eta(B_j\times(s_{k-1},s_k])|^p \mid \mathcal{F}_{t_{k-1}}\right]\right)\right)^{\frac1p}\right\}\\
&\quad +\mathbb{E}\Phi\left\{\left(\sum_{k=1}^K\sum_{j=1}^J\sum_{i=1}^I 1_{A^k_{ji}} \left|\xi^k_{ji}\right|^p\right.\right. \qquad (3.7)\\
&\qquad\left.\left. \times\, \mathbb{E}\left[1_{\{\eta(B_j\times(s_{k-1},s_k])\ge m_0\}}\, |\eta(B_j\times(s_{k-1},s_k])|^p \mid \mathcal{F}_{t_{k-1}}\right]\right)^{\frac1p}\right\}. \qquad (3.8)
\end{aligned}
$$

The last summand can be treated in the same way as the third term in inequality 3.6. To be more precise, let us first recall the following facts. Since the sets $B_j\times(s_{k-1},s_k]$ are disjoint for $k=1,\dots,K$ and $j=1,\dots,J$, the random variables $\eta(B_j\times(s_{k-1},s_k])$ are independent for $k=1,\dots,K$ and $j=1,\dots,J$. Now, by Proposition 3.2 it follows for $k=1,\dots,K$ and $j=1,\dots,J$

$$
\mathbb{E}\left[1_{\{\eta(B_j\times(s_{k-1},s_k])\ge m_0\}}\, |\eta(B_j\times(s_{k-1},s_k])|^p \mid \mathcal{F}_{t_{k-1}}\right]\le C\,\tau^p\nu(B_j)^p\,.
$$

Hence,

$$\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p \mathbb{E}\left[1_{\{\eta(B_j\times(s_{k-1},s_k])\geq m_0\}}\,|\eta(B_j\times(s_{k-1},s_k])|^p \mid \mathcal{F}_{t_{k-1}}\right]\right)^{\frac{1}{p}}\right\}$$

$$\leq C\,\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p \tau^p \nu(B_j)^p\right)^{\frac{1}{p}}\right\}. \tag{3.9}$$

Comparing the RHS of inequality 3.9 with the third summand of Eq. 3.6, we see that both are the same.

It remains to look at the first summand of Eq. 3.7. Here, note that the summands in the inner part are martingale differences in E and E is UMD. In order to apply Theorem 2.2 of [22] we have to construct a sequence of martingale differences having the same law (on $\mathbb{R}$) as

$$\Big\{1_{\{\eta(B_j\times(s_{k-1},s_k])\geq m_0\}}\,|\eta(B_j\times(s_{k-1},s_k])|^p$$
$$-\,\mathbb{E}\left[1_{\{\eta(B_j\times(s_{k-1},s_k])\geq m_0\}}\,|\eta(B_j\times(s_{k-1},s_k])|^p \mid \mathcal{F}_{t_{k-1}}\right],\quad m=(J-1)k+j\Big\}_{m=1}^{JK}. \tag{3.10}$$

Therefore, we introduce a second probability space $(\check{\Omega},\check{\mathcal{F}},\check{P})$ over which a time homogeneous Poisson random measure $\check{\eta}$ on Z with intensity measure ν is defined. Since for any $j=1,\ldots,J$ and $k=1,\ldots K$, the law of $\eta(B_j\times(s_{k-1},s_k])$ has the same law as $\check{\eta}(B_j\times(s_{k-1},s_k])$ and both random variables are independent of $\mathcal{F}_{t_{k-1}}$, the sequence of martingale differences given by

$$\Big\{1_{\{\check{\eta}(B_j\times(s_{k-1},s_k])\geq m_0\}}\,|\check{\eta}(B_j\times(s_{k-1},s_k])|^p$$
$$-\,\mathbb{E}\left[1_{\{\check{\eta}(B_j\times(s_{k-1},s_k])\geq m_0\}}\,|\check{\eta}(B_j\times(s_{k-1},s_k])|^p \mid \mathcal{F}_{t_{k-1}}\right],\quad m=(J-1)k+j\Big\}_{m=1}^{JK}.$$

and the sequence of martingale differences given by Eq. 3.10 have the same laws. Thus we can apply Theorem 2.2 of [22] and conclude that

$$\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p \left(1_{\{\eta(B_j\times(s_{k-1},s_k])\geq m_0\}}\,\eta(B_j\times(s_{k-1},s_k])^p\right.\right.\right.$$
$$\left.\left.\left.-\mathbb{E}\left[1_{\{\eta(B_j\times(s_{k-1},s_k])\geq m_0\}}\,\eta(B_j\times(s_{k-1},s_k])^p \mid \mathcal{F}_{t_{k-1}}\right]\right)\right)^{\frac{1}{p}}\right\}$$

$$=\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p \left(1_{\{\check{\eta}(B_j\times(s_{k-1},s_k])\geq m_0\}}\,\check{\eta}(B_j\times(s_{k-1},s_k])^p\right.\right.\right.$$
$$\left.\left.\left.-\mathbb{E}\left[1_{\{\check{\eta}(B_j\times(s_{k-1},s_k])\geq m_0\}}\,\check{\eta}(B_j\times(s_{k-1},s_k])^p \mid \mathcal{F}_{t_{k-1}}\right]\right)\right)^{\frac{1}{p}}\right\}.$$

Since $\mathbb{R}_+ \ni x \mapsto \Phi(x^{\frac{1}{p}})$ is convex, we can continue and write

$$II \le C_1 \mathbb{E}\Phi\left\{ \left(\sum_{k=1}^K \sum_{j=1}^J \sum_{i=1}^I 1_{A_{ji}^k} \left|\xi_{ji}^k\right|^p 1_{\{\check{\eta}(B_j\times(s_{k-1},s_k])\ge m_0\}}\, \check{\eta}(B_j \times (s_{k-1},s_k])^p \right)^{\frac{1}{p}} \right\}$$
$$+ C_2 \mathbb{E}\Phi\left\{ \left(\sum_{k=1}^K \sum_{j=1}^J \sum_{i=1}^I 1_{A_{ji}^k} \left|\xi_{ji}^k\right|^p \right.\right.$$
$$\left.\left. \times \mathbb{E}\left[1_{\{\check{\eta}(B_j\times(s_{k-1},s_k])\ge m_0\}}\, \check{\eta}(B_j \times (s_{k-1},s_k])^p \mid \mathcal{F}_{t_{k-1}} \right] \right)^{\frac{1}{p}} \right\}.$$

Again, after applying Proposition 3.2, the second summand can be treated in the same way as the third term in inequality 3.6. Now, let us recall that the $\mathbb{N}_0$ valued random variables $\{\check{\eta}(B_j \times (s_{k-1}, s_k]) : k = 1, \ldots, K \text{ and } j = 1, \ldots, J\}$ are Poisson distributed with parameter $\nu(B_j)\tau$. Therefore, the explicit formula of the expectation gives

$$II = \sum_{\mathfrak{l}\in\tilde{\Omega}} \mathbb{P}\left(\check{\eta}(B_j \times (s_{k-1}, s_k]) = l_{k,j}, 0 \le j \le J, 0 \le k \le K\right)$$
$$\times \mathbb{E}\Phi\left\{ \left(\sum_{k=1}^K \sum_{j=1}^J \sum_{i=1}^I \sum_{l=m_0}^\infty 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}}\, l_{k,j}^p \left|\xi_{ji}^k\right|^p \right)^{\frac{1}{p}} \right\}$$
$$= \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_{k=1}^K \prod_{j=1}^J e^{-\nu(B_j)\tau} \frac{\nu(B_j)^{l_{kj}} \tau^{l_{kj}}}{l_{kj}!}$$
$$\times \mathbb{E}\Phi\left\{ \left(\sum_{k=1}^K \sum_{j=1}^J \sum_{i=1}^I \sum_{l=m_0}^\infty 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}}\, l_{k,j}^p \left|\xi_{ji}^k\right|^p \right)^{\frac{1}{p}} \right\}$$

where we put $\tilde{\Omega} = \otimes_{k=1}^K \otimes_{j=1}^J \mathbb{N}_0$. Since the sum over l starts at m_0, the set $\tilde{\Omega}_0 := \{l_{kj} < m_0, \forall k = 1, \ldots, K \text{ and } j = 1, \ldots, J\}$ does not contribute to the sum and, hence, can be omitted. Therefore, for any $\mathfrak{l}$ which contributes to the sum we can put at least one time $\tau^{m_0-1}\nu(B_j)^{m_0-1}$ in front of the outer sum. To realise this step, let $\mathfrak{e}(\mathfrak{l})_{jk} := l_{kj} - (m_0 - 1)$ if $l_{jk} \ge m_0$ and $\mathfrak{e}(\mathfrak{l})_{jk} := l_{kj}$ otherwise. In addition, let $\#\mathfrak{l} := \{\mathfrak{l}_{k,j} \ge m_0 : 0 \le k \le K, 0 \le j \le J\}$. Property B.7 gives the following inequality

$$r\,\Phi(u) \le \Phi\left(u r^{1/c_\Phi}\right), \quad 0 < r \le 1. \tag{3.11}$$

Thus, we get

$$
\begin{aligned}
II &\leq \max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\tau^{m_0-1} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_{k=1}^{K}\prod_{j=1}^{J} e^{-\nu(B_j)\tau}\frac{\nu(B_j)^{\mathfrak{e}(\mathfrak{l})_{kj}}\tau^{\mathfrak{e}(\mathfrak{l})_{ki}}}{\mathfrak{l}_{ki}!} \\
&\quad \times \mathbb{E}\left[\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=m_0}^{\infty} 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}}\, l^p \left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}\right] \\
&= \max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\tau^{m_0-1} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_{k=1}^{K}\prod_{j=1}^{J} e^{-\nu(B_j)\tau}\frac{\nu(B_j)^{\mathfrak{e}(\mathfrak{l})_{kj}}\tau^{\mathfrak{e}(\mathfrak{l})_{ki}}}{\mathfrak{e}(\mathfrak{l})_{ki}!} \\
&\quad \times \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=m_0}^{\infty} 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}}\right.\right. \\
&\qquad \left.\left.\times \frac{\mathfrak{l}_{k,j}^p}{\mathfrak{l}_{k,j}^{1/c_\Phi}(\mathfrak{l}_{kj}-1)^{1/c_\Phi}\cdots(\mathfrak{l}_{kj}-(m_0-1))^{1/c_\Phi}}\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}.
\end{aligned}
$$

Since $p\leq 1+(m_0-1)/c_\Phi$, there exists a constant $C>0$ such that for all $l\geq m_0$ we have

$$
\frac{l^p}{l^{1/c_\Phi}(l-1)^{1/c_\Phi}\cdots(l-(m_0-1))^{1/c_\Phi}} \leq C\,(l-(m_0-1)).
$$

Hence, remunerating and taking into account that all summands are positive, give

$$
\begin{aligned}
II &\leq C\max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\tau^{m_0-1} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_{k=1}^{K}\prod_{j=1}^{J} e^{-\nu(B_j)\tau}\frac{\nu(B_j)^{\mathfrak{e}(\mathfrak{l})_{kj}}\tau^{\mathfrak{e}(\mathfrak{l})_{ki}}}{\mathfrak{e}(\mathfrak{l})_{ki}!} \\
&\quad \times \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=m_0}^{\infty} 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}}\,(l-m_0+1)\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\} \\
&\leq C\max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\tau^{m_0-1} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_{k=1}^{K}\prod_{j=1}^{J} e^{-\nu(B_j)\tau}\frac{\nu(B_j)^{\mathfrak{e}(\mathfrak{l})_{kj}}\tau^{\mathfrak{e}(\mathfrak{l})_{ki}}}{\mathfrak{e}(\mathfrak{l})_{ki}!} \\
&\quad \times \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=m_0}^{\infty} 1_{A_{ji}^k} 1_{\{l-m_0+1=\mathfrak{l}_{k,j}-m_0+1\}}\,(l-m_0+1)\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\} \\
&\leq C\max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\tau^{m_0-1} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_{k=1}^{K}\prod_{j=1}^{J} e^{-\nu(B_j)\tau}\frac{\nu(B_j)^{\mathfrak{e}(\mathfrak{l})_{kj}}\tau^{\mathfrak{e}(\mathfrak{l})_{ki}}}{\mathfrak{e}(\mathfrak{l})_{ki}!} \\
&\quad \times \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=1}^{\infty} 1_{A_{ji}^k} 1_{\{l=\mathfrak{e}(\mathfrak{l})_{k,j}\}}\, l\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}.
\end{aligned}
$$

Note that for any $\mathfrak{l} \in \tilde{\Omega}$ there exists at most $\mathfrak{k}_1, \ldots, \mathfrak{k}_n$ different indices belonging to $\tilde{\Omega} \setminus \tilde{\Omega}_0$ with $n = KJ$ such that $\mathfrak{e}(\mathfrak{k}_i) = \mathfrak{l}$, $i = 1, \ldots, n$. Therefore, we get

$$II \le C K J \max_{1\le j\le J} \nu(B_j)^{m_0-1}\tau^{m_0-1} \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_{k=1}^{K}\prod_{j=1}^{J} e^{-\nu(B_j)\tau} \frac{\nu(B_j)^{\mathfrak{l}_{kj}}\tau^{\mathfrak{l}_{ki}}}{\mathfrak{l}_{ki}!}$$
$$\times \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=0}^{\infty} 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}}\, l \left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}.$$

Substituting the original probability of the Poisson random measure $\check{\eta}$ gives

$$II \le C K J \max_{1\le j\le J} \nu(B_j)^{m_0-1}\tau^{m_0-1}$$
$$\times \sum_{\mathfrak{l}\in\tilde{\Omega}} \prod_{k=1}^{K}\prod_{j=1}^{J} \mathbb{P}\left(\check{\eta}(B_j\times(s_{k-1},s_k]) = \mathfrak{l}_{k,j}, 0\le j\le J, 0\le k\le K\right)$$
$$\times \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=0}^{\infty} 1_{A_{ji}^k} 1_{\{l=\mathfrak{l}_{k,j}\}}\, l \left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}$$
$$\le C K J \max_{1\le j\le J} \nu(B_j)^{m_0-1}\tau^{m_0-1}\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=0}^{\infty} 1_{A_{ji}^k}\check{\eta}(B_j\times(s_{k-1},s_k])\left|\xi_{ji}^k\right|^p\right)^{\frac{1}{p}}\right\}.$$

Again we want to apply Theorem 2.2 [22] and again, before applying the Theorem 2.2 [22], we have to write the inner part as a sequence of martingale differences. In particular, we write

$$II \le C K J \max_{1\le j\le J} \nu(B_j)^{m_0-1}\tau^{m_0-1}$$
$$\times \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=0}^{\infty} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p\left(\check{\eta}(B_j\times(s_{k-1},s_k]) - \nu(B_j)\tau\right)\right)^{\frac{1}{p}}\right\}$$
$$+ \max_{1\le j\le J} \nu(B_j)^{m_0-1}\tau^{m_0-1}\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=0}^{\infty} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p \nu(B_j)\tau\right)^{\frac{1}{p}}\right\}.$$

In the first term we can replace $\check{\eta}(B_j\times(s_{k-1},s_k])$ by $\eta(B_j\times(s_{k-1},s_k])$. To tackle the second term we apply Lemma A.4. Moreover, we use the fact that $\nu(B_j)\tau =$

$\mathbb{E}\left[\eta(B_j \times (s_{k-1}, s_k])\right]$ and that $\eta(B_j \times (s_{k-1}, s_k])$ is $\mathcal{F}_{s_{k-1}}$-measurable for any $k = 1 \ldots K$. Thus we can write

$$II \leq C\,K\,J \max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\tau^{m_0-1}$$
$$\times \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=0}^{\infty} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p \left(\eta(B_j \times (s_{k-1}, s_k]) - \nu(B_j)\tau\right)\right)^{\frac{1}{p}}\right\}$$
$$+ \max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\tau^{m_0-1}\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=0}^{\infty} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p \eta(B_j \times (s_{k-1}, s_k])\right)^{\frac{1}{p}}\right\}.$$

Using the convexity we can split the first term into two and apply again Lemma A.4. Thus we have

$$II \leq C\,K\,J \max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\tau^{m_0-1}\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=0}^{\infty} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p \eta(B_j \times (s_{k-1}, s_k])\right)^{\frac{1}{p}}\right\}$$
$$+2 \max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\tau^{m_0-1}\mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=0}^{\infty} 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p \eta(B_j \times (s_{k-1}, s_k])\right)^{\frac{1}{p}}\right\}.$$

Using the definition of the Itô integral, we finally write

$$II \leq C\,(K\,J + 2) \max_{1\leq j\leq J} \nu(B_j)^{m_0-1}\,\tau^{m_0-1}\,\mathbb{E}\Phi\left\{\left(\int_0^t \int_Z |\xi(z,s)|^p\ \eta(dz,ds)\right)^{\frac{1}{p}}\right\}.$$

Note that $K\tau^{-1}$=T. Using the assumption $\nu(B_j) \leq 1$ for all $j = 1, \ldots, J$, we obtain

$$II \leq C\,T\,\tau\,\tau J\mathbb{E}\Phi\left\{\left(\int_0^t \int_Z |\xi(z,s)|\,\eta(dz,ds)\right)^{\frac{1}{p}}\right\}.$$

It remains to investigate the last summand in Eq. 3.6. First, observe that, since $\nu(B_j) \leq 1$, $\nu(B_j)^p \leq \nu(B_j)$. Now, by the following calculations we can give the desired estimate.

$$III = \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\left|1_{A_{ji}^k}\,\xi_{ji}^k\nu(B_j)\tau\right|^p\right)^{\frac{1}{p}}\right\}$$
$$\leq \mathbb{E}\Phi\left\{\left(\sum_{k=1}^{K}\sum_{j=1}^{J}\sum_{i=1}^{I}\tau^p 1_{A_{ji}^k}\left|\xi_{ji}^k\right|^p\ \nu(B_j)^p\right)^{\frac{1}{p}}\right\}$$

$$
\begin{aligned}
&\le \tau^{\frac{p-1}{p}}\, \mathbb{E}\Phi \left\{ \left(\sum_{k=1}^K \tau \sum_{j=1}^J \sum_{i=1}^I 1_{A_{ji}^k} \left|\xi_{ji}^k\right|^p \nu(B_j)^p \right)^{\frac{1}{p}} \right\} \\
&\le \tau^{\frac{p-1}{p}} \sum_{k=1}^K \tau\, \mathbb{E}\Phi \left\{ \left(\sum_{j=1}^J \sum_{i=1}^I 1_{A_{ji}^k} \left|\xi_{ji}^k\right|^p \nu(B_j)^p \right)^{\frac{1}{p}} \right\} \\
&\le \tau^{(p-1)/p} \max_j \nu(B_j)^{(p-1)/p}\, \mathbb{E} \sum_{k=1}^K \tau\, \Phi \left\{ \left(\sum_{j=1}^J \sum_{i=1}^I 1_{A_{ji}^k} \left|\xi_{ji}^k\right|^p \nu(B_j) \right)^{\frac{1}{p}} \right\} \\
&\le \tau^{(p-1)/p} \max_j \nu(B_j)^{(p-1)/p}\, \nu(Z_J)^{c_\Phi/p}\, \mathbb{E} \sum_{k=1}^K \tau\, \Phi \left\{ \left(\sum_{j=1}^J \sum_{i=1}^I 1_{A_{ji}^k} \left|\xi_{ji}^k\right|^p \frac{\nu(B_j)}{\nu(Z_J)} \right)^{\frac{1}{p}} \right\} \\
&\le \tau^{(p-1)/p} \max_j \nu(B_j)^{(p-1)/p}\, \nu(Z_J)^{c_\Phi/p}\, \mathbb{E} \sum_{k=1}^K \tau \sum_{j=1}^J \Phi \left\{ \left(\sum_{i=1}^I 1_{A_{ji}^k} \left|\xi_{ji}^k\right|^p \right)^{\frac{1}{p}} \right\} \frac{\nu(B_j)}{\nu(Z_J)} \\
&\le \tau^{(p-1)/p} \max_j \nu(B_j)^{(p-1)/p}\, \nu(Z_J)^{c_\Phi/p-1} \mathbb{E} \sum_{k=1}^K \tau \sum_{j=1}^J \sum_{i=1}^I 1_{A_{ji}^k} \Phi \left\{ \left|\xi_{ji}^k\right| \right\} \nu(B_j).
\end{aligned}
$$

Here, we used in the first step the convexity of Φ and put $\tau^{(p-1)/p}$ and $\max_j \nu(B_j)^{(p-1)/p}$ in front of the expectation value. Then we multiplied the sum with $\nu(Z_J)$. Note, since $\sum_j^J \nu(B_j) = \nu(Z_J)$, $\nu(B_j)/\nu(Z_J)$ is a probability measure. Again using the convexity, we applied the Jensen inequality and put Φ into the inner sum. Next, again, taking into account that $\{A_{ji}^k, 1 \le i \le I\}$ are disjoint, we put the sum running over the index i in front of the function Φ. Hence, the RHS of Eq. 3.12 is bounded by

$$
\tau^{(p-1)/p} \max_j \nu(B_j)^{(p-1)/p}\, \nu(Z_J)^{c_\Phi/p-1} \int_0^T \int_Z \mathbb{E}\Phi\left(|\xi(s,z)|\right) \nu(dz)\, ds.
$$

Taking everything into account we arrive at

$$
\begin{aligned}
&\mathbb{E} \sup_{0\le t\le T} \Phi \left\{ \left| \int_0^t \int_Z \xi(s,x)\, \tilde{\eta}(dx,ds) \right| \right\} \\
&\quad \le C_p(E) 2^{c_\Phi} \left(1 + \tau T(\tau J)\right) \mathbb{E}\Phi \left\{ \left(\int_0^T \int_Z |\xi(s,x)|^p\, \eta(dx,ds) \right)^{\frac{1}{p}} \right\} \\
&\qquad + C_p(E) \tau^{(p-1)/p} \max_j \nu(B_j)^{(p-1)/p}\, \nu(Z_J)^{c_\Phi/p-1} \int_0^T \int_Z \mathbb{E}\Phi\left(|\xi(s,z)|\right) \nu(dz)\, ds,
\end{aligned}
$$

which is the assertion. □

4 Proof of Corollary 2.14

The proof is a generalisation of the proofs of Bass and Cranston, Lemma 5.2 [6] or Protter and Talay, Lemma 4.1 [26]. The proof here differs only in the beginning. Assume η is a $\mathbb{R}^d$-valued Poisson random measure. Then the quadratic variation $[L] = \{[L]_t : 0 \le t < \infty\}$ of the process $L = \{L(t) : 0 \le t < \infty\}$, where

$$L(t) = \int_0^t \int_Z \xi(s;z)\tilde{\eta}(dz;ds), \quad t \ge 0,$$

is given by

$$[L]_t = \int_0^t \int_Z |\xi(s;z)|^2 \, \eta(dz;ds), \quad t \ge 0.$$

If $\Phi(x) = x^r$ for $r \ge 2$, then, in this setting, the generalised Burkholder Davis Gundy inequality reads

$$\mathbb{E} \sup_{0<s\le t} \left| \int_0^t \int_Z \xi(s;z)\tilde{\eta}(dz;ds) \right|^r \le C\,\mathbb{E} \left(\int_0^t \int_Z |\xi(s;z)|^2 \, \eta(dz;ds) \right)^{\frac{r}{2}}, \quad t \ge 0, \quad (4.1)$$

where the constant C depends only on r, $2 \le r < \infty$.

Starting from this inequality, Bass and Cranston, Lemma 5.2 [6], resp. Protter and Talay, Lemma 4.1 [26] have proved the Corollary 2.14 for $p = 2$, $E = \mathbb{R}^d$, $\mathbb{R}_+ \ni x \mapsto \Phi(x) = x^{p^n}$ for $n \ge 1$.

The proof of Corollary 2.14 follows along the same lines as the proof of Lemma 5.2, rep. Lemma 4.1 in the articles mentioned above. Also, as mentioned before, the only difference of the proof of Lemma 5.2, rep. Lemma 4.1 and the proof of Corollary 2.14 is in the first step. Here, using the inequality 2.5 given in Proposition 2.13 for Φ instead of the inequality 4.1 gives

$$\mathbb{E} \sup_{0<s\le t} \left| \int_0^t \int_Z \xi(s;z)\tilde{\eta}(dz;ds) \right|^{p^n} \le C_p(E)\, 2^{p^n + p^{n-1}} (m_0 - 1)^{(p-1)\frac{r}{p}} \times \mathbb{E} \left(\int_0^t \int_Z |\xi(s;z)|^p \, \eta(dz;ds) \right)^{p^{n-1}}.$$

Observe that the process

$$[0,\infty) \ni t \mapsto \int_0^t \int_Z |\xi(s;z)|^p \, \eta(dz;ds)$$

is a real valued process of finite variation. That means, we are in the real valued case and can proceed as the in Lemma 5.2, rep. Lemma 4.1. Therefore we omit the proof.

Appendix A: Discrete Inequalities of Burkholder-Davis-Gundy Type

In this section we verify, that the Burkholder Gundy inequality also holds in Banach spaces of martingale type p, $p \in (1, 2]$. But first, let us state some basic facts about maximal inequalities.

A useful tool in the theory of martingales is Doob's maximal inequality. The simplest version says that all real valued non-negative submartingales $\{M_n\}_{n=0}^N$ satisfy the inequality,

$$\lambda \mathbb{P}\left(\sup_{0\le k\le n} |M_k| > \lambda \right) \le \mathbb{E}\, 1_{\max_{k\le n} M_k \ge \lambda} |M_n|, \quad 1 \le n \le N,$$

and, hence, satisfy

$$\lambda^p\, \mathbb{P}\left(\sup_{0\le k\le n} |M_k|^p > \lambda \right) \le \mathbb{E}|M_n|^p, \quad 1 \le n \le N. \tag{A.1}$$

Now, one sees immediately that all real valued non-negative submartingales $\{M_n\}_{n=0}^N$ satisfy

$$\mathbb{E}|M_n|^p \le \mathbb{E} \sup_{0\le k\le n} |M_k|^p \le q^p\, \mathbb{E}|M_n|^p, \tag{A.2}$$

where q is the conjugate exponent to p. From the last version of Doob's maximal inequality we can derive the following corollary.

Corollary A.1 *Let $p \in (1,2]$ and let E be a Banach space of martingale type p. Then there exists a constant $C = C_p(E)$ such that for all E-valued finite martingales $\{M_n\}_{n=0}^N$ the following inequality holds*

$$\mathbb{E} \sup_{0\le n\le N} |M_n|_E^p \le C\, \mathbb{E} \sum_{k=0}^N |M_n - M_{n-1}|_E^p, \tag{A.3}$$

where as usual, we put $M_{-1} = 0$.

Nevertheless, in the proof of inequality (ii) we use a stronger inequality, namely, we suppose that there exists a constant C such that for all E-valued finite martingales $\{M_n\}_{n=0}^N$ the following inequality holds

$$\mathbb{E}|M_n|^2 \le C\mathbb{E} \left(\sum_{n=0}^N |M_n - M_{n-1}|^p \right)^{\frac{2}{p}}. \tag{A.4}$$

We can derive this stronger inequality from a generalisation of Doob's maximal inequality. Since it is interesting on its own, before showing inequality A.4, we state a generalisation of Doob's maximal inequality. To be more precise, in the Doob's maximal inequality we can replace the square by a convex, non decreasing and continuous function $\Phi : [0,\infty) \to \mathbb{R}$ with $\Phi(0) = 0$. In addition, Φ has to satisfy *the growth condition* (see Appendix B).

Proposition A.2 (Garsia, p. 173 [15]) *For all convex, non decreasing, and continuous function $\Phi : [0,\infty) \to \mathbb{R}$ with $\Phi(0) = 0$ and satisfying the growth condition, there* *exists a constant* $C > 0$ *such that for all non-negative real valued sub-martingales* $\{X_n\}_{n=0}^N$ *with* $X_0 = 0$ *we have*

$$\mathbb{E}\Phi\Big(\sup_{1\le n\le N} X_n\Big) \le C\,\mathbb{E}\Phi\big(X_N\big).$$

To be precise, $C = 4\,(C^*_\Psi - 1)$ *where* Ψ *is the conjugate convex function to* Φ.

Assume E is of martingale type p. Now, from the Definition 2.10 and the generalised Doob maximal inequality, i.e. Proposition A.2, we can show that the Burkholder-Davis-Gundy inequality is also valid on E.

Theorem A.3 *Let* $\Phi : [0,\infty) \to \mathbb{R}$ *be a non decreasing, convex and continuous function with* $\Phi(0) = 0$ *and satisfying the growth condition (for definition we refer to Appendix B).*

Let $p \in [1, 2]$ *be fixed and let* E *be a Banach space* E *of martingale type* p. *Then, there exists a constant* $C_p(E,\phi) > 0$ *such that for all* E*-valued finite martingale* $\{M_n\}_{n=0}^N$ *the following inequality holds*

$$\mathbb{E}\Phi\left(\sup_{0\le n\le N} |M_n|_E\right) \le C_p(E,\phi)\,\mathbb{E}\Phi\left(\left(\sum_{n=0}^N |M_n - M_{n-1}|_E^p\right)^{\frac{1}{p}}\right),$$

where as usual, we put $M_{-1} = 0$.

Before proving Theorem A.3 we have to do some preparatory work. First we will introduce the following notion. For any E-valued martingale over a probability space $(\Omega,\mathcal{F},\mathbb{P})$, let $(\mathcal{F}_n^M)_{n\in\mathbb{N}}$ be the natural filtration induced by $\{M_n\}_{n=0}^N$, i.e. $\mathcal{F}_n^M := \sigma(M_k, k \le n)$, $n \in \mathbb{N}$. By $\{m_n\}_{n=0}^N$ we denote the sequence of martingale differences, i.e. $m_n := M_{n+1} - M_n$, $n \in \mathbb{N}$, and by M^* the sequence $\{M_n^*\}_{n=0}^N$ given by $M_n^* := \sup_{k\le n} |M_n|$. Let

$$S_{n,p}(M) := \left(\sum_{k\le n} |m_k|_E^p\right)^{\frac{1}{p}},$$

and $S_p(M) := S_{\infty,n}$. We will need the following Lemmata. Since the Lemmata are valid for real valued random variables, we omit their proofs and give only the reference.

Lemma A.4 (Burkholder et al., Theorem 3.2 [11]; Garsia, Theorem 0.1 [15]) *Let* Φ *be a convex function satisfying the conditions of Theorem* A.3 *and* $(\Omega,\mathcal{F},\{\mathcal{F}_n\}_{n=0}^N,\mathbb{P})$ *be a filtered probability space. Then there exists a constant* $C > 0$*, only depending on* Φ*, such that for all sequence* $\{z_n\}_{n=0}^N$ *of real-valued, non negative and measurable functions* $(\Omega,\mathcal{F},\mathbb{P})$ *the following inequality holds*

$$\mathbb{E}\,\Phi\Big(\sum_{n=0}^N \mathbb{E}[z_n \mid \mathcal{F}_{n-1}]\Big) \le C\,\mathbb{E}\Phi\Big(\sum_{n=0}^N z_n\Big).$$

To be more precise, $C = (c^*_\Phi)^{2c^*_\Phi}$*, where* c^*_Φ *is defined in Eq.* B.5.

By means of the generalised Doob's maximal inequality and the Lemmata before, the following Proposition can be verified.

Proposition A.5 *There exists a constant $C < \infty$ such that for all E-valued martingales $\{M_n\}_{n=0}^N$ and all M-previsible processes $\{w_m\}_{m=0}^N$ satisfying $|M_n - M_{n-1}| \le w_n$ for all $1 \le n \le N$, we have*

$$\mathbb{E}\Phi(M_n^*) \le C\mathbb{E}\phi(S_{n,p}(M)) + C\mathbb{E}\phi(w_n^*), \quad 1 \le n \le N.$$

An estimate of the constant is given by $C_{\Phi,p}(E) = \min\{2\delta^{-c_\Phi^*}\beta^{c_\Phi^*} : \beta > 1, 0 < \delta < 1 - \beta$ such that $2L_p(E)\frac{\delta^p\beta^{c_\Phi^*}}{(\beta-\delta-1)^p} = \frac{1}{2}\}$.

Proof of Proposition A.5 This proof follows the proof of Theorem 15.1 [10], where only the real valued case is considered. Therefore, we had to modify the original proof of Burkholder in some points. Without loss of generality we set $n = N$. Similarly, we will show that the random variables M_N^* and $S_{N,p}(M) \vee w_N^*$ satisfy the assumption of Lemma 7.1 [10]. I.e. We will show that for $\beta > 1$ and $0 < \delta < \beta - 1$ the following holds

$$\mathbb{P}\left(M_N^* > \beta\lambda,\ S_{N,p}(M) \vee w_N^* \le \delta\lambda\right) \le 2\,L_p(X)\,\frac{\delta^p}{(\beta-\delta-1)^p}\,\mathbb{P}\left(M_N^* > \lambda\right), \quad \lambda > 0. \tag{A.5}$$

To prove Eq. A.5, we introduce the following stopping times. Let $\mu := \inf\{1 \le n \le N : |M_n| > \lambda\}$, $\nu := \inf\{1 \le n \le N : |M_n| > \beta\lambda\}$, and $\sigma := \inf\{1 \le n \le N : |S_{n,p}(M)| > \delta\lambda$ or $w_n > \delta\lambda\}$. If the infimum will not be attained, we set the stopping time to ∞. Let $H = \{H_n,\ 1 \le n \le N\}$ be defined by

$$H_n := \sum_{k \le n} h_k = \sum_{k \le n} 1_{\{\mu < k \le \nu \vee \sigma\}}\,(M_k - M_{k-1}), \quad 0 \le n \le N.$$

Since w is previsible, $\{\mu < k \le \nu \vee \sigma\} \in \mathcal{F}_k^M$, and the process H is a martingale. Moreover, on $\{\mu = \infty\} = \{M_N^* \le \lambda\}$, $S_{N,p}(H) = 0$. On $\{0 < \sigma < \infty\}$, the assumption on w leads to

$$\begin{aligned} S_{N,p}^p(H) \le S_{\sigma,p}^p(H) \le S_{\sigma,p}^p(M) &= S_{\sigma-1,p}^p(M) + (M_\sigma - M_{\sigma-1})^p \\ &\le S_{\sigma-1,p}^p(M) + w_\sigma^p \le 2\delta^p\lambda^p. \end{aligned} \tag{A.6}$$

This inequality can also be extended to the case where $\sigma = \infty$. Therefore, since E is a Banach space of martingale type p, by Definition 2.10, there exists a constant $L_p(E)$ such that

$$\mathbb{E}|H_N|^p \le L_p(E)\,\mathbb{E}\sum_{k=0}^N |h_k|^p \le L_p(E)\,\mathbb{E}S_{N,p}^p(H). \tag{A.7}$$

Substituting Eq. A.6, we get

$$\mathbb{E}|H_N|^p \le 2\, L_p(E)\, \delta^p \lambda^p \mathbb{P}\left(M_N^* > \lambda\right).$$

Since $\{M_N^* > \beta\lambda,\ S_{N,p}(M) \vee w_N^* \le \delta\lambda\} \subset \{H_N^* > \beta\lambda - \lambda - \delta\lambda\}$,

$$\mathbb{P}\left(M_N^* > \beta\lambda,\ S_{N,p}(M) \vee w_N^* \le \delta\lambda\right) \le \mathbb{P}\left(H_N^* > \beta\lambda - \lambda - \delta\lambda\right).$$

The simple version of Doob's maximal inequality, i.e. Eqs. A.1, and A.7 give

$$\begin{aligned}\mathbb{P}\left(M_N^* > \beta\lambda,\ S_{N,p}(M) \vee w_N^* \le \delta\lambda\right) &\le \mathbb{P}\left(H_N^* > \beta\lambda - \lambda - \delta\lambda\right)\\ &\le \frac{1}{(\beta-\delta-1)^p\lambda^p}\mathbb{E}|H_N|^p\\ &\le 2\, L_p(E)\, \frac{\delta^p}{(\beta-\delta-1)^p}\mathbb{P}\left(M_N^* > \lambda\right).\end{aligned}$$

Since δ can be chosen arbitrary small, the assumptions of Lemma 7.1 [10] are satisfied and we apply the Lemma to verify the assertion. The exact constant can be verified by using inequality B.7 and Definition B.5.

Proof of Theorem A.3 The proof works in analogy to the proof of Davis, Burkholder and Gundy (see e.g. Theorem 1.1 [11], or Theorem 15.1 [10]). For an E-valued martingale $\{M_n\}_{n=0}^N$ and its sequence of martingale differences $\{m_n\}_{n=0}^N$, let $\mathcal{A}_n := \{|m_n| \le 2m_{n-1}^*\}$, $0 \le n \le N$. Davis introduced in [12] the following decomposition of M, where $M = G + H$, and $G = \{G_n\}_{n=0}^N$ and $H = \{H_n\}_{n=0}^N$ are defined by

$$G_n = \sum_{k=1}^n g_k := \sum_{k=1}^n \left[y_k - \mathbb{E}[y_k \mid \mathcal{F}_{k-1}]\right], \quad 0 \le n \le N,$$

$$H_n = \sum_{k=1}^n h_k := \sum_{k=1}^n \left[z_k + \mathbb{E}[y_k \mid \mathcal{F}_{k-1}]\right], \quad 0 \le n \le N,$$

with $y_k = m_k 1_{\mathcal{A}_k}$ and $z_k = m_k 1_{\mathcal{A}_k^C}$, $0 \le k \le N$.

Now, since

$$M_n^* \le G_n^* + H_n^*,$$

by Eq. B.6, there exists a constant $c_\Phi < \infty$, depending only on Φ, such that

$$\mathbb{E}\Phi(M_n^*) \le c_\Phi\, \mathbb{E}\Phi\big(G_n^*\big) + c_\Phi\, \mathbb{E}\Phi\big(H_n^*\big). \tag{A.8}$$

First, we will investigate the last term, i.e. $\mathbb{E}\Phi(H^*)$ and then we will investigate $\mathbb{E}\Phi(G^*)$. Observe that, since $\{m_n\}_{n=0}^N$ is a sequence of martingale differences,

$$\mathbb{E}\left[y_k \mid \mathcal{F}_{k-1}\right] + \mathbb{E}\left[z_k \mid \mathcal{F}_{k-1}\right] = \mathbb{E}\left[m_k \mid \mathcal{F}_{k-1}\right] = 0, \quad 1 \le k \le N.$$

Therefore, $H_n = \sum_{k=0}^{n} \big[z_k - \mathbb{E}[z_k \mid \mathcal{F}_{k-1}] \big]$, $1 \le n \le N$. Applying the Jensen inequality and Lemma A.4 we get

$$\mathbb{E}\Phi\big(H_n^*\big) \le \mathbb{E}\Phi\Big(\sum_{k=0}^{n} |z_k - \mathbb{E}[z_k \mid \mathcal{F}_{k-1}]| \Big)$$
$$\le \mathbb{E}\Phi\Big(\sum_{k=0}^{n} |z_k| + \sum_{k=0}^{n} \mathbb{E}[|z_k| \mid \mathcal{F}_{k-1}] \Big) \le 2\,\mathbb{E}\Phi\Big(\sum_{k=0}^{n} |z_k| \Big).$$

Since $|m_k| > 2m_{k-1}^*$ implies $|m_k| < 2(m_k^* - m_{k-1}^*)$, and, hence, $|z_k| \le 2(m_k^* - m_{k-1}^*)$. Moreover, since $m_n^* = \sup_{k \le n} |m_k| = \sum_{k=1}^{n} (m_k^* - m_{k-1}^*)$, it follows that $\sum_{k=1}^{n} |z_k| \le 2m_n^*$. Therefore,

$$\mathbb{E}\Phi(H_n^*) \le 4\,\mathbb{E}\Phi(m_n^*), \quad 1 \le n \le N.$$

Since $\{|m_n|\}_{n=1}^{N}$ is a non negative real valued sub-martingale, we get by a generalisation of Doob's maximal inequality, Proposition A.2

$$\mathbb{E}\Phi(H_n^*) \le C\,2\,\mathbb{E}\Phi(|m_n|).$$

From $m_n = M_n - M_{n-1}$, it follows that $|m_n| \le S_{n,p}(M)$, and, hence,

$$\mathbb{E}\Phi(H_n^*) \le C'\,2\,\mathbb{E}\Phi(S_{n,p}(M)), \quad 1 \le n \le N. \tag{A.9}$$

In the next paragraph, we will give an upper estimate of the term $\mathbb{E}\Phi(G_n^*)$. Observe, first, that $|m_k| \le 2m_{k-1}^*$ implies $|g_k| \le 4m_{k-1}^*$, $0 \le k \le N$. This means, that g_k is controlled by a $\mathcal{F}_{k-1}$-measurable random variable, and, therefore, we can apply Proposition A.5 to get a control of G^*. In particular, there exists a constant $c_{\Phi,E} < \infty$, only depending on Φ and E, such that

$$\mathbb{E}\Phi(G_n^*) \le c_{\Phi,E}\mathbb{E}\Phi(S_{n,p}(G)) + c_{\Phi,E}\mathbb{E}\Phi(m_n^*). \tag{A.10}$$

The term $\mathbb{E}\Phi(m_n^*)$ can be estimated by the generalised Doob maximal inequality A.2. So, we obtain

$$\mathbb{E}\Phi(G_n^*) \le c'_{\Phi,E}\mathbb{E}\Phi(S_{n,p}(G)) + c'_{\Phi,E}\mathbb{E}\Phi(m_n). \tag{A.11}$$

From $m_n = M_n - M_{n-1}$, it follows that $|m_n| \le S_{n,p}(M)$. It remains to investigate $\mathbb{E}\Phi(S_{n,p}(G))$. Note, that

$$\mathbb{E}\Phi(S_{n,p}(G)) \le c''_{E,\Phi}\,\mathbb{E}\Phi(S_{n,p}(M)) + c''_{E,\Phi}\,\mathbb{E}\Phi(S_{n,p}(H)).$$

Now, since

$$\sum_{k\le n}|z_k|^p \le \sum_{k\le n}|m_k^* - m_{k-1}^*|^p$$
$$\le |m_n^*|^{p-1}\sum_{k\le n}|m_k^* - m_{k-1}^*| \le |m_n^*|^p,$$

we have by Lemma A.4 applied to $\{|z_n|^p\}_{n=1}^N$

$$\mathbb{E}\Phi(S_{n,p}(H)) \le C\,\mathbb{E}\Phi\big(m_n^*\big), \quad 1 \le n \le N.$$

Again applying Proposition A.2 leads to

$$\mathbb{E}\Phi(S_{n,p}(G)) \le \quad c''_{E,\Phi}\,\mathbb{E}\Phi(S_{n,p}(M)) + c'''_{E,\Phi}\mathbb{E}\Phi\big(|m_n|\big) \le c''''_{E,\Phi}\,\mathbb{E}\Phi(S_{n,p}(M)). \quad \text{(A.12)}$$

Therefore, substituting Eqs. A.9, A.11 and A.12 in Eq. A.8 gives

$$\mathbb{E}\Phi(M_n^*) \le \bar{c}_{E,\Phi}\mathbb{E}\Phi(S_{n,p}(M)), \quad 1 \le n \le N. \quad \text{(A.13)}$$

□

Appendix B: Preliminaries About Convex Functions

We summarise in this paragraph some facts about convex functions, which we have used in the proof of Claim 3.1. For a more detailed summary, we refer to Appendix [19, p. 218, (6)] or book [27]).

Let $\Phi : [0,\infty) \to \mathbb{R}$ be a strictly increasing convex function with $\Phi(0) = 0$. By e.g. Theorem A [27] it follows that there exists a strictly increasing function $\phi : [0,\infty) \to \mathbb{R}_+$, such that

$$\Phi(t) = \int_0^t \phi(s)\,ds, \quad t \ge 0.$$

To such a convex function Φ we can associate another convex function Ψ of the same type such that

$$\Psi(t) = \int_0^t \psi(s)\,ds, \quad t \ge 0.$$

and $\phi(s) = \inf\{t \ge 0 : \psi(t) \ge s\}$ and $\psi(s) = \inf\{t \ge 0 : \phi(t) \ge s\}, s \ge 0$. This function is usually called the to Φ conjugate function (in the sense of Young) (see e.g. Chapter I.15 [27]). Particularly, the following holds (see e.g. Chapter I.15, p. 30 [27]).

Proposition B.1 *Let $\Phi : [0,\infty) \to \mathbb{R}$ be a strictly increasing convex function and $\Psi : [0,\infty) \to \mathbb{R}$ its conjugate. Then*

$$u\,\phi(u) = \Phi(u) + \Psi(\phi(u)), \quad \text{(B.1)}$$

$$\int_0^v t\,d\psi(t) = \Phi(\psi(v)), \quad \text{(B.2)}$$

$$u\,v \le \Phi(u) + \Psi(v), \quad \text{(B.3)}$$

$$\Phi(ua) \le a\Phi(a), \quad \forall 0 < a \le 1, \quad \text{(B.4)}$$

Furthermore, we say Φ satisfies *the growth condition*, iff there exists a constant $C < \infty$ with

$$\Phi(2\lambda) \le C\,\Phi(\lambda), \quad \lambda \in [0,\infty).$$

If the growth condition holds, then (see e.g. Appendix, p. 218, (7) [19])

$$c_\Phi := \sup_{u>0} \frac{u\phi(u)}{\Phi(u)} \tag{B.5}$$

is finite and we get

$$\Phi(t_1 \vee t_2) \le \Phi(t_1) + \Phi(t_2), \quad t_1, t_2 \ge 0, \tag{B.6}$$

$$\Phi(rt) \le r^{c_\Phi}\,\Phi(t), \quad t \ge 0,\ r \ge 1, \tag{B.7}$$

$$\Psi(t) \le (c_\Phi - 1)\,\Phi(\psi(t)), \quad t \ge 0. \tag{B.8}$$

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