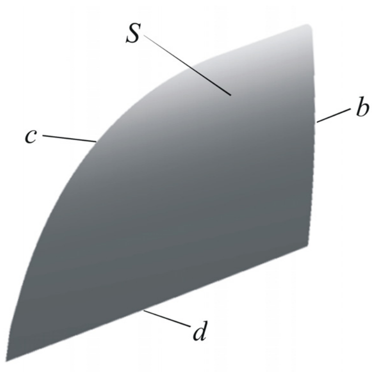
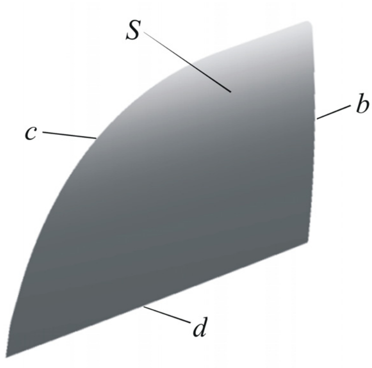
Computational Line Geometry as a Tool for Solving Engineering Problems

International Workshop on Line Geometry and Kinematics

$\Pi\acute{\alpha}\varphi o\varsigma$, 2011 - 04 -29

Car Side Windows: The Task


S
b
c
d



S ... window surface
d ... daylight line
b ... b-pillar
c ... roof line

Given: A surface S;
(side window suggested by the stylist)

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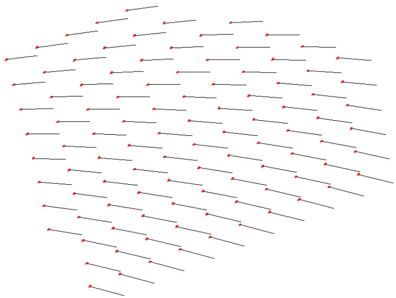
(side window suggested by the stylist)



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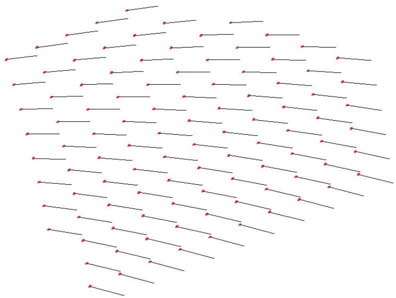
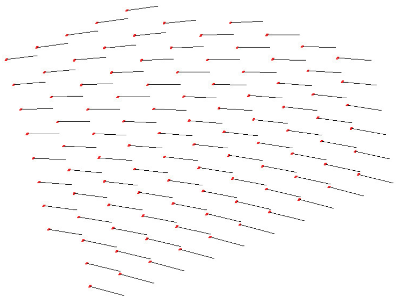


Can S be moved in itself?

Given: A surface S;
(side window suggested by the stylist)



Can S be moved in itself?

In general: no!

Which spatial curves can be moved in themselves?

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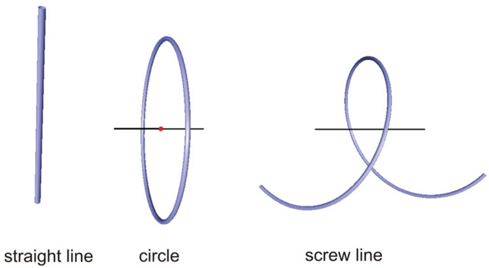
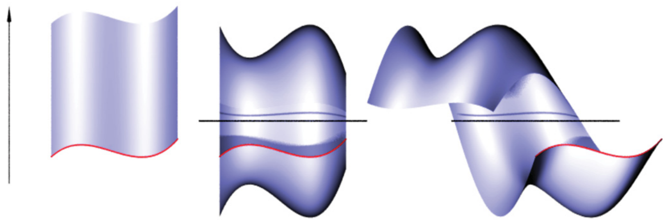
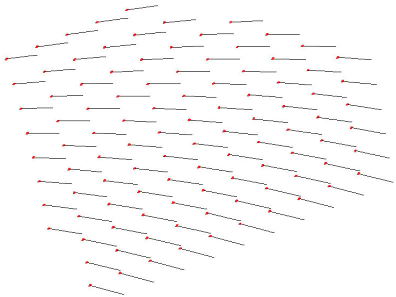


Which surfaces can be moved in themselves?

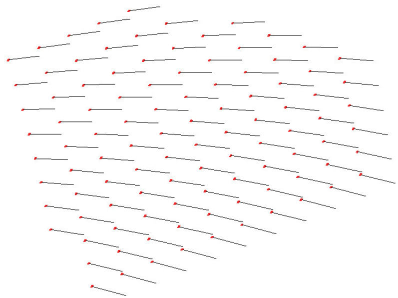
Which surfaces can be moved in themselves?



general cylinder surface of revolution screw surface

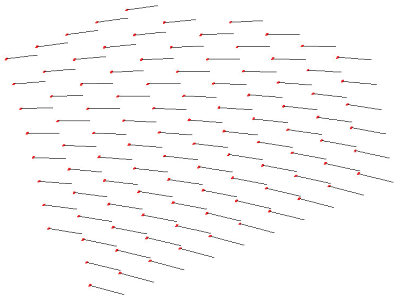


Which screw motion best fits the surface S?



Which screw motion best fits the surface S?

axis a, screw parameter p

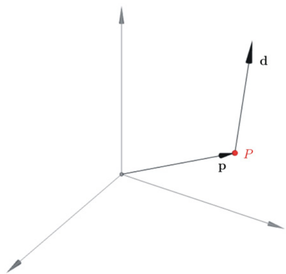


Which screw motion best fits the surface S?

axis a, screw parameter p

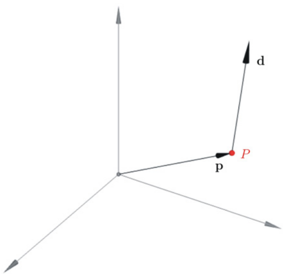
How to find this screw motion?

Car Side Windows: Obtaining the Optimal Screw Motion



straight line g:

$$\mathbf{p} = \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} d_{i1} \\ d_{i2} \\ d_{i3} \end{bmatrix}$$



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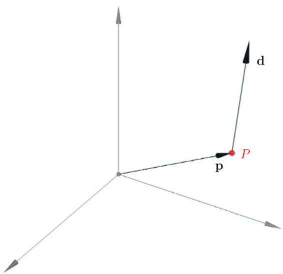
point G: $\begin{bmatrix} \mathbf{g} \\ \overline{\mathbf{g}} \end{bmatrix}$, $\mathbf{g} = \mathbf{d}$, $\overline{\mathbf{g}} = \mathbf{p} \times \mathbf{d}$

3-dimensional space $\mathbb{E}^3$



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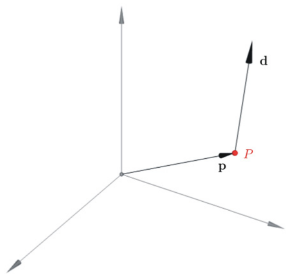

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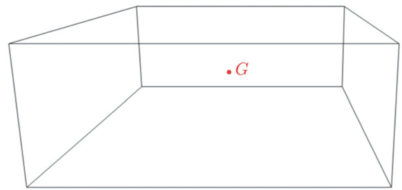
5-dimensional space $\mathbb{P}^5$



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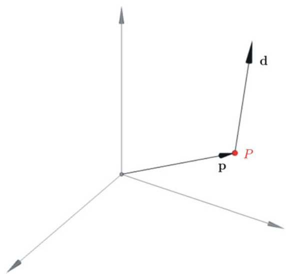

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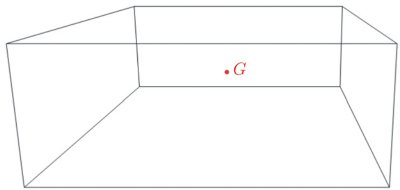

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point G: $\begin{bmatrix} \mathbf{g} \\ \overline{\mathbf{g}} \end{bmatrix}$, $\mathbf{g} = \mathbf{d}$, $\overline{\mathbf{g}} = \mathbf{p} \times \mathbf{d}$

Plücker condition: $\langle \mathbf{g}, \overline{\mathbf{g}} \rangle = g_1\overline{g}_1 + g_2\overline{g}_2 + g_3\overline{g}_3 = 0$

Klein quadric $\mathcal{M}_2^4 \subset \mathbb{P}^5$

3-dimensional space $\mathbb{E}^3$

screw motion M

3-dimensional space $\mathbb{E}^3$

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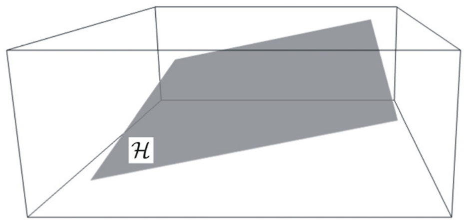
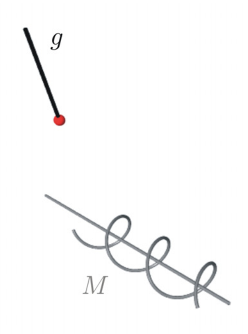
5-dimensional space $\mathbb{P}^5$



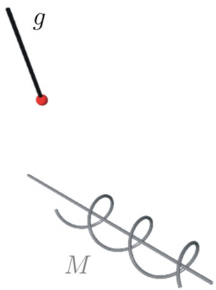
hyperplane $\mathcal{H}$

3-dimensional space $\mathbb{E}^3$



normal g belongs to the linear complex $\mathcal{L}_M$

3-dimensional space $\mathbb{E}^3$



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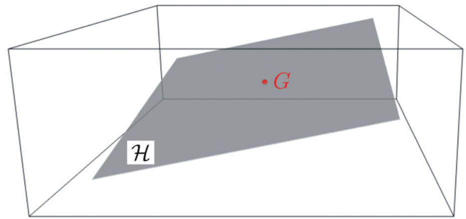
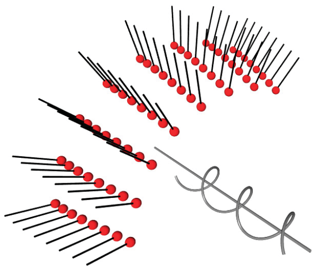
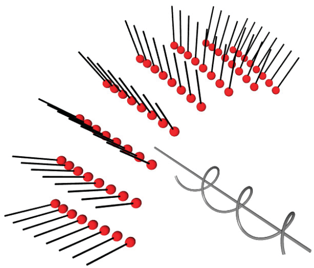


Point G lies in the hyperplane $\mathcal{H}$

3-dimensional space $\mathbb{E}^3$



3-dimensional space $\mathbb{E}^3$



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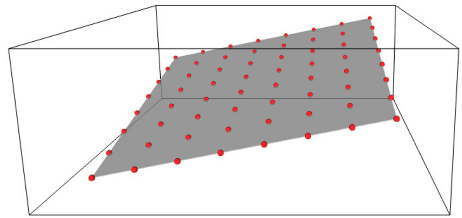

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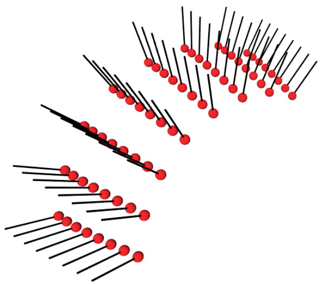

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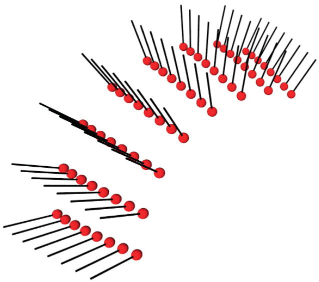

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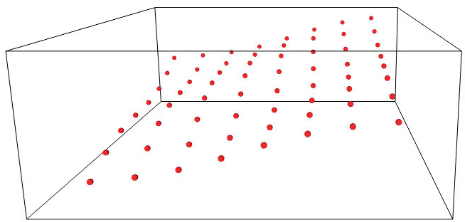

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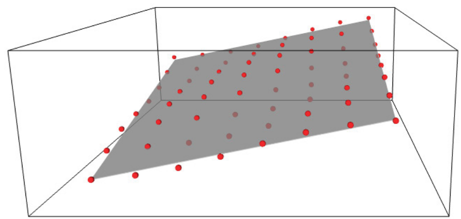

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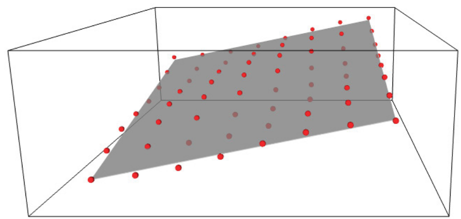
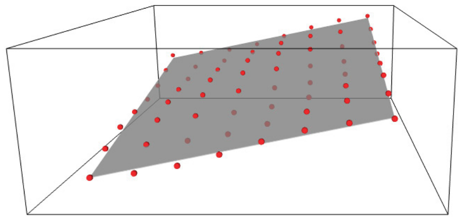


hyperplane of regression $\mathcal{H}$

3-dimensional space $\mathbb{E}^3$

5-dimensional space $\mathbb{P}^5$



yields the optimal screw motion

hyperplane of regression $\mathcal{H}$

$$\mathbf{v} \cdot \mathbf{g_i} + \mathbf{w} \cdot \overline{\mathbf{g_i}} = 0, \quad i = 1, \ldots, n$$

$$\mathbf{v} \cdot \mathbf{g_i} + \mathbf{w} \cdot \overline{\mathbf{g_i}} \quad = \quad 0, \;\; i = 1, \ldots, n$$

minimize the squared error function

$$e(\mathbf{v}, \mathbf{w}) \quad := \quad \left[\mathbf{v}^\top, \mathbf{w}^\top\right] \cdot \mathbf{S} \cdot \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}$$

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where **S** is the positive semidefinite 6×6-matrix *"scatter matrix"*:

$$\mathbf{S} \quad = \quad \begin{bmatrix} \mathbf{g}_1 & \cdots & \mathbf{g}_n \\ \overline{\mathbf{g}}_1 & \cdots & \overline{\mathbf{g}}_n \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_1^\top & \overline{\mathbf{g}}_1^\top \\ \vdots & \vdots \\ \mathbf{g}_n^\top & \overline{\mathbf{g}}_n^\top \end{bmatrix} \quad = \quad \begin{bmatrix} \sum_{i=1}^n \mathbf{g}_i \mathbf{g}_i^\top & \sum_{i=1}^n \mathbf{g}_i \overline{\mathbf{g}}_i^\top \\ \sum_{i=1}^n \overline{\mathbf{g}}_i \mathbf{g}_i^\top & \sum_{i=1}^n \overline{\mathbf{g}}_i \overline{\mathbf{g}}_i^\top \end{bmatrix}$$

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subject to the normalizing constraint:

$$\left[\mathbf{v}^\top, \mathbf{w}^\top\right] \cdot \mathbf{E}_6 \cdot \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} \quad = \quad w_1^2 + w_2^2 + w_3^2 \quad = \quad 1$$

$$\mathbf{v} \cdot \mathbf{g_i} + \mathbf{w} \cdot \overline{\mathbf{g_i}} \quad = \quad 0, \quad i = 1, \ldots, n$$

minimize the squared error function

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where $\mathbf{E}_6 = \begin{bmatrix} \mathbf{I}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \mathbf{O}_3 \end{bmatrix}$

Lagrangian multiplier method yields

$$(\mathbf{S} - \lambda \mathbf{E}_6) \cdot \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

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generalized eigenvalue problem

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generalized eigenvalue problem

determine the eigenvector space for the smallest eigenvalue λ_0:
$[\mathbf{w}_0, \mathbf{v}_0]$ solves the optimization problem

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screw parameter: $p = \dfrac{\langle \mathbf{w}_0, \mathbf{v}_0 \rangle}{\langle \mathbf{w}_0, \mathbf{w}_0 \rangle}$

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direction vector of the screw axis a: $\mathbf{d} = \mathbf{w}_0$

Lagrangian multiplier method yields

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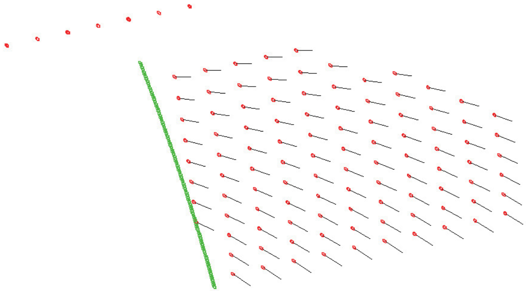
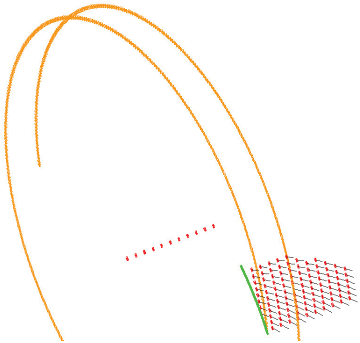
direction vector of the screw axis a: $\mathbf{d} = \mathbf{w}_0$

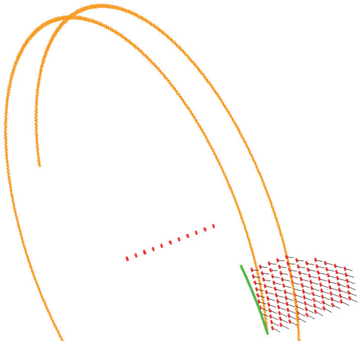
point on a: $\mathbf{a} = \dfrac{\mathbf{w}_0 \times \mathbf{v}_0}{\langle \mathbf{w}_0, \mathbf{w}_0 \rangle}$

Car Side Windows: Additional Constraints

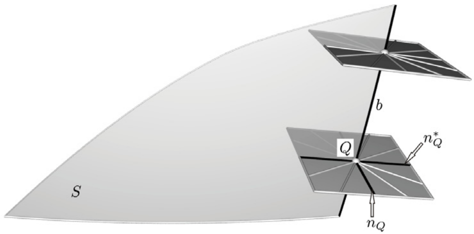


boundary curve b (b-pillar)

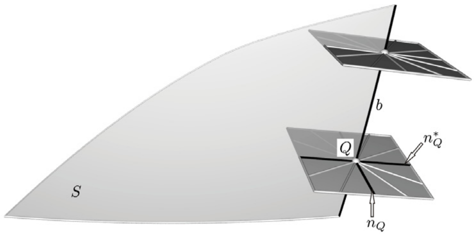




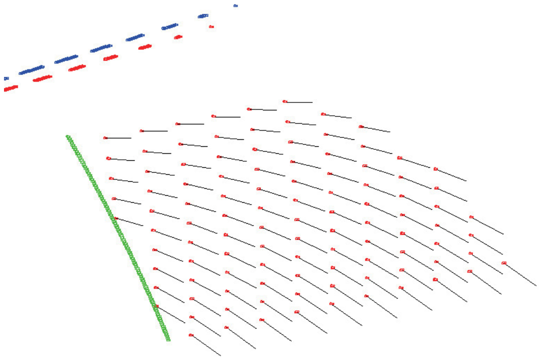
b is not! a trajectory of the optimal screw motion M



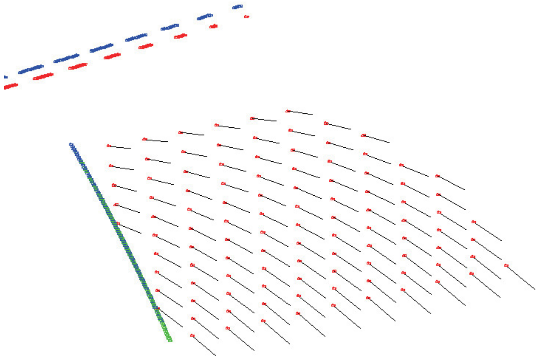
b
Q
n_Q^*
S
n_Q



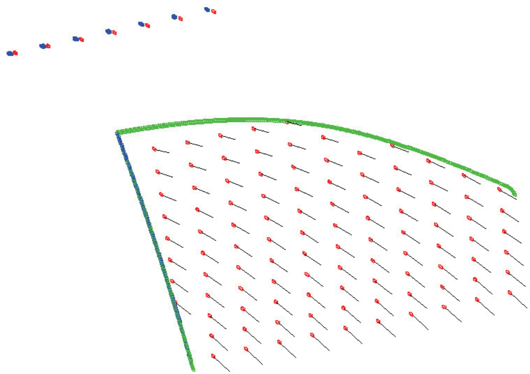
add some normals of the prescribed b-pillar *b* to the interpolation problem input



computing the optimal screw motion for both, S and b



delivers some replacement for the b-pillar b



the optimal screw motion
delivers the optimal side window sheet (screw surface S)
out of the roofline c

Car Side Windows: Benefits

Car Side Windows: Benefits

engineering workload
    constructing the side window

Car Side Windows: Benefits

engineering workload
    constructing the side window ↓

Car Side Windows: Benefits

engineering workload
    constructing the side window ⬇

quality of the outcome

Car Side Windows: Benefits

engineering workload
    constructing the side window ↓

quality of the outcome ↑

Car Side Windows: Benefits

engineering workload ⬇
constructing the side window

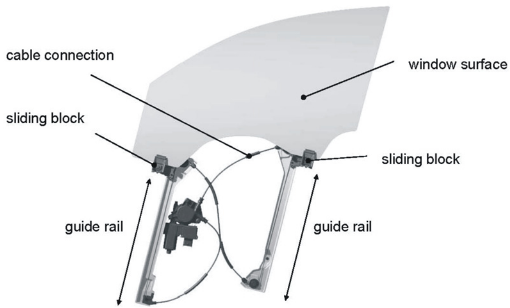
quality of the outcome ⬆

serial production cost

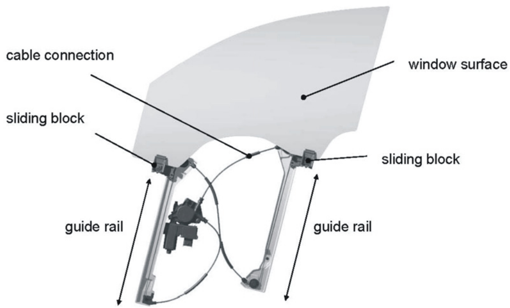
Car Side Windows: Benefits

engineering workload constructing the side window	↓
quality of the outcome	↑
serial production cost	→

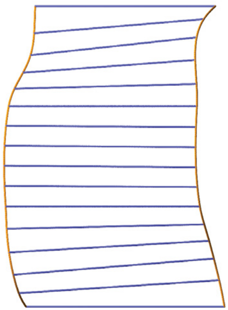
Rope Ladder Problem: Problem Formulation

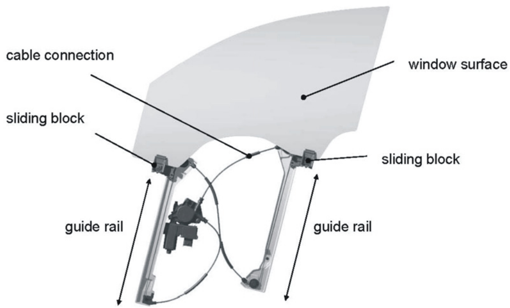

cable connection
window surface
sliding block
sliding block
guide rail
guide rail

How can you move a rod so that its endpoint paths have equal length?



How can you move a rod so that its endpoint paths have equal length?





$$A \dots \vec{OA} = \mathbf{a}(t),\ B \dots \vec{OB} = \mathbf{b}(t)$$

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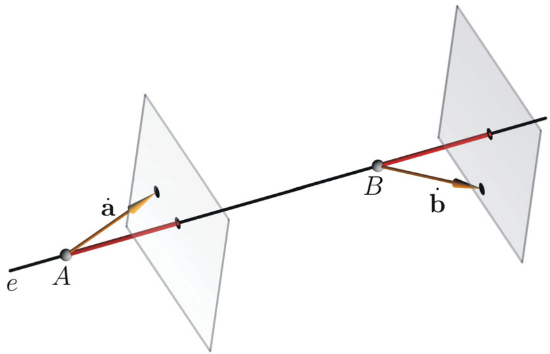

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$\mathrm{dist}^2(A, B) = \langle \mathbf{b} - \mathbf{a}, \mathbf{b} - \mathbf{a} \rangle = d^2 = const.$

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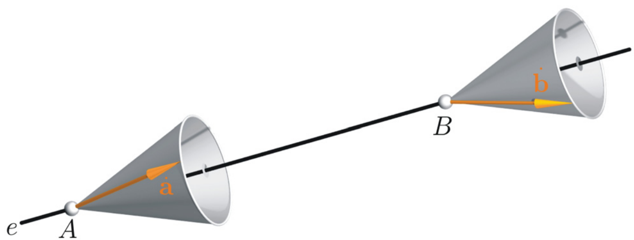
$|\dot{\mathbf{a}}| = |\dot{\mathbf{b}}| \implies \angle(\vec{AB}, \dot{\mathbf{a}}) = \angle(\vec{AB}, \dot{\mathbf{b}})$



Σ . . . moving system

Σ ... moving system

Σ^* ... fixed system

Σ ... moving system

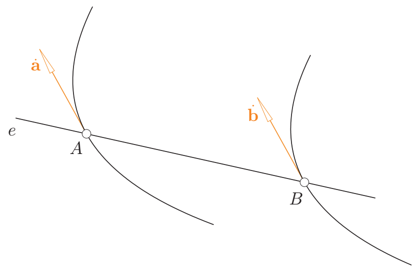
Σ^* ... fixed system

Σ/Σ^* ... motion

Rope Ladder Problem: The Planar Case

planar case A

$\angle(\vec{AB}, \dot{\mathbf{a}}) = \angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$



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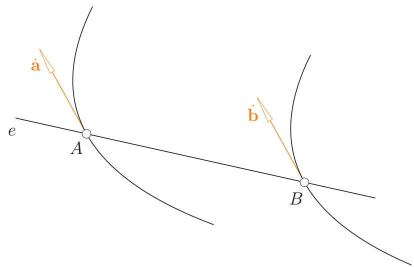
$\dot{\mathbf{a}} = \dot{\mathbf{b}}$ for all $t \in [t_0, t_1]$



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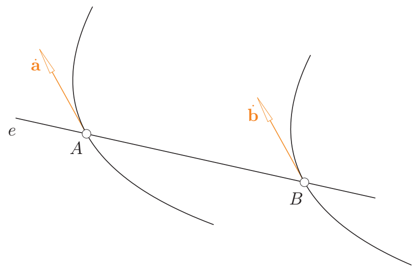
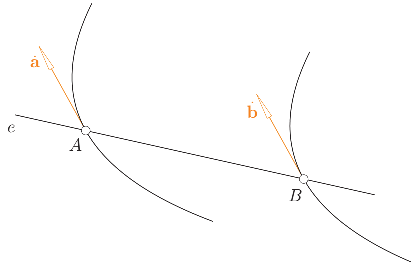


$\dot{\mathbf{a}} = \dot{\mathbf{b}}$ for all $t \in [t_0, t_1]$

curved translation



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$\dot{\mathbf{a}} = \dot{\mathbf{b}}$ for all $t \in [t_0, t_1]$

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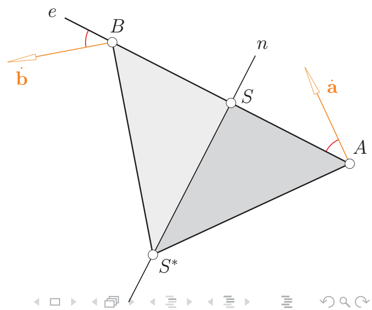
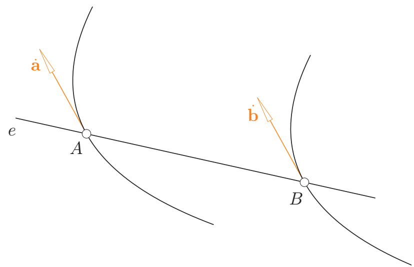


planar case B

$\angle(\vec{AB}, \dot{\mathbf{a}}) = -\angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$



planar case A

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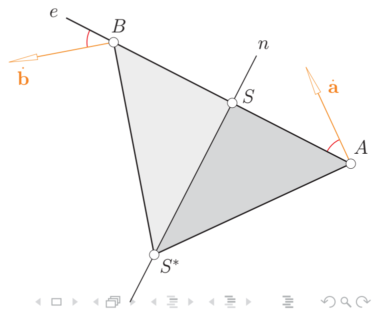
curved translation

planar case B

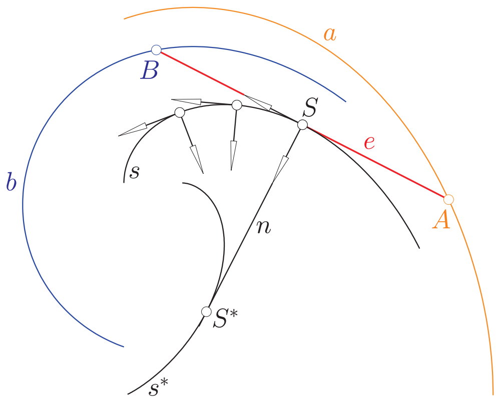
$\angle(\vec{AB}, \dot{\mathbf{a}}) = -\angle(\vec{AB}, \dot{\mathbf{b}})$ for all $t \in [t_0, t_1]$

bisector n of AB = moving polhode



planar case B

Σ/Σ^* ... motion of a straight line n rolling on a curve s^*

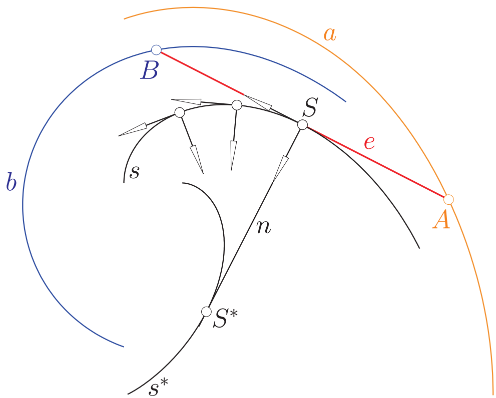


planar case B

Σ/Σ^* ... motion of a straight line n rolling on a curve s^*

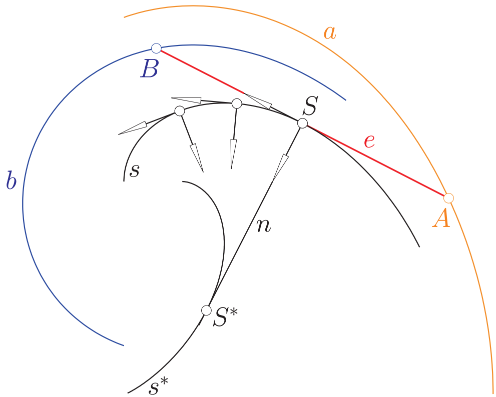
S ... midpoint of AB

Σ/Σ^* ... Frenet motion along the path s of S

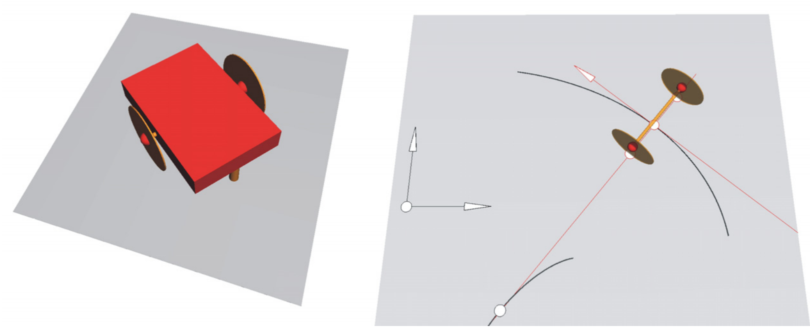


planar case B

s is a tractrix with respect to *a* and *b*



mobile robot with two wheels



Rope Ladder Problem: The Spatial Case

ruled surface Φ generated by the motion of $e = AB$:

$\mathbf{y}(t, u) = \mathbf{x}(t) + u\mathbf{e}(t)$ with $\langle \mathbf{e}, \mathbf{e} \rangle = 1$

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$B \ldots\ \mathbf{b}(t) = \mathbf{y}(t,a+d)$

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$\mathbf{y}(t, u) = \mathbf{x}(t) + u\mathbf{e}(t)$ with $\langle \mathbf{e}, \mathbf{e} \rangle = 1$

$A \ldots \; \mathbf{a}(t) = \mathbf{y}(t, a)$
$B \ldots \; \mathbf{b}(t) = \mathbf{y}(t, a + d)$

$|\dot{\mathbf{a}}| = |\dot{\mathbf{b}}| \implies (2a + d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$

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spatial case A

$$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$$

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Φ is a cylinder (trivial case)

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spatial case A

$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$

Φ is a cylinder (trivial case)

spatial case B

$\dot{\mathbf{x}} = 0 \iff \mathbf{x} = \mathbf{c} = \mathbf{s}$

$$(2a+d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

spatial case A

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Φ is a cylinder (trivial case)

spatial case B

$$\dot{\mathbf{x}} = 0 \iff \mathbf{x} = \mathbf{c} = \mathbf{s}$$

Φ is a cone and A, B are symmetric w.r.t. its vertex S (trivial case)

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spatial case C

$\dot{\mathbf{e}}, \dot{\mathbf{x}} \neq 0$

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spatial case C

$\dot{\mathbf{e}}, \dot{\mathbf{x}} \neq 0$

$$a + \frac{d}{2} = -\frac{\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle}{\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle}$$

$$(2a + d)\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle = -2\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle$$

spatial case A

$\dot{\mathbf{e}} = 0 \iff \mathbf{e} = \mathbf{c}$

Φ is a cylinder (trivial case)

spatial case B

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The midpoint S of AB is the striction (cuspidal) point on $e = AB$.

An Example: screw motion

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all points on a common right cylinder around the screw axis have paths of equal length

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all points on a common right cylinder around the screw axis have paths of equal length

striction curve $\mathbf{s}$ is the helix generated by S

Rope Ladders: An interpolation problem

Given: series of positions $A_i B_i$, $i = 1, \ldots, n$ of the rod AB

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Construction:

Step 1: Find a suitable curve s interpolating the midpoints S_i of $A_i B_i$

Step 2: Find a ruled surface Φ that interpolates $e_i = A_i B_i$ and whose striction curve is s

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$\tau \ldots$ arclength on s: $\mathbf{s} = \mathbf{s}(\tau)$;

$\langle \mathbf{s}', \mathbf{s}' \rangle = 1$;

$\mathbf{s}(\tau_i) = \mathbf{s}_i$

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$$\langle \mathbf{s}', \mathbf{e}' \rangle = 0 \qquad \text{(S)}$$

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$$\langle \mathbf{s}', \mathbf{e}' \rangle = 0 \qquad \text{(S)}$$

$$\langle \mathbf{e}, \mathbf{e} \rangle = 1 \qquad \text{(U)}$$

$\sigma := \angle(\mathbf{e}, \mathbf{s}')$ ... striction of Φ

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$\mathbf{t} = \mathbf{s}'$, $\mathbf{h} = \frac{1}{|\mathbf{s}''|}\mathbf{s}''$, $\mathbf{b} = \mathbf{t} \times \mathbf{p}$ ... Frenet frame of s

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$$\begin{array}{lllll} \varepsilon & \ldots & \langle \mathbf{s}', \mathbf{e} \rangle & - & \cos\sigma & = & 0 \\ \varepsilon_1 & \ldots & \langle \mathbf{s}'', \mathbf{e} \rangle & + & \sigma' \sin\sigma & = & 0 \end{array}$$

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developable surface Γ with generators g parallel to $\mathbf{b}$

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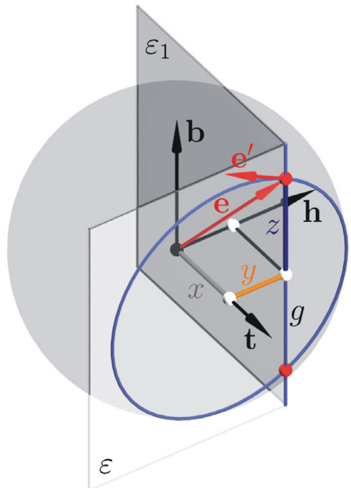
developable surface Γ with generators g parallel to $\mathbf{b}$

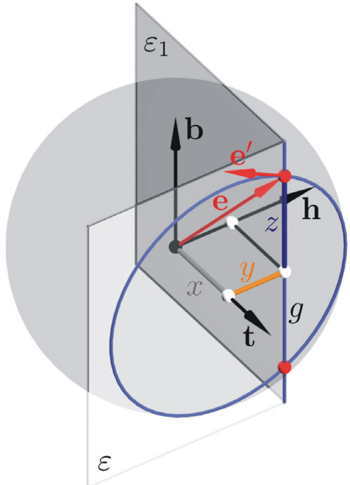
$$\varepsilon_1 \quad \ldots \quad \langle \mathbf{h}, \mathbf{e} \rangle \quad + \quad \frac{\sigma' \sin\sigma}{\kappa} \quad = \quad 0$$

The intersection of Γ with the unit sphere contains the spherical generator image $\mathbf{e} = \mathbf{e}(s)$ of Φ



ε_1
b
e′
e
h
z
y
x
g
t
ε

$$x = \cos\sigma$$
$$y = -\frac{\sigma' \sin\sigma}{\kappa}$$
$$z = \pm\sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}$$



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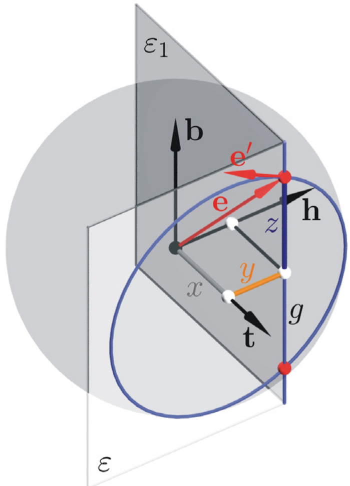
$$y = -\frac{\sigma' \sin\sigma}{\kappa}$$
$$z = \pm\sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}$$



$$\mathbf{e} = \cos\sigma\cdot\mathbf{t} - \sigma'\sin\sigma\cdot\mathbf{h} \pm \sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

$$\mathbf{e} \;=\; \cos\sigma\cdot\mathbf{t} \;-\; \sigma'\sin\sigma\cdot\mathbf{h} \;\pm\; \sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

$$\mathbf{e} \;=\; \cos\sigma\cdot\mathbf{t} \;-\; \sigma'\sin\sigma\cdot\mathbf{h} \;\pm\; \sin\sigma\sqrt{1-\tfrac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

Special Cases

$$\mathbf{e} \;=\; \cos\sigma\cdot\mathbf{t} \;-\; \sigma'\sin\sigma\cdot\mathbf{h} \;\pm\; \sin\sigma\sqrt{1-\tfrac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

Special Cases

- $\sigma' = 0$: Φ is a ruled surface of constant striction; $\mathbf{e} \in [\mathbf{t}, \mathbf{b}]$

$$\mathbf{e} \;=\; \cos\sigma\cdot\mathbf{t} \;-\; \sigma'\sin\sigma\cdot\mathbf{h} \;\pm\; \sin\sigma\sqrt{1-\tfrac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

Special Cases

- $\sigma' = 0$: Φ is a ruled surface of constant striction; $\mathbf{e} \in [\mathbf{t}, \mathbf{b}]$
- $\sigma = 0$: Φ is the tangent surface of s

$$\mathbf{e} \;=\; \cos\sigma \cdot \mathbf{t} \;-\; \sigma' \sin\sigma \cdot \mathbf{h} \;\pm\; \sin\sigma\sqrt{1 - \tfrac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

Special Cases

- $\sigma' = 0$: Φ is a ruled surface of constant striction;
  $\mathbf{e} \in [\mathbf{t}, \mathbf{b}]$

- $\sigma = 0$: Φ is the tangent surface of *s*

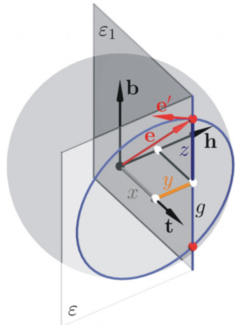
- $\kappa = 0$ (striction curve *s* is a straight line):
  a solution is possible only if $\sigma' = 0$, i.e.; $\sigma = const$.
  Φ is a ruled surface of constant slope and a straight line as striction curve.

$$\mathbf{e} \;=\; \cos\sigma\cdot\mathbf{t} \;-\; \sigma'\sin\sigma\cdot\mathbf{h} \;\pm\; \sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

$$\mathbf{e} = \cos\sigma\cdot\mathbf{t} - \sigma'\sin\sigma\cdot\mathbf{h} \pm \sin\sigma\sqrt{1-\frac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

$$\varepsilon \quad \ldots \quad \langle\mathbf{t},\mathbf{e}\rangle - \cos\sigma = 0$$

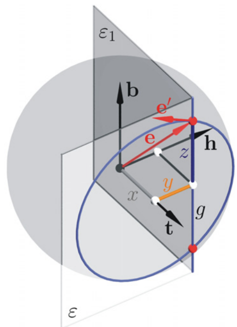
$$\varepsilon_1 \quad \ldots \quad \langle\mathbf{h},\mathbf{e}\rangle + \frac{\sigma'\sin\sigma}{\kappa} = 0$$



$$\mathbf{e} \quad = \quad \cos\sigma \cdot \mathbf{t} \quad - \quad \sigma' \sin\sigma \cdot \mathbf{h} \quad \pm \quad \sin\sigma \sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

$$\varepsilon \quad \dots \quad \langle \mathbf{t}, \mathbf{e} \rangle \quad - \quad \cos\sigma \quad = \quad 0$$

$$\varepsilon_1 \quad \dots \quad \langle \mathbf{h}, \mathbf{e} \rangle \quad + \quad \frac{\sigma' \sin\sigma}{\kappa} \quad = \quad 0$$

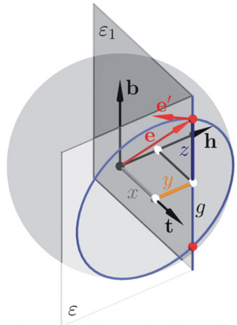


Construct a striction function σ with

$$\mathbf{e} \quad = \quad \cos\sigma \cdot \mathbf{t} \quad - \quad \sigma' \sin\sigma \cdot \mathbf{h} \quad \pm \quad \sin\sigma \sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

$$\varepsilon \quad \ldots \quad \langle \mathbf{t}, \mathbf{e} \rangle \quad - \quad \cos\sigma \quad = \quad 0$$

$$\varepsilon_1 \quad \ldots \quad \langle \mathbf{h}, \mathbf{e} \rangle \quad + \quad \frac{\sigma' \sin\sigma}{\kappa} \quad = \quad 0$$

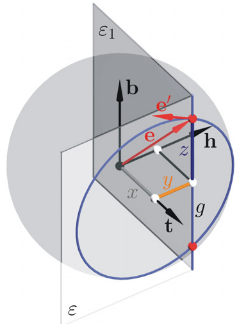


Construct a striction function σ with

$$\sigma'(\tau)^2 \quad \leq \quad \kappa^2(\tau)$$

$$\mathbf{e} \;=\; \cos\sigma\cdot\mathbf{t} \;-\; \sigma'\sin\sigma\cdot\mathbf{h} \;\pm\; \sin\sigma\sqrt{1-\tfrac{\sigma'^2}{\kappa^2}}\cdot\mathbf{b}$$

$$\begin{aligned} \varepsilon \quad &\ldots \quad \langle \mathbf{t},\mathbf{e}\rangle - \cos\sigma = 0 \\ \varepsilon_1 \quad &\ldots \quad \langle \mathbf{h},\mathbf{e}\rangle + \frac{\sigma'\sin\sigma}{\kappa} = 0 \end{aligned}$$



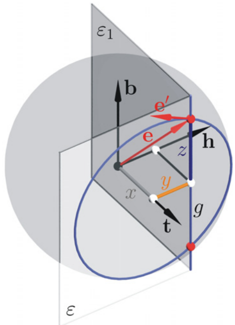
Construct a striction function σ with

$$\sigma'(\tau)^2 \;\leq\; \kappa^2(\tau)$$

$$\left.\begin{aligned} \sigma(\tau_i) &= \arccos\langle \mathbf{s}'(\tau_i),\mathbf{e}_i\rangle \\ \sigma'(\tau_i) &= -\frac{\langle \mathbf{s}''(\tau_i),\mathbf{e}_i\rangle}{\sin\sigma(\tau_i)} \end{aligned}\right\},\; i=1,\ldots,n$$

$$\mathbf{e} \quad = \quad \cos\sigma \cdot \mathbf{t} \; - \; \sigma' \sin\sigma \cdot \mathbf{h} \; \pm \; \sin\sigma\sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot \mathbf{b}$$

$$\begin{array}{llcrcl} \varepsilon & \dots & \langle \mathbf{t}, \mathbf{e} \rangle & - & \cos\sigma & = 0 \\ \varepsilon_1 & \dots & \langle \mathbf{h}, \mathbf{e} \rangle & + & \dfrac{\sigma' \sin\sigma}{\kappa} & = 0 \end{array}$$



Construct a striction function σ with

$$\sigma'(\tau)^2 \quad \leq \quad \kappa^2(\tau)$$

$$\left.\begin{array}{rcl} \sigma(\tau_i) & = & \arccos\langle \mathbf{s}'(\tau_i), \mathbf{e}_i \rangle \\ \sigma'(\tau_i) & = & -\dfrac{\langle \mathbf{s}''(\tau_i), \mathbf{e}_i \rangle}{\sin\sigma(\tau_i)} \end{array}\right\}, \; i = 1, \dots, n$$

Construct σ as a Hermite interpolant.

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